South East Asian J. of Mathematics and Mathematical Sciences
Vol. 19, No. 1 (2023), pp. 399-412

DOI: 10.56827/SEAJMMS.2023.1901.31 **ISSN (Online): 2582-0850**

ISSN (Print): 0972-7752

THE GEODETIC FAULT TOLERANT DOMINATION NUMBER OF A GRAPH

D. Stalin and J. John*

Department of Mathematics,
St. Alphonsa College of Arts and Science,
Karungal - 629159, Tamil Nadu, INDIA

E-mail : stalindd@gmail.com

*Department of Mathematics,
Government College of Engineering, Tirunelveli, INDIA

E-mail : john@gcetly.ac.in

(**Received:** *Oct. 14, 2021* **Accepted:** *Apr. 17, 2023* **Published:** *Apr. 30, 2023)*

Abstract: For a connected graph $G = (V, E)$, a set $F \subseteq V$ of vertices in G is called dominating set if every vertex not in F has at least one neighbor in F. A dominating set $F \subseteq V$ is called fault tolerant dominating set if $F - \{v\}$ is dominating set for every $v \in F$. A fault tolerant dominating set is said to be geodetic fault tolerant dominating set if $I[F] = V$. The minimum cardinality of a geodetic fault tolerant dominating set is called geodetic fault tolerant domination number and is denoted by $\gamma_{gft}(G)$. The minimum geodetic fault tolerant dominating set is denoted by γ_{gft}-set. The geodetic fault tolerant domination number of certain classes of graphs are determined. Some general properties satisfied by this concept are studied. It is shown that for every positive integer $2 < a \leq b$ there is a connected graph G such that $\gamma(G) = a$, $\gamma_g(G) = b$ and $\gamma_{gft}(G) = a + b - 2$, where $\gamma(G)$ and $\gamma_g(G)$ are the domination number and geodetic domination number of G respectively.

Keywords and Phrases: Domination number, Fault Tolerant domination number, Geodetic number, Geodetic fault tolerant number.

2020 Mathematics Subject Classification: 05C69, 05C12.

1. Introduction

By a graph $G = (V, E)$, we mean a finite undirected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. For basic graph theoretic terminology we refer to Harary [6]. If $e = \{u, v\}$ is an edge of a graph G with $d(u) = 1$ and $d(v) > 1$, then we call e a *pendent edge*, u a leaf and v a support vertex. Let L be the set of all leaves of a graph G. The vertex of degree $p-1$ is called an universal vertex. We denote by P_p, C_p and $K_{m,n}$, the path on p vertices, the cycle on p vertices and complete bipartite graph in which one partite set has m vertices and the other partite set has n vertices. For any set M of vertices of G, the *induced subgraph* $\langle M\rangle$ is the maximal subgraph of G with vertex set M. $N(v) = \{u \in V(G) : uv \in E(G)\}$ is called the *neighbourhood* of the vertex v in G. A vertex v in a connected graph G is said to be a *semi simplicial* vertex of G if $\Delta(< N(v) >) = |N(v)| - 1$. A vertex v is a *simplicial* vertex of a graph G if $< N(v) >$ is complete.

A set $D \subseteq V$ of vertices in G is called a dominating set if every vertex not in D has at least one neighbour in D. A vertex in a graph G dominates itself and its neighbors. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. A dominating set F is said to be a *fault tolerant dominating* set if $F - \{v\}$ is also a dominating for all $v \in F$. The minimum cardinality of F is called *fault tolerant domination number* and is denoted by $\gamma_{ft}(G)$. It is observed that $\gamma_{ft}(G) \geq 1 + \gamma(G)$. For references on domination parameters in graphs see [1, 5, 7, 9-14, 16, 17-20].

