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FIXED POINT THEOREMS WITH $\mathcal{P}$-C-CONTRACTION IN PARTIALLY ORDERED MODULAR METRICS SPACES

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Abstract: This study aims to broaden the understanding of $\mathcal{P}$-C contracts in the context of partially ordered modular metric spaces. In both monotonic and non-monotonic mappings in these spaces, fixed point results have been obtained using this idea. An illustration is provided to demonstrate our key findings. All findings are new and generalize the results of Chaipuniya et al. [12] and Amor et al. [8].

Keywords and Phrases: Fixed point, $\mathcal{P}$-C-contraction, Modular metric space, Partially ordered set.

2020 Mathematics Subject Classification: 47H09, 47H10, 46A80.

1. Introduction

Fixed point theorems are useful in the investigation of existence of solutions of differential equations, integral equations, partial differential equations, linear and non-linear simultaneous equations and difference equations. Fixed points are therefore of paramount importance in many areas of mathematical and physical sciences. Due to the wide applications of fixed point theory, this topic has become the keen interest of research in the field of pure mathematics as well as in the field of applied mathematics (see [19, 31]). The notion of modular spaces has been introduced by Chistyakov [17] and outlined their theory for an arbitrary non-empty set ([14, 15, 16, 17]). He also demonstrated that how the metric function may be generalized into the metric modular function. For each $\lambda > 0$, the distance function $\varsigma(p, q)$ has been substituted by $\omega_\lambda(p, q)$. It is easier to apply to use these spaces in variety of research areas because the modular convergence, modular limit, and modular completeness are "weaker" in modular metric space than in metric space which improves the applicability of such spaces. Karapinar et al. [4, 5, 9] used this metric space to find fixed points of various contractive mappings. Following that, the generalisation of such spaces includes modular b-metric space, modular A-metric space, generalised metric space, etc., and fixed point and linked fixed point solutions are demonstrated (see [1, 2, 11, 18, 20, 32]).

The concept of contractive mapping known as $\mathcal{C}$-contraction was first described by Chatterjea in [13]. Alber and Guerre-Delabriere generalized the famous Banach contraction principle [10], known as weakly-contraction in the setting of Hilbert spaces. Pacurar and Rus [28] gave a characterization of ϕ-contraction mappings in the context of cyclic operators and proved some fixed point results for such mappings on a complete metric space. As a natural next step, Karapinar [23] generalized the results in [28] by replacing the notion of cyclic ϕ-contraction mappings with cyclic weak ϕ-contraction mappings. Many more results for cyclic mapping analysis have been proved (see in [26, 25]). Choudhury [18] also extended the idea of $\mathcal{C}$-contraction to weakly $\mathcal{C}$-contraction and demonstrated a few fixed point theorems. Numerous authors defined the relationships on spaces using terms like, if M is a partially ordered set and is denoted by $(M, \sqsubseteq)$. Any two elements in M are said to be similar if $(M, \sqsubseteq)$ is a totally ordered set. Additionally, if each component of a convergent sequence in M is equivalent to its limit, the set is said to be sequentially ordered. A partially ordered metric space is one where $(M, \sqsubseteq)$ is a partially ordered set and (M, ς) is a metric space. Ran and Reurings [29] established various fixed point theorems for a monotone mapping on full partially ordered metric spaces. For outstanding work on fixed point outcomes on partially ordered metric spaces see [3, 7, 21, 22, 24, 30].

Chaipuniya et al. [12] introduced a new type of contractive condition, which is defined on an ordered space and named it as $\mathcal{P}$-contraction. Amor et al. [8] generalized this contractive condition and studied a new class of mappings and introduced the concept of a $\mathcal{P}$-$\mathcal{C}$-contraction. We extended this concept and introduced the idea of $\mathcal{P}$-$\mathcal{C}$-contraction in partially ordered modular metric spaces, which generalised the findings of Amor et al. [8] in this new context. The definitions and principles of partially ordered metric spaces, modular metric spaces, and several kinds of mappings are now covered.

2. Preliminaries

Chistyakov in [17], introduced the notion of modular metric spaces as follows:

Definition 1. ([17]) *Consider a nonempty set M. A function $\omega\colon (0,+\infty)\times M\times M\to[0,+\infty]$ is called a metric modular on M, if for all $p,q,r\in M$ it satisfies the following conditions:*

(i) $\omega_\lambda(p,q)=0$ *for all* $\lambda>0$ *if and only if* $p=q$;

(ii) $\omega_\lambda(p,q)=\omega_\lambda(q,p)$ *for all* $\lambda>0$;

(iii) $\omega_{\lambda+\mu}(p,q)\le\omega_\lambda(p,r)+\omega_\mu(r,q)$ *for all* $\lambda,\mu>0$.

And $(M_\omega,\omega_\lambda)$ *is called a modular metric spaces.*
If **(i)** is replaced by the following condition
(i') $\omega_\lambda(p,p)=0$ for all $\lambda>0, p\in M$, then ω is called pseudomodular (metric) on M.
A modular ω on a set M is called regular if the following weaker version of (i) is satisfied:
$p=q$ if and only if $\omega_\lambda(p,q)=0$ for some $\lambda>0$. Finally, ω is said to be convex if, instead of (iii), for all $\lambda,\mu>0$ and $p,q,r\in M$ it satisfies the inequality

$$\textbf{(iv)}\ \ \omega_{\lambda+\mu}(p,q)\le\frac{\lambda}{\lambda+\mu}\omega_\lambda(p,r)+\frac{\mu}{\lambda+\mu}\omega_\mu(r,q).$$

A convex psedomodular ω on M has the following additional property: given $p,q\in M$, we have

$$\text{if }\ 0<\mu\le\lambda,\ \text{ then }\ \omega_\lambda(p,q)\le\omega_{\lambda-\mu}(p,p)+\omega_\mu(p,q)=\omega_\mu(p,q).$$

An important property of the (metric) psedomodular on set M is that the mapping $\lambda\mapsto\omega_\lambda(p,q)\in[0,+\infty]$ is non-increasing on $(0,+\infty)$ for all $p,q\in M$.