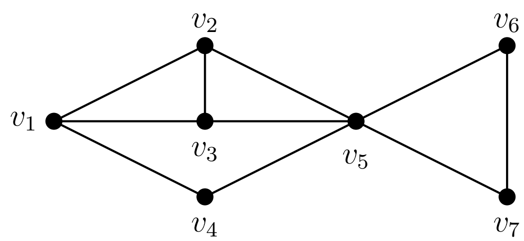
The *distance* $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest u-v path in G. The *eccentricity* $e(v)$ of a vertex v in G is the maximum distance from v and a vertex of G. The minimum eccentricity among the vertices of G is the *radius*, *rad* G or $r(G)$ and the maximum eccentricity is its *diameter*, *diam*(G) of G. A vertex v of G is said to be *peripheral vertex* if $e(v) = diam G$. A vertex x is said to lie on a $u - v$ geodesic P if x is a vertex of P including the vertices u and v. For $S \subseteq V$, $I[S] = \cup_{u,v\in S} I[u, v]$. A set $S \subseteq V$ is called an *geodetic set* of G if $I[S] = V$. The *geodetic number* $g(G)$ of G is the minimum order of its geodetic sets and any geodetic set of order $g(G)$ is an *geodetic basis* of G or g-set of G. A set of vertices S in G is called a geodetic dominating set of G if S is both geodetic set and a dominating set. The minimum cardinality of geodetic dominating set of G is its geodetic domination number and is denoted by $\gamma_g(G)$. A geodetic dominating set of size $\gamma_g(G)$ is said to be a γ_g-set. Geodetic number was introduced in [8] and further studied in [2, 3, 4, 6, 15, 12]. Moreover recently geodetic number was studied in [3, 7, 10]. Consider a client-server architecture based network in which any client must be able to communicate to one of the servers. If anyone of the server is fault or busy the system will not affect. A smallest group of servers with these properties is a fault tolerant dominating set for the graph representing the computer network.

Theorem 1.1. *Every extreme vertex of a connected graph G belongs to every minimum geodetic set of G. In particular every end vertex of a connected graph G belongs to every minimum geodetic set of G.*

Theorem 1.2. *For every connected graph G, $g(G) = 2$ if and only if G has peripheral vertices u and v such that every vertex of G lies on the diametral path of G.*

2. The Geodetic Fault Tolerant Domination Number of a Graph

Definition 2.1. *A Dominating set $F \subseteq V$ is said to be geodetic fault tolerant dominating set if F is both geodetic and fault tolerant dominating set of G. The minimum cardinality of a geodetic fault tolerant dominating set is called geodetic Fault tolerant domination number and is denoted by $\gamma_{gft}(G)$. The minimum Fault tolerant dominating set is denoted by γ_{gft}-set.*



G

Figure 2.1

Example 2.2. For the graph G given in Figure 2.1, $F = \{v_1, v_2, v_5, v_6, v_7\}$ is a minimum dominating set of G so that $\gamma_{gft}(G) \leq 5$. It is easily verified that G has no geodetic fault tolerant dominating set of cardinality less than 5. Hence $\gamma_{gft}(G) = 5$.

Remark 2.3. *For the graph G given in Figure 2.1, $D = \{v_1, v_5\}$ is a γ- set of G so that $\gamma(G) = 2$. Also $F_1 = \{v_1, v_2, v_5, v_6\}$ is a minimum fault tolerant dominating set, $S = \{v_1, v_5, v_6, v_7\}$ is a geodetic dominating set and $F = \{v_1, v_2, v_5, v_6, v_7\}$ is a minimum fault tolerant geodetic dominating set of G. Thus dominating set, fault tolerant dominating set, geodetic dominating set and geodetic fault tolerant dominating set of G are different.*

Remark 2.4. *There can be more than one fault tolerant dominating set for a graph. For the graph G given in Figure 2.1,* $F_1 = \{v_1, v_2, v_5, v_6, v_7\}$ *,* $F_2 = \{v_1, v_3, v_5, v_6, v_7\}$ *and* $F_3 = \{v_1, v_4, v_5, v_6, v_7\}$ *are the three* γ_{gft}*- sets of* G*.*

Observation 2.5. *(i) Let* G *be a connected graph with cut-vertices* F *be a geodetic fault tolerant dominating set of* G*. If* v *is a cut vertex of* G*, then* F *contains at least one vertex of every component of* $G - v$*.*
(ii) Every extreme vertex of a connected graph G *belongs to every geodetic fault tolerant dominating set of* G*.*
(iii)For any connected graph G*,* $2 \leq max\{\gamma(G), g(G)\} \leq \gamma_g(G) \leq \gamma_{gft}(G) \leq p$*.*

Theorem 2.6. *Let* v *be an end vertex and* u *be its adjacent vertex of a connected graph* G*. Then* $\{u, v\}$ *is a subset of every geodetic fault tolerant dominating set of* G*.*

Proof. Let v be an end vertex and u be its adjacent vertex of a connected graph G, S is a geodetic fault tolerant dominating set of G. If $\{u, v\}$ is not a subset of S, then S is not a dominating set of G, which is a contradiction. Moreover if $u \in S$ and $v \notin S$, then by Observation 2.5(ii), S is not a geodetic fault tolerant dominating set of G, which is a contradiction. Suppose that $u \notin S$. Then $S - \{v\}$ is not a dominating set of G and so S is not a geodetic fault tolerant dominating set of G, which is a contradiction.

Corollary 2.7. *Let* G *be a connected graph with* ℓ *end vertices. Then* $\gamma_{gft}(G) \geq \ell + 1$*.*

Proof. The proof follows from Theorem 2.6.

Theorem 2.8. *Let* G *be a connected graph of order* $p \geq 3$ *with at least one cut-vertex. Then* $\gamma_{gft}(G) \geq 3$*.*

Proof. Let u be a cut vertex of G and S be a geodetic fault tolerant dominating set of G. Let $G_1, G_2, ..., G_n (n \geq 2)$ be the components of $G - \{u\}$. Then by Observation 2.5(i), S contains at least two vertices v and w (say) of G. Since v and w are not adjacent, $S - \{v\}$ and $S - \{w\}$ are not a dominating set of G. This implies that $\gamma_{gft}(G) \geq 3$.

Theorem 2.9. *Let* G *be a connected graph. Then* $\gamma_{gft}(G) = 3$ *if there exists a* $\gamma-$ *set* $D = \{x, y\}$ *such that* $N(x) = N(y)$*.*

Proof. Let $D = \{x, y\}$ be a $\gamma-$ set of G and $N(x) = N(y) = \{v_1, v_2, ..., v_n\}$. Then $dim(G) = 2$. Moreover x and y are peripheral vertices of G, and so every vertex of G lies on the diametral path of x and y. Since $D - \{x\}$ and $D - \{y\}$ are not a dominating set of G, $\gamma_{gft}(G) \geq 3$. Let $F = D \cup \{v_i\} (1 \leq i \leq n)$. Then F is a γ_{gft}-set of G so that $\gamma_{gft}(G) = 3$.

In the following we determine the geodetic fault tolerant domination number of some standard graphs.

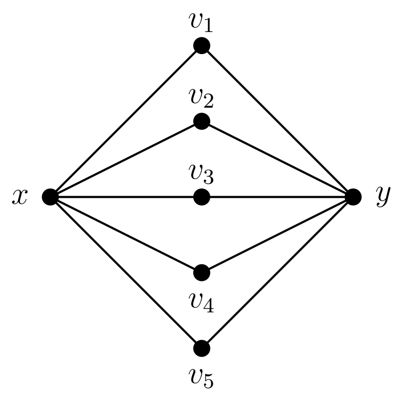


Figure 2.2

Theorem 2.10. *For the complete graph $K_p (p \geq 3)$,$\gamma_{gft}(K_p) = p$.*
Proof. Since every vertex of complete graph is extreme vertex, the result follows from Observation 2.5(ii).

Theorem 2.11. *For the complete bipartite graph $G = K_{m,n}$,*

$$\gamma_{gft}(K_{m,n}) = \begin{cases} p, & \text{if} \quad m = 1 \\ 3, & \text{if} \quad m = 2 \\ 4 & \text{if} \quad m, n \geq 3. \end{cases}$$

Proof. Let $U = \{u_1, u_2, ..., u_m\}$ and $W = \{w_1, w_2....w_n\}$ be the bipartite sets of G. (i) If $m = 1$,then the proof follows from Theorem 2.6.
(ii)If $m = 2$, then the proof follows from Theorem 2.9.
(iii)Let $m \geq 3$. Clearly $F = \{u_i, w_j\}(1 \leq i \leq m), (1 \leq j \leq n)$ is a minimum dominating set of G. However F is not a geodetic set of G. Let $F_1 = F \cup \{u_s, w_k\}$,$(1 \leq i \neq s \leq m), (1 \leq j \neq k \leq n)$. Then F_1 is a geodetic fault tolerant dominating set of G and so $\gamma_{ft}(G) \leq 4$. Suppose $\gamma_{gft}(G) = 3$. Let F' be a fault tolerant dominating set of G such that $|F'| = 3$. If $F' \subseteq U$, then $F' - \{u_i\}$, where $u_i \in F'$, is not a dominating(as well as geodetic) set of G. This implies that F' is not a geodetic fault tolerant dominating set of G, which is a contradiction. If $F' \subseteq W$, then $F' - \{w_i\}$ is not a dominating(as well as geodetic) set of G. Hence F' is not a geodetic fault tolerant dominating set of G, where $w_i \in F'$, which is