Definition 2. ([17]) *Consider a psedomodular ω on M for a fixed $p_0\in M$. The sets*

$$M^*_\omega=M^*_\omega(p_0)=\{p\in M:\omega_\lambda(p,p_0)\to0\ \textit{as}\ \lambda\to+\infty\}.$$

$$M_\omega=M_\omega(p_0)=\{p\in M:\exists\,\lambda=\lambda(p)>0\ \textit{and}\ \omega_\lambda(p,p_0)<+\infty\}.$$

are called modular metric spaces (around p_0*).*
Also, the modular space M_ω can be equipped with a (nontrivial) metric ς_ω, generated by ω, which is given by

$$\varsigma_\omega(p,q) = \inf\{\lambda > 0 : \omega_\lambda(p,q) \leq \lambda\}, \quad p,q \in M_\omega.$$

If a convex modular ω on M, the two modular spaces coincide, $M_\omega = M^*_\omega$, and this common set can be endowed with a metric ς_ω given by

$$\varsigma_\omega(p,q) = \inf\{\lambda > 0 : \omega_\lambda(p,q) \leq 1\}, \quad p,q \in M_\omega.$$

And for a non-convex modular ω on M, $\varsigma^*_\omega(p,p) = 0$ and $\varsigma_\omega(p,q) = \varsigma_\omega(q,p)$.

Definition 3. ([17]) *Consider a modular metric space* M_ω *and a sequence* $\{p_n\}_{n\in\mathbb{N}}$ *of* M_ω*. Then*

(i) $\{p_n\}_{n\in\mathbb{N}}$ *from* M_ω *is called a modular convergent to an element* $p \in M_\omega$ *if* $\omega_\lambda(p_n,p) \to 0$ *as* $n \to \infty$ *for all* $\lambda > 0$, *and any such element* p *will be called a modular limit of the sequence* $\{p_n\}$.

(ii) $\{p_n\}_{n\in\mathbb{N}} \subset M_\omega$ *is a modular Cauchy sequence (ω-Cauchy) if there exists a number* $\lambda = \lambda(\{p_n\}) > 0$ *such that* $\omega_\lambda(p_n,p_m) \to 0$ *as* $n,m \to \infty$, *i.e., for all* $\epsilon > 0$ *there exists* $n_0(\epsilon) \in \mathbb{N}$ *such that, for all* $n,m \geq n_0(\epsilon)$: $\omega_\lambda(p_n,p_m) \leq \epsilon$.

(iii) *A modular space* M_ω *is called modular complete if every modular Cauchy sequence* $\{p_n\}$ *in* M_ω *is modular convergent in the following sence - if* $\{p_n\} \subset X_\omega$ *and there exists* $\lambda = \lambda(\{p_n\}) > 0$ *such that* $\lim_{n,m\to+\infty} \omega_\lambda(p_n,p_m) = 0$, *then there exists* $p \in M_\omega$ *such that* $\lim_{n\to\infty} \omega_\lambda(p_n,p) = 0$.

Definition 4. ([17]) *Consider* ω *a metric modular on* M*. Then it is said that* ω *satisfies the* $\triangle_2$*-condition. If, given a sequence* $\{p_n\}_{n\in\mathbb{N}} \subset M_\omega$, $p \in M_\omega$ *and* $\lambda > 0$ *such that* $\lim_{n\to+\infty} \omega_\lambda(p_n,p) = 0$, *we have* $\lim_{n\to\infty} \omega_{\frac{\lambda}{2}}(p_n,p) = 0$.

Mongkolkeha et al. [27] defined the contractions condition in modular metric spaces.

Definition 5. ([27]) *Consider a metric modular* ω *and* M_ω, *a modular metric space induced by* ω *and* $\Omega\colon M_\omega \to M_\omega$, *an arbitrary mapping. Then a mapping* Ω *is called contractive if for each* $p,q \in M_\omega$ *and for all* $\lambda > 0$ *there exists* $0 \leq k < 1$ *such that*

$$\omega_\lambda(\Omega p, \Omega q) \leq k\omega_\lambda(p,q). \tag{1}$$

Chatterjea [13] came with the concept of C-contraction, defined as follows:

Definition 6. ([13]) *A mapping $\Omega : M \to M$, where (M, ς), is a metric space is called a C-contraction if there exists $\alpha \in [0, \frac{1}{2})$ such that for all $p, q \in M$, the following condition holds:*

$$\varsigma(\Omega p, \Omega q) \leq \alpha[\varsigma(p, \Omega q) + \varsigma(q, \Omega p)],$$

for all $p, q \in M$.

In 1997, the notion of weakly ϕ-contraction has been introduced by Alber and Guerre-Delabriere [6] in the context of Hilbert spaces.

Definition 7. ([6]) *A self mapping $\Omega : M \to M$, on a metric space M is called weakly ϕ-contraction if there exists a continuous non-decreasing function $\phi : [0, +\infty) \to [0, +\infty)$ with $\phi(t) = 0$ if and only if $t = 0$ such that*

$$\varsigma(\Omega p, \Omega q) \leq \varsigma(p, q) - \phi(\varsigma(p, q))$$

for all $p, q \in M$.

By using the concept of C-contraction, Choudhury [18] generalized weakly ϕ-contraction, named as weakly C-contraction.

Definition 8. ([18]) *A mapping $\Omega : M \to M$ where (M, ς) is a metric space is called weakly C-contractive if, the following condition holds:*

$$\varsigma(\Omega p, \Omega q) \leq \tfrac{1}{2}[\varsigma(p, \Omega q) + \varsigma(q, \Omega p)] - \phi[\varsigma(p, \Omega q), \varsigma(q, \Omega p)],$$

where $\phi : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ is a continuous function such that $\phi(p, q) = 0$ if and only if $p = q = 0$ for $p, q \in M$.

Chaipunya et al. [12] has introduced the notion of $\mathcal{P}$-function.

Definition 9. ([12]) *Consider a partially ordered metric space $(M, \sqsubseteq, \varsigma)$. Then, a function $\varrho : M \times M \to \mathbb{R}$ is called a $\mathcal{P}$- function with respect to '$\sqsubseteq$' in M, if the following conditions hold:*

(i) $\varrho(p, q) \geq 0$ *for every comparable* $p, q \in M$;

(ii) *for any sequences* $\{p_n\}_{n\in\mathbb{N}}$, $\{q_n\}_{n\in\mathbb{N}}$ *in* M *such that* p_n *and* q_n *are comparable for each* $n \in \mathbb{N}$, *if* $\lim_{n\to+\infty} p_n = p$ *and* $\lim_{n\to+\infty} q_n = q$, *then* $\lim_{n\to+\infty} \varrho(p_n, q_n) = \varrho(p, q)$;

(iii) *for any sequences* $\{p_n\}_{n\in\mathbb{N}}$, $\{q_n\}_{n\in\mathbb{N}}$ *in* M *such that* p_n *and* q_n *are comparable for each* $n \in \mathbb{N}$, *if* $\lim_{n\to+\infty} \varrho(p_n, q_n) = 0$, *then* $\lim_{n\to+\infty} \varsigma(p_n, q_n) = 0$.