a contradiction. Therefore $F' \subsetneq U \cup W$. Let $F' = \{u_s, w_j, w_k\}$,$(j \neq k)$. Then $F' - \{u_s\}$ is not a dominating set of G and hence F' is not a geodetic fault tolerant dominating set of G, which is a contradiction.Therefore $\gamma_{gft}(G) = 4$.

Theorem 2.12. *For the wheel graph* $G = W_{1,p-1}$ $\gamma_{gft}(G) = \lceil \frac{p-1}{2} \rceil + 1$.

Proof. Let $V = \{x, v_1, v_2, ..., v_{p-1}\}$ be the vertex set of G, where x is the central vertex of G. Since the diameter of G is two and $|V - \{x\}|$ vertices form a cycle, every vertex of G lies in the $v_i - v_j$ geodesic of $\lceil \frac{p-1}{2} \rceil$ vertices of $V - \{x\}$. Moreover $\{x\}$ dominates G. Clearly $\{x\}$ and $\lceil \frac{p-1}{2} \rceil$ vertices (alternative vertex of $< V - \{x\} >$) of $V - \{x\}$ form a γ_{gft}-set. Hence $\gamma_{gft}(G) \leq \lceil \frac{p}{2} \rceil + 1$. Suppose that $\gamma_{gft}(G) \leq \lceil \frac{p-1}{2} \rceil$. Let F' be a geodetic fault tolerant dominating set of G such that $|F'| < |F|$. Then there exist a vertex $v_i \in F$ such that $v_i \notin F'$. This implies that there exist a shortest $v_{i-1}, v_{i+1}(v_{i-1}, v_{i+1} \in F')$ path including x such that v_i not lies on any geodesic path of vertices of F'. Thus F' is not a geodetic fault tolerant dominating set of G, which is a contradiction. Therefore $\gamma_{gft}(G) = \lceil \frac{p}{2} \rceil + 1$.

Theorem 2.13. *For any cycle* $G = C_p$, $\gamma_{ft}(C_p)(p \geq 3) = 2k + r(0 \leq r \leq 2)$ *where* $p = 3k + r$.

Proof. Let $v_1, v_2, ..., v_p$ be the vertices of G. Since G is 2-regular, every vertex of G dominates exactly three vertices of G, $\gamma = \lceil \frac{p}{3} \rceil$. Then by Observation 2.5(iii), $\gamma_{gft}(G) > \lceil \frac{p}{3} \rceil$. Let $p = 3k + r, (0 \leq r \leq 2)$.

Case i. Let $r = 0$. Then p=3k. Let $F = \{v_2, v_3, v_5, v_6......, v_{3k-1}, v_{3k}\}$. Then F is both geodetic and dominating set of G. Thus F is a geodetic fault tolerant dominating set of G and so $\gamma_{gft} \leq 2k$. Suppose that $\gamma_{gft}(G) < 2k$. Then there exists a geodetic fault tolerant dominating set F' such that $|F'| < |F|$. Therefore there exists a vertex $v_i \in F$ such that $v_i \notin F'$. Since $v_i \in F$ either v_{i-1} or v_{i+1} belongs to F'. Suppose $v_{i-1} \in F'$ then $F' - \{v_{i-1}\}$ is not a dominating set of G, which is a contradiction. Similarly if $v_{i+1} \in F'$, then F' is not a geodetic fault tolerant dominating set of G, which is a contradiction. Therefore $\gamma_{ft}(G) = 2k$.

Case ii. Let $r = 1$. Then $p = 3k + 1$. Let $V = \{v_1, v_2, ..., v_{3k+1}\}$ be the set of vertices of G. From the case(i), F is a geodetic fault tolerant dominating set of $\langle V - \{v_{3k+1}\}\rangle$. Since only one neighbor v_{3k} of v_{3k+1} belongs to F, $F - \{v_{3k}\}$ is not a geodetic fault tolerant dominating set of G. Therefore either $F_1 = F \cup \{v_{3k+1}\}$ or $F_1 = F \cup \{v_1\}$ is a γ_{gft}-set of G. Thus $\gamma_{gft}(G) = 2k + 1$.