If, in addition the following condition is also satisfy:

(A) for any sequences $\{p_n\}_{n\in\mathbb{N}}$, $\{q_n\}_{n\in\mathbb{N}}$ in M such that p_n and q_n are comparable for each $n \in \mathbb{N}$, if $\lim_{n\to+\infty} \varsigma(p_n, q_n)$ exists, then $\lim_{n\to+\infty} \varrho(p_n, q_n)$ also exists.

Then ϱ is called type(A) of $\mathcal{P}$-function with respect to '$\sqsubseteq$' in M.
Chaipuniya et al. [12] introduced the following concept.

Definition 10. ([12]) *Let $(M, \sqsubseteq, \varsigma)$ be a partially ordered metric space, a mapping $\Omega\colon M \to M$, is called a $\mathcal{P}$-contraction with respect to '$\sqsubseteq$', if there exists a $\mathcal{P}$-function $\varrho\colon M \times M \to \mathbb{R}$ with respect to '$\sqsubseteq$' in M such that*

$$\varsigma(\Omega p, \Omega q) \leq \varsigma(p, q) - \varrho(p, q) \tag{2}$$

for any comparable $p, q \in M$.

Amor et al. [8] introduced the concept of a $\mathcal{P}$-C-contraction defined as follows:

Definition 11. ([8]) *Let $(M, \sqsubseteq, \varsigma)$ be a partially ordered metric space, a mapping $\Omega : M \to M$ is called a $\mathcal{P}$-C-contraction with respect to '$\sqsubseteq$', if there exists a $\mathcal{P}$-function $\varrho\colon M \times M \to \mathbb{R}$ with respect to '$\sqsubseteq$' in M such that the following inequality holds for any comparable $p, q \in M$:*

$$\varsigma(\Omega p, \Omega q) \leq \frac{\varsigma(p, \Omega q) + \varsigma(q, \Omega p)}{2} - Q(p, q) \tag{3}$$

where

$$Q(p, q) = \max(\varrho(p, \Omega q), \varrho(q, \Omega p)). \tag{4}$$

And proved the following theorems.

Theorem 1. ([8]) *Consider a complete partially ordered metric space $(M, \sqsubseteq, \varsigma)$ and a continuous non-decreasing self mapping $\Omega : M \to M$, which is a $\mathcal{P}$-$\mathcal{C}$-contraction of type (A) with respect to '$\sqsubseteq$'. If there exists $p_0 \in M$ with $p_0 \sqsubseteq \Omega p_0$, then $\{\Omega^n p_0\}_{n\in\mathbb{N}}$ converges to a fixed point of Ω in M.*

Next theorem was proved by dropping the continuity of Ω.

Theorem 2. ([8]) *Consider a complete sequentially ordered metric space $(M, \sqsubseteq, \varsigma)$ and a non-decreasing $\mathcal{P}$-$\mathcal{C}$-contraction of type (A) with respect to '$\sqsubseteq$', $\Omega : M \to M$. If there exists $p_0 \in M$ with $p_0 \sqsubseteq \Omega p_0$, then $\{\Omega^n p_0\}_{n\in\mathbb{N}}$ converges to a fixed point of Ω in M.*

Amor et al. [8] also proved the uniqueness of fixed point in the following result.

Theorem 3. ([8]) *Consider a complete sequentially ordered metric space $(M, \sqsubseteq, \varsigma)$ and a non-decreasing $\mathcal{P}$-$\mathcal{C}$-contraction of type (A) with respect to '$\sqsubseteq$,' $\Omega : M \to M$.*

Suppose that for each $p, q \in M$, there exists $w \in M$ which is comparable to both p and q. If there exists $p_0 \in M$ with $p_0 \sqsubseteq \Omega p_0$, then $\{\Omega^n p_0\}_{n\in\mathbb{N}}$ converges to a fixed point of Ω in M.

Zhao et al. [32] generalized Chaterjea type contraction in ω-complete modular metric space.

Definition 12. ([32]) *Consider a metric modular ω on M and a modular metric space M_ω be induced by ω. Then $\Omega\colon M_\omega \to M_\omega$ is called a weakly $\mathcal{C}$-contraction in M_ω, if for all $p, q \in M_\omega$ and for all $\lambda > 0$, the following condition holds :*

$$\omega_\lambda(\Omega p, \Omega q) \leq \frac{1}{2}\left[\omega_{2\lambda}(p, \Omega q) + \omega_{2\lambda}(q, \Omega p)\right] - \phi[\omega_\lambda(p, \Omega q), \omega_\lambda(q, \Omega p)] \tag{5}$$

where, $\phi\colon [0, +\infty)\times[0, +\infty) \to [0, +\infty)$ is a continuous mapping such that $\phi(p, q) = 0$ if and only if $p = q$.

Now, we define continuity of $\{p_n\}_{n\in\mathbb{N}}$, $\mathcal{P}$-function in modular metric space and extend the notion of $\mathcal{P}$-C-contraction in modular metric space as follows:

3. Main Results

Definition 13. *Consider a partially ordered modular metric space $(M_\omega, \sqsubseteq, \omega_\lambda)$ and a self mapping $\Omega : M_\omega \to M_\omega$, then the mapping Ω is called continuous, if for any sequence $\{p_n\}$ in M_ω, $\lim_{n\to+\infty} \omega_\lambda(p_n, p) = 0$, implies $\lim_{n\to+\infty} \omega_\lambda(\Omega p_n, \Omega p) = 0$ for all $\lambda > 0$ and $n \in \mathbb{N}$.*

Definition 14. *Consider a partially ordered modular metric space $(M, \sqsubseteq, \omega_\lambda)$, then a function $\varrho : M_\omega \times M_\omega \to \mathbb{R}$ is called a $\mathcal{P}$- function with respect to '$\sqsubseteq$' in M_ω if, it satisfies the following conditions:*

(i) *$\varrho(p, q) \geq 0$ for every comparable $p, q \in M_\omega$;*

(ii) *for any sequences $\{p_n\}_{n\in\mathbb{N}}$, $\{q_n\}_{n\in\mathbb{N}}$ in M_ω such that p_n and q_n are comparable for every $n \in \mathbb{N}$, if $\lim_{n\to+\infty} p_n = p$ and $\lim_{n\to+\infty} q_n = q$, then $\lim_{n\to+\infty} \varrho(p_n, q_n) = \varrho(p, q)$;*

(iii) *for any sequences $\{p_n\}_{n\in\mathbb{N}}$, $\{q_n\}_{n\in\mathbb{N}}$ in M_ω such that p_n and q_n are comparable for every $n \in \mathbb{N}$, if $\lim_{n\to+\infty} \varrho(p_n, q_n) = 0$, then $\lim_{n\to+\infty} \omega_\lambda(p_n, q_n) = 0$, for all $\lambda > 0$.*

If, in addition the following condition is also satisfy

(A) *for any sequences* $\{p_n\}_{n\in\mathbb{N}}$, $\{q_n\}_{n\in\mathbb{N}}$ *in* M *such that* p_n *and* q_n *are comparable for each* $n \in \mathbb{N}$, *if* $\lim_{n\to+\infty} \omega_\lambda(p_n, q_n)$ *exists, then* $\lim_{n\to+\infty} \varrho(p_n, q_n)$ *also exists.*

Then ϱ *is said to be a type (A) of* $\mathcal{P}$*-function with respect to '*$\sqsubseteq$*' in* M_ω.

Proposition 1. *Consider a partially ordered modular metric space* $(M_\omega, \sqsubseteq, \omega_\lambda)$ *and a* $\mathcal{P}$*-function* $\varrho : M_\omega \times M_\omega \to \mathbb{R}$ *with respect to '*$\sqsubseteq$*' in* M_ω*. If* $p, q \in M_\omega$ *are comparable and* $\varrho(p, q) = 0$*, then* $p = q$.

Proof. Consider $p, q \in M_\omega$ such that both are comparable and $\varrho(p, q) = 0$. Define two constant sequences $\{p_n\}_{n\in\mathbb{N}}$ and $\{q_n\}_{n\in\mathbb{N}}$ such that $p_n = p$ and $q_n = q$ for all $n \in M_\omega$. It follows from condition (iii) of a $\mathcal{P}$-function, since p and q are comparable, that $\omega_\lambda(p, q) = 0$ i.e. $p = q$. Hence, $\varrho(p, q) = 0$ implies $p = q$.

Corollary 4. *Consider a totally ordered modular metric space* $(M_\omega, \sqsubseteq, \omega_\lambda)$ *and a* $\mathcal{P}$*-function* $\varrho : M_\omega \times M_\omega \to \mathbb{R}$ *with respect to '*$\sqsubseteq$*' in* M_ω*. If* $p, q \in M_\omega$ *are comparable and* $\varrho(p, q) = 0$*, then* $p = q$.

Definition 15. *Consider a partially ordered modular metric space* $(M_\omega, \sqsubseteq, \omega_\lambda)$*. A mapping* $\Omega : M_\omega \to M_\omega$ *is called* $\mathcal{P}$*-*C*-contraction with respect to '*$\sqsubseteq$*', if there exists a* $\mathcal{P}$*-function* $\varrho: M_\omega \times M_\omega \to \mathbb{R}$ *with respect to '*$\sqsubseteq$*' in* M_ω *such that for every comparable* $p, q \in M_\omega$*, the following condition holds:*

$$\omega_\lambda(\Omega p, \Omega q) \le \frac{\omega_{2\lambda}(p, \Omega q) + \omega_{2\lambda}(q, \Omega p)}{2} - Q(p, q) \tag{6}$$

where

$$Q(p, q) = \max(\varrho(p, \Omega q), \varrho(q, \Omega p))$$

Now, we prove some fixed point results for monotonic mappings.

Theorem 5. *Consider a complete partially ordered modular metric space* $(M_\omega, \sqsubseteq)$ *and a continuous and non-decreasing* $\mathcal{P}$*-*C*-contraction* $\Omega: M_\omega \to M_\omega$ *of type (A) with respect to '*$\sqsubseteq$*'. If there exists* $p_0 \sqsubseteq \Omega p_0$*, then* $\{\Omega^n p_0\}_{n\in\mathbb{N}}$ *converges to a fixed point of* Ω *in* M_ω.

Proof. Select $p_0 \in M_\omega$ such that $p_0 \sqsubseteq \Omega p_0$ to test for the fixed point's existence. the proof is finished, if $\Omega p_0 = p_0$. Hence on the contrary assume that $\Omega p_0 \neq p_0$, then define a sequence $\{p_n\}_{n\in\mathbb{N}}$ such that $p_n = \Omega^n p_0$. Since $p_0 \sqsubseteq \Omega p_0$ and Ω is non-decreasing with respect to '$\sqsubseteq$'. It follows:

$$p_0 \sqsubseteq p_1 \sqsubseteq p_2 \sqsubseteq \cdots \sqsubseteq p_n \sqsubseteq p_{n+1} \sqsubseteq \cdots$$

Assume that $\varrho(p_n, p_{n+1}) \neq 0$ for all $n \in \mathbb{N}$. Otherwise there exist $n_0 \in \mathbb{N}$ such that $p_{n_0} = p_{n_0+1}$, that is $p_{n_0} = \Omega p_{n_0}$ and there is nothing to prove. Hence, consider only the case for which $0 < \varrho(p_n, p_{n+1})$ for all $n \in \mathbb{N}$.
Since $p_n \sqsubseteq p_{n+1}$ for all $n \in \mathbb{N}$, then

$$\begin{aligned}\omega_\lambda(p_{n+1}, p_n) &= \omega_\lambda(\Omega p_n, \Omega p_{n-1}) \\ &\leq \frac{1}{2}[\omega_{2\lambda}(p_n, \Omega p_{n-1}) + \omega_{2\lambda}(p_{n-1}, \Omega p_n)] - Q(p_n, p_{n-1}) \\ &\leq \frac{1}{2}[\omega_{2\lambda}(p_n, p_n) + \omega_{2\lambda}(p_{n-1}, p_{n+1})] - Q(p_n, p_{n-1}) \\ &= \frac{1}{2}(\omega_{2\lambda}(p_{n-1}, p_{n+1})) - Q(p_n, p_{n-1}) \qquad (7)\end{aligned}$$