Case iii. Let r=2. Let $V = \{v_1, v_2, ..., v_{3k}, v_{3k+1}, v_{3k+2}\}$ be the set of vertices of G. From case(ii), $F_2 = F_1 \cup \{v_{3k+1}\}$ or $F_1 \cup \{v_1\}$ is a geodetic fault tolerant dominating set of $\langle V - v_{3k+2}\rangle$. Moreover $F_2 - \{v_{3k+1}\}$ is not a dominating set of G. This implies that F_2 is not a geodetic fault tolerant dominating set of G, so that $|F| > 2k + 1$. Then $F_2 \cup \{v_{3k+2}\}$ or $F_2 \cup \{v_1\}$ is a γ_{gft}-set of C_p and so that $\gamma_{gft}(C_p) = 2k + 2$.

Theorem 2.14. *For the path $G = P_p$,*

$$\gamma_{gft(P_p)} = \begin{cases} p, & \text{if} \quad p \leq 4 \\ 4 + \lfloor \frac{2(p-4)}{3} \rfloor, & \text{if} \quad (p-4)\, mod\, 3 \leq 1 \\ 5 + \lfloor \frac{2(p-4)}{3} \rfloor, & otherwise. \end{cases}$$

Proof. Case (i). Let $p \leq 4$.Then the proof follows from Theorem 2.6
Suppose $p \geq 5$. Let $\{u, v, v_1, v_2, v_3, v_4, ..., v_{p-3}, v_{p-4}, y, z\}$ be the set of vertices of P_p, where $\{u, v\}$ is the set of end vertices and $\{y, z\}$ is the set of its adjacent vertices in G. Take $H = \{u, v, y, z\}$. By Theorem 2.6, H contained in every γ_{gft}-set of G. Let $H_1 = V - H = \{v_1, v_2, ..., v_{p-4}\}$ be the set of $p-4$ vertices of G. Let us assume that $p - 4 = 3k + r, (0 \leq r \leq 2)$.
Case (ii). Let $(p-4)\, mod\, 3 \leq 1$. Then $r = 0$ or $r = 1$
Subcase (i). Let $r = 0$. Then $p-4 = 3k$ and $H_1 = V - H = \{v_1, v_2, ..., v_{3k-1}, v_{3k}\}$ Let $F = H \cup \{v_2, v_3, v_5, v_6, ..., v_{3k-1}, v_{3k}\}$(from every path P_3 two vertices belongs to F except {u,v,y,z}). Since for every vertex $v \in F$, $F - \{v\}$ is a dominating set of G and every vertex of G lies in the diametral path of u and v so that F is a geodetic fault tolerant dominating set of G. Hence $|\gamma_{gft}| \leq 2k + 4$. Suppose $|\gamma_{gft}| < 4 + 2k$, then there exists a geodetic fault tolerant dominating set F' such that $|F'| < |F|$. Hence there exists a vertex $u_i \in F$ such that $u_i \notin F'$. Since $u_i \in F$, either u_{i-1} or u_{i+1} belongs to F'. Suppose that $u_{i-1} \in F'$ then $F' - \{u_{i-1}\}$ is not a dominating set of G, which is a contradiction. Also the same if $u_{i+1} \in F'$. Hence F is a γ_{gft}-set of G and so that $|\gamma_{gft}| = 4 + 2k$, where k is the integer part of $\frac{(p-4)}{3}$.
Subcase (ii). Suppose r=1. Then $p - 4 = 3k + 1$. Let $\{u, v, u_1, u_2, ..., u_{3k+1}, y, z\}$ be the set of vertices of P_p and $H_1 = V - H = \{u_1, u_2, ..., u_{3k}, u_{3k+1}\}$. Let $F = H \cup \{u_2, u_3, u_5, u_6, ..., u_{3k-1}, u_{3k}\}$. By subcase(i), geodetic fault tolerant dominating set of $\langle V - \{u_{3k+1}\rangle\}$ is F. Moreover u_{3k+1} is dominated by u_{3k}, v_3. Hence F is a γ_{gft}-set of G and so that $|\gamma_{gft}| = 4 + 2k$.
Case (iii). Let $r = 2$. Then $p-4 = 3k+2$. Let $\{v_1, v_2, u_1, u_2, ..., u_{3k+1}, u_{3k+2}, v_3, v_4\}$ be the set of vertices of P_p. Then $H_1 = V - H = \{u_1, u_2, ..., u_{3k+1}, u_{3k+2}\}$. From the subcase(ii) $V - \{u_{3k+2}\}$ is dominated by $F = H \cup \{u_2, u_3, u_5, u_6, ..., u_{3k-1}, u_{3k}\}$. Since $N(v_{3k+2}) = \{v_{3k+1}, v_3\}$ and $v_3 \in F$, either $F \cup \{v_{3k+1}\}$ or $F \cup \{v_{3k+2}\}$ is a γ_{gft}-set of G. Otherwise $F' - \{v_3\}$ is not a dominating set of G and then $|\gamma_{gft}| = 5 + 2k$,where k is the integer part of $\frac{(p-4)}{3}$. Hence the result.