where

$$Q(p_n, p_{n-1}) = \max(\varrho(p_n, p_n), \varrho(p_{n-1}, p_{n+1}))$$

for all $n \in \mathbb{N}$.
For every comparable $p, q \in M_\omega$, since $\varrho(p, q) \geq 0$, therefore
$Q(p_n, p_{n-1}) \geq 0$, and the condition (7) becomes

$$\begin{aligned}\omega_\lambda(p_{n+1}, p_n) &\leq \frac{1}{2}(\omega_{2\lambda}(p_{n-1}, p_{n+1})) \\ &\leq \frac{1}{2}[\omega_\lambda(p_{n-1}, p_n) + \omega_\lambda(p_n, p_{n+1})] \qquad (8)\end{aligned}$$

for all $\lambda > 0$. Thus, from the condition (8)

$$\omega_\lambda(p_n, p_{n+1}) < \omega_\lambda(p_{n-1}, p_n)$$

for all $\lambda > 0$ and for all $n \in \mathbb{N}$.
Therefore, the sequence of non-negative real numbers $\{\omega_\lambda(p_n, p_{n+1})\}_{n=1}^{+\infty}$, is non-increasing and hence it is convergent. It follows that, for each $\lambda > 0$, there exists $l \geq 0$

$$\lim_{n\to\infty} \omega_\lambda(p_n, p_{n+1}) = l. \qquad (9)$$

Letting $n \to +\infty$ in (8) follows that

$$l \leq \lim_{n\to+\infty} \frac{1}{2}[\omega_\lambda(p_{n-1}, p_n) + \omega_\lambda(p_n, p_{n+1})] \leq \frac{l+l}{2} \qquad (10)$$

or

$$\lim_{n\to+\infty} \omega_{2\lambda}(p_{n-1}, p_{n+1}) = 2l. \qquad (11)$$

It follows that

$$\lim_{n\to+\infty} \omega_\lambda(p_{n-1}, p_{n+1}) = l. \tag{12}$$

Thus, there exists $t \geq 0$ such that $\lim_{n\to+\infty} \varrho(p_{n-1}, p_{n+1}) = t$ and then

$$\lim_{n\to+\infty} Q(p_{n-1}, p_n) \geq t. \tag{13}$$

Again, taking $n \to +\infty$ in (7) and using (10), (12) and (13) obtain

$$l \leq \frac{1}{2} 2l - t.$$

Assume that $l > 0$, then $t = 0$ and so, $\lim_{n\to+\infty} \omega_\lambda(p_{n-1}, p_{n+1}) = 0$ which implies from (12) that $l = 0$, a contradiction. Therefore

$$\lim_{n\to+\infty} \omega_\lambda(p_n, p_{n+1}) = 0. \tag{14}$$

Thus, for each $\lambda > 0$ and $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\omega_\lambda(p_n, p_{n+1}) < \epsilon \tag{15}$$

for all $n \in \mathbb{N}$ with $n \geq n_0$.

Next to prove that the sequence $\{p_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Assume that $\{p_n\}_{n=1}^{+\infty}$ is not a Cauchy sequence, then there exists $\epsilon > 0$ such that for all $k \geq 1$, there exists $n_{(k)} > m_{(k)} > k$ with $\omega_{2\lambda}(p_{m_{(k)}}, p_{n_{(k)}}) \geq \epsilon$.

$$\omega_\lambda(p_{m_{(k)}}, p_{n_{(k)}}) \geq \omega_{2\lambda}(p_{m_{(k)}}, p_{n_{(k)}})$$

then we have

$$\omega_\lambda(p_{m_{(k)}}, p_{n_{(k)}}) \geq \epsilon.$$

Consider a smallest number $n_{(k)}$, satisfying the above condition. Hence

$$\omega_\lambda(p_{m_{(k)}}, p_{n_{(k-1)}}) < \epsilon.$$

$$\lim_{k\to+\infty} \omega_\lambda(p_{m_{(k)}}, p_{n_{(k-1)}}) = \epsilon. \tag{16}$$

Applying the condition (iii) of modular metric space

$$\begin{aligned} \epsilon \leq \omega_{2\lambda}(p_{m_{(k)}}, p_{n_{(k)}}) \;\; \leq \;\; & \omega_\lambda(p_{m_{(k)}}, p_{n_{(k)-1}}) \\ & + \omega_\lambda(p_{n_{(k)-1}}, p_{n_{(k)}}) \\ & \leq \epsilon + \omega_\lambda(p_{n_{(k)-1}}, p_{n_{(k)}}). \end{aligned} \tag{17}$$

Taking $k \to +\infty$ in (17), following is obtained

$$\lim_{k\to+\infty} \omega_{2\lambda}(p_{m_{(k)}}, p_{n_{(k)}}) = \epsilon. \tag{18}$$

Since

$$\omega_{2\lambda}(p_{m_{(k)}}, p_{n_{(k)}}) \leq \omega_{\lambda}(p_{m_{(k)}}, p_{n_{(k)-1}}) + \omega_{\lambda}(p_{n_{(k)-1}}, p_{n_{(k)}})$$

implies

$$\epsilon \leq \lim_{k\to+\infty} \omega_{2\lambda}(p_{m_{(k)}}, p_{n_{(k)-1}}) \leq \epsilon. \tag{19}$$

It implies that

$$\lim_{k\to+\infty} \omega_{2\lambda}(p_{m_{(k)}}, p_{n_{(k)-1}}) = \epsilon. \tag{20}$$

Similarly

$$\lim_{k\to+\infty} \omega_{\lambda}(p_{m_{(k)-1}}, p_{n_{(k)}}) = \epsilon. \tag{21}$$

From conditions (16) and (21), it is clear that $\lim_{k\to+\infty} \varrho(p_{m_{(k)}-1}, p_{n_{(k)}})$ and $\lim_{k\to+\infty} \varrho(p_{m_{(k)}}, p_{n_{(k)}-1})$ also exists. By $\mathcal{P}$-$\mathcal{C}$-contractive condition

$$\begin{aligned} \omega_{\lambda}(p_{m_{(k)}}, p_{n_{(k)}}) &= \omega_{\lambda}(\Omega p_{m_{(k)}-1}, \Omega p_{n_{(k)}-1}) \\ &\leq \frac{1}{2}[\omega_{2\lambda}(p_{m_{(k)}-1}, p_{n_{(k)}}) \\ &+\omega_{2\lambda}(p_{n_{(k)}-1}, p_{m_{(k)}})] - \\ &Q((p_{m_{(k)}-1}, p_{n_{(k)}-1})). \end{aligned} \tag{22}$$

Taking $k \to +\infty$ in (22), following is obtained
$0 \leq -\lim_{k\to+\infty} \max\{\varrho(p_{m_{(k)-1}}, p_{n_{(k)}}), \varrho(p_{n_{(k)-1}}, p_{m_{(k)}})\}$,
which further implies that $\lim_{k\to+\infty} \varrho(p_{m_{(k)-1}}, p_{n_{(k)}}) = 0$, following this it will have $\lim_{k\to+\infty} \omega_{\lambda}(p_{m_{(k)-1}}, p_{n_{(k)}}) = 0$ which is not possible, as $\epsilon > 0$. So, $\{p_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence.

By the completeness of M_ω, there exists point $p^* \in M_\omega$, such that $p_n = \Omega^n p_0 \to p^*$ as $n \to +\infty$.

Since, Ω is continuous, therefore $\Omega\Omega^n p_0 = \Omega^{n+1} p_0 \to p^*$, which implies that $\Omega p^* = p^*$. Thus p^* is a fixed point of Ω.

Next, we replace the continuity of Ω by $\triangle_2$-condition in the Theorem 5 and find out that we can still guarantee a fixed point if the partially ordered set is strengthened to a sequentially ordered set.

Theorem 6. *Consider a complete sequentially ordered modular metric space* $(M_\omega, \sqsubseteq$

) and a non-decreasing $\mathcal{P}$-C-contraction $\Omega\colon M_\omega \to M_\omega$ of type (A) with respect to '$\sqsubseteq$', satisfying the $\triangle_2$-condition. If there exists $p_0 \sqsubseteq \Omega p_0$, then $\{\Omega^n p_0\}_{n\in\mathbb{N}}$ converges to a fixed point of Ω in M_ω.

Proof. Following the proof of Theorem 5, the sequence $\{p_n\}_{n\in\mathbb{N}}$ is convergent and it's limit is p^* in M_ω.

Next, the existence of a fixed point p^* of Ω in M_ω is proved. Indeed, suppose that p^* is not a fixed point of Ω. i.e. $\omega_\lambda(p^*, \Omega p^*) \neq 0$. Since $(M_\omega, \sqsubseteq)$ be a complete sequentially ordered modular metric space, then p^* is comparable with p_n for all $n \in \mathbb{N}$, then by the notion of metric modular ω and the $\mathcal{P}$-C-contraction of Ω

$$\begin{aligned}\omega_\lambda(\Omega p^*, p^*) &\leq \omega_{\frac{\lambda}{2}}(\Omega p^*, \Omega p_n) + \omega_{\frac{\lambda}{2}}(\Omega p_n, p^*)\\ &\leq \frac{1}{2}[\omega_\lambda(p^*, \Omega p_n) + \omega_\lambda(p_n, \Omega p^*)] - Q(p^*, p_n) + \omega_{\frac{\lambda}{2}}(\Omega p_n, p^*)\\ &= \frac{1}{2}[\omega_\lambda(p^*, p_{n+1}) + \omega_\lambda(p_n, \Omega p^*)]\\ &\quad - \max[\varrho(p^*, \Omega p_n), \varrho(p_n, \Omega p^*)] + \omega_{\frac{\lambda}{2}}(p_{n+1}, p^*) \qquad (23)\end{aligned}$$

for all $\lambda > 0$ and for all $n \in \mathbb{N}$. Taking $n \to +\infty$ in (23) and by using Δ_2-condition, obtain that

$$\begin{aligned}\omega_\lambda(\Omega p^*, p^*) &\leq \frac{1}{2}\omega_\lambda(\Omega p^*, p^*) - \max(\varrho(p^*, \Omega p^*), \varrho(\Omega p^*, p^*))\\ &\leq \frac{1}{2}\omega_\lambda(\Omega p^*, p^*) < \omega_\lambda(\Omega p^*, p^*).\end{aligned}$$

This is a contradiction, therefore it must have

$$\omega_\lambda(\Omega p^*, p^*) = 0.$$

This proves that $p^* = \Omega p^*$. Thus p^* is a fixed point of Ω.

In the next theorem the sufficient condition for the uniqueness of the fixed point is included.

Theorem 7. *Consider a complete partially ordered modular metric space $(M_\omega, \sqsubseteq)$ and a continuous and non-decreasing $\mathcal{P}$-C-contraction $\Omega\colon M_\omega \to M_\omega$ of type (A) with respect to '$\sqsubseteq$'. Suppose that for each $p, q \in M_\omega$, there exists $w \in M_\omega$ which is comparable to both p and q. If there exists $p_0 \sqsubseteq \Omega p_0$, then $\{\Omega^n p_0\}_{n\in\mathbb{N}}$ converges to a unique fixed point of Ω in M_ω.*

Proof. A self mapping Ω has a fixed point, it is proved in Theorem 5, now the uniqueness of fixed point of Ω is proved. Assume that u and v be two distinct fixed points of Ω. i.e. $\omega_\lambda(u, v) \neq 0$. Here, we consider two cases as follows:

Case 1. Taking the comparability of u and v, we get the comparability of $\Omega^n u$ and to $\Omega^n v$ for all $n \in \mathbb{N}$ and

$$\begin{aligned}
\omega_\lambda(u,v) &= \omega_\lambda(\Omega^n u, \Omega^n v)\\
&\leq \frac{1}{2}[\omega_{2\lambda}(u, \Omega^n v) + \omega_{2\lambda}(v, \Omega^n u)] - Q(u,v)\\
&\leq \frac{1}{2}[\omega_{\lambda}(u, \Omega^n v) + \omega_{\lambda}(v, \Omega^n u)] - Q(u,v)\\
&= \frac{1}{2}[\omega_{\lambda}(u, v) + \omega_{\lambda}(u, v)] - Q(u,v)\\
&= \omega_\lambda(u,v) - \max(\varrho(u,v), \varrho(v,u))\\
&= \omega_\lambda(u,v) - \varrho(u,v). \qquad (24)
\end{aligned}$$

Since $\varrho(u,v) \geq 0$, here for $\varrho(u,v) > 0$, then the condition (24) is not possible. Therefore, it must have $\varrho(u,v) = 0$ and by Preposition 1 $u = v$.
Case 2. Assume that u is not comparable to v then there exist $w \in M_\omega$, which is comparable to u and v. Monotonicity of Ω implies that $\Omega^n w$ is comparable to $\Omega^n u = u$ and $\Omega^n v = v$ for all $n \in \mathbb{N}$. Therefore

$$\begin{aligned}
\omega_\lambda(u, \Omega^n w) &\leq \frac{1}{2}[\omega_{2\lambda}(u, \Omega^n w) + \omega_{2\lambda}(\Omega^{n-1} w, u)] - Q(u, \Omega^{n-1} w)\\
&= \frac{1}{2}[\omega_{2\lambda}(u, \Omega^n w) + \omega_{2\lambda}(\Omega^{n-1} w, u)] - \max(\varrho(u, \Omega^n w), \varrho(\Omega^{n-1} w, u))\\
&\leq \frac{1}{2}[\omega_{2\lambda}(u, \Omega^n w) + \omega_{2\lambda}(\Omega^{n-1} w, u)] - \varrho(u, \Omega^n w)\\
&\leq \frac{1}{2}[\omega_{\lambda}(u, \Omega^n w) + \omega_{\lambda}(\Omega^{n-1} w, u)] - \varrho(u, \Omega^n w)\\
&< \frac{1}{2}[\omega_{\lambda}(u, \Omega^n w) + \omega_{\lambda}(\Omega^{n-1} w, u)]. \qquad (25)
\end{aligned}$$

From the above condition it is found that

$$\omega_\lambda(u, \Omega^n w) \quad < \quad \omega_\lambda(u, \Omega^{n-1} w). \qquad (26)$$

If the sequences $s_n = \omega_\lambda(u, \Omega^n w)$ and $t_n = \varrho(u, \Omega^n w)$ are defined, then from condition (26) it may be obtained that $\{s_n\}_{n=1}^{+\infty}$ is non-increasing and there exists $l, q \geq 0$ such that $\lim_{n\to\infty} s_n = l$ and $\lim_{n\to+\infty} t_n = q$.
Assume that $l > 0$. Since Ω is $\mathcal{P}$-C-contractive, then from (25)

$$l \leq l - q.$$

Which is a contradiction. This implies that $l = 0$.
Hence $\lim_{n\to+\infty} s_n = 0$ i.e. $\lim_{n\to+\infty} \omega_\lambda(u, \Omega^n w) = 0$. In the same way, it can be shown that $\lim_{n\to+\infty} \omega_\lambda(v, \Omega^n w) = 0$. That is, $\{\Omega^n w\}_{n\in\mathbb{N}}$ converges to both u and v. Since the limit of a convergent sequence in a modular metric space is unique, it concludes that $u = v$. Hence, this yields the uniqueness of the fixed point.

By using the above condition of uniqueness, following Theorems can also be proved.

Theorem 8. *Consider $(M_\omega, \sqsubseteq)$ be a complete sequentially ordered modular metric space and $\Omega\colon M_\omega \to M_\omega$ be a non-decreasing $\mathcal{P}$-C-contraction of type (A) with respect to '$\sqsubseteq$'. Suppose that for each $p, q \in M_\omega$, there exists $w \in M_\omega$ which is comparable to both p and q. If there exists $p_0 \sqsubseteq \Omega p_0$, satisfying the Δ_2-condition then $\{\Omega^n p_0\}_{n\in\mathbb{N}}$ converges to a unique fixed point of Ω in M_ω.*

Proof. If $p_n = \Omega^n p_0$, then according to Theorem 6, $\{p_n\}_{n\in\mathbb{N}}$ converges to a fixed point of Ω in M_ω. The proof of uniqueness of fixed point is similar to the proof of Theorem 7.

Remark 1. *Notice that, when $(M_\omega, \sqsubseteq)$ is a totally ordered set, and $(M_\omega, \sqsubseteq, \omega_\lambda)$ be a partially ordered modular metric space. Then also uniqueness of the fixed point can be obtained.*

Following example justify the findings of Theorem 5.

Example 1. Let $M_\omega = \{(0,1), (1,0), (1,1)\} \subset \mathbb{R}^2$ and suppose that $p = (p_1, p_2)$ and $q = (q_1, q_2)$ for $p, q \in M$. Let '$\sqsubseteq$' be an ordering in M_ω given by $\mathbb{R} = \{(p,q) : p \in M_\omega\} \cup \{((0,1),(1,1))\}$. Then $(M_\omega, \sqsubseteq)$ is a partially ordered modular metric space with ϱ as a $\mathcal{P}$-function of type (A) with respect to '$\sqsubseteq$' in M_ω. Now, let Ω be a mapping $\Omega : M_\omega \to M_\omega$ define by $\Omega(0,1) = (0,1)$, $\Omega(1,1) = (0,1)$ and $\Omega(1,0) = (1,0)$. Then Ω is a continuous and non-decreasing mapping with respect to '$\sqsubseteq$'. Since $(0,1) \sqsubseteq (1,1)$ and $\Omega(0,1) = (0,1) \sqsubseteq \Omega(1,1) = (0,1)$.
Define $\omega_\lambda, \varrho : M_\omega \times M_\omega \to \mathbb{R}$ by

$$\omega_\lambda(p,q) = \begin{cases} 0 & \text{if } p = q \\ \dfrac{2\max\{p_1+q_1, p_2+q_2\}}{\lambda} & \text{otherwise,} \end{cases}$$

and

$$\varrho(p,q) = \begin{cases} 0 & \text{if } p = q \\ \dfrac{\max\{p_1, p_2+q_2\}}{\lambda} & \text{otherwise.} \end{cases}$$

Let $p, q \in M_\omega$ be comparable with respect to '$\sqsubseteq$'. Consider the following four cases.
Case I : If $p = q = (0,1)$, then

$$\omega_\lambda(\Omega(0,1),\Omega(0,1)) = \omega_{2\lambda}((0,1),\Omega(0,1)) = \varrho((0,1),\Omega(0,1)) = 0.$$

Case II : If $p = q = (1,0)$, then

$$\omega_\lambda(\Omega(1,0),\Omega(1,0)) = \omega_{2\lambda}((1,0),\Omega(1,0)) = \varrho((1,0),\Omega(1,0)) = 0.$$

Case III : If $p = q = (1,1)$, then

$$\omega_\lambda(\Omega(1,1),\Omega(1,1)) = 0,\ \omega_{2\lambda}((1,1),\Omega(1,1)) < \omega_\lambda((1,1),\Omega(1,1)) = \tfrac{2}{\lambda},(as\ 2\lambda < \lambda)$$
$$\varrho((1,1),\Omega(1,1)) = \tfrac{2}{\lambda}.$$

Case IV : If $p = (0,1)$ and $q = (1,1)$, then

$$\omega_\lambda(\Omega(0,1),\Omega(1,1)) = 0,\ \omega_{2\lambda}((0,1),\Omega(1,1)) = 0,$$
$$\omega_{2\lambda}(\Omega(0,1),(1,1)) < \omega_\lambda(\Omega(0,1),(1,1)) = \tfrac{4}{\lambda},(as\ 2\lambda < \lambda)\ \varrho((0,1),\Omega(1,1)) = 0 \text{ and}$$
$$\varrho(\Omega(0,1),(1,1)) = \tfrac{2}{\lambda}.$$

Therefore, in each case condition (6) and all the conditions of Theorem 5 for every comparable $p, q \in M_\omega$ are satisfied, also $(0,1) \sqsubseteq \Omega(0,1)$ and $(1,0) \sqsubseteq \Omega(1,0)$. Therefore, $(0,1)$ and $(1,0)$ are the fixed points of mapping Ω.

Fixed Point Theorems for Non-monotonic Mappings

In this section, to confirm the existence and uniqueness of fixed point of Ω the monotonicity of Ω is replaced with alternate condition - The comparability of $p, q \in M_\omega$ implies the comparability of $\Omega p, \Omega q \in \Omega M_\omega$.

Theorem 9. *Consider a complete partially ordered modular metric space $(M_\omega, \sqsubseteq)$ and a continuous self mapping $\Omega\colon M_\omega \to M_\omega$ which is a $\mathcal{P}$-C-contraction of type(A) with respect to '$\sqsubseteq$' such that the comparability for each $p, q \in M_\omega$, implies the comparability of $\Omega p, \Omega q \in \Omega M_\omega$. If there exists $p_0 \sqsubseteq \Omega p_0$, then $\{\Omega^n p_0\}_{n\in\mathbb{N}}$ converges to a fixed point of Ω in M_ω.*

Proof. Choose $p_0 \in M_\omega$ such that $p_0 \sqsubseteq \Omega p_0$. If $\Omega p_0 = p_0$, then the proof is finished. Suppose that $\Omega p_0 \neq p_0$, there exists a sequence $\{p_n\}_{n\in\mathbb{N}}$ such that $p_n = \Omega^n p_0$. Since $p_0 \sqsubseteq \Omega p_0$, then $p_n \sqsubseteq p_{n+1}$ for all $n \in \mathbb{N}$. Rest of the proof is similar to the proof of Theorem 5.

We are ignoring the proofs of following results because they can be demonstrated using the plots of the earlier theorems in this study.

Theorem 10. *Consider a complete sequentially ordered modular metric space $(M_\omega, \sqsubseteq)$ be and a $\mathcal{P}$-C-contraction $\Omega\colon M_\omega \to M_\omega$ of type(A) with respect to '$\sqsubseteq$' such that the comparability for each $p, q \in M_\omega$ implies the comparability of $\Omega p, \Omega q \in \Omega M_\omega$. If there exists $p_0 \sqsubseteq \Omega p_0$, the $\{\Omega^n p_0\}_{n\in\mathbb{N}}$ converges to a fixed point of Ω in M_ω.*

Theorem 11. *Consider a complete partially ordered modular metric space $(M_\omega, \sqsubseteq)$ and a $\mathcal{P}$-C-contraction $\Omega\colon M_\omega \to M_\omega$ of type(A) with respect to '$\sqsubseteq$' such that the comparability for each $p, q \in M_\omega$, implies the comparability of $\Omega p, \Omega q \in \Omega M_\omega$. Also assume that for each $p, q \in M_\omega$, there exists $r \in M_\omega$ which is comparable to both p and q. If there exists $p_0 \in M_\omega$ such that $p_0 \sqsubseteq \Omega p_0$, then $\{\Omega^n p_0\}_{n\in\mathbb{N}}$ converges to a unique fixed point of Ω in M_ω.*

Theorem 12. *Consider a complete sequentially ordered modular metric space $(M_\omega, \sqsubseteq)$ and a $\mathcal{P}$-C-contraction $\Omega\colon M_\omega \to M_\omega$ of type(A) with respect to '$\sqsubseteq$' such that the comparability for each $p, q \in M_\omega$, implies the comparability of $\Omega p, \Omega q \in \Omega M_\omega$. Suppose that for each $p, q \in M_\omega$, there exists $r \in M_\omega$ which is comparable to both p and q. If there exists $p_0 \in M_\omega$ such that $p_0 \sqsubseteq \Omega p_0$, the $\{\Omega^n p_0\}_{n\in\mathbb{N}}$ converges to a unique fixed point of Ω in M_ω.*

Corollary 13. *Consider a complete partially ordered modular metric space $(M_\omega, \sqsubseteq)$ be and a continuous and non-decreasing mapping $\Omega\colon M_\omega \to M_\omega$ with respect to '$\sqsubseteq$' such that*

$$\omega_\lambda(\Omega p, \Omega q) \le k(\omega_{2\lambda}(p, \Omega q) + \omega_{2\lambda}(q, \Omega p)).$$

If there exists $p_0 \sqsubseteq \Omega p_0$, then $\{\Omega^n p_0\}_{n\in\mathbb{N}}$ converges to a fixed point of Ω in M_ω.

Proof. By considering the contractive condition of Chaterjea et al. [13] and replacing $Q(p, q) = 0$ in (6), the above result can be proved.

4. Conclusion

In this manuscript, we defined $\mathcal{P}$-function in the sence of modular metric space and generalized the $\mathcal{P}$-$\mathcal{C}$-contraction in this new setting. Also, all the proved results are new and if we take $\omega_\lambda(p, q) = \dfrac{d(p, q)}{\lambda}$ for all $\lambda \in (0, \infty)$ and $Q(p, q) = \varrho(p, q)$, the main results of Chaipuniya et al. [12] are obtained and for $\omega_\lambda(p, q) = \dfrac{d(p, q)}{\lambda}$ for all $\lambda \in (0, \infty)$ the Theorems 1, 2 and 3 of Amor et al. [8] can be proved.

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