3. Some Results on Geodetic Fault Tolerant Domination Number of a Graph

Theorem 3.1. *For any connected graph G, $\gamma_{ft}(G) \geq 1 + \gamma(G)$. Moreover $\gamma_{ft}(G) = 1 + \gamma(G)$ if and only if D is a geodetic set and there exists a vertex*

$v \in V - D$ such that $u \in N(v)$ for all $u \in D$, where D is a γ-set of G.

Proof. Let S be a γ_{gft}-set of G and D be a γ- set of G. Then it is clear that $D - \{v\}$ for all $v \in D$ is not a dominating set of G so that $\gamma_{ft}(G) > \gamma(G)$. Assume that $\gamma_{ft}(G) = 1 + \gamma(G)$. Then $S = D \cup \{x\}, x \in V - D$. To prove $u \in N(x)$ for all $u \in D$. Assume the contrary that there exists at least one vertex $v \in D$ such that $v \notin N(x)$. Then $S - \{x\}$ is not a dominating set of G, which is a contradiction. Conversely suppose that D is a geodetic set and there exists a vertex $v \in V - D$ such that $u \in N(v)$ for all $u \in D$. Then $S = D \cup \{x\}, x \in V - D$. Since every vertex in D is adjacent with x, $S - \{u\}$, for all $u \in S$ is a dominating set and S is geodetic set of G. This implies that S is a geodetic fault tolerant dominating set of G and so $\gamma_{ft}(G) = 1 + \gamma(G)$. Hence $\gamma_{ft}(G) \geq 1 + \gamma(G)$.

Theorem 3.2. *For any connected graph G, $2 \leq max\{1 + \gamma(G), g(G)\} \leq \gamma_g(G) \leq \gamma_{gft}(G) \leq p$.*

Proof. By Observation 2.5(iii), $\gamma_g(G) \geq g(G)$ and from the definition every $g-$set is a subset of every geodetic fault tolerant dominating set of G. Then by Theorem 3.1, $max\{1 + \gamma(G), g(G)\} \leq \gamma_{gft}(G)$. Moreover a geodetic dominating set is a subset of geodetic fault tolerant dominating set of G. This implies that $\gamma_g(G) \leq \gamma_{gft}(G)$. Also by Theorem 2.6, $\gamma_{ft}(K_2) = 2$. This conclude that $2 \leq max\{1 + \gamma(G), g(G)\} \leq \gamma_g(G) \leq \gamma_{gft}(G) \leq p$.

Theorem 3.3. *Let S be a minimum geodetic set of G. Then $\gamma_{gft}(G) \geq \gamma_g(G)$ and $\gamma_{gft}(G) = \gamma_g(G)$ if and only if $S - \{v\}$, for all $v \in S$ is a dominating set of G.*

Proof. Let S be a geodetic fault tolerant dominating set of G. Since every geodetic fault tolerant dominating set of G contains a geodetic set of G, $\gamma_{gft}(G) \geq \gamma_g(G)$. Since S is a minimum geodetic set of G and $S - \{v\}$, for all $v \in S$ is a dominating set of G, S is a γ_{gft}- set of G. This implies that $\gamma_{gft}(G) = \gamma_g(G)$.

Theorem 3.4. *For any tree T, $\ell + 1 \leq \gamma_{gft}(T) \leq p$. Moreover $\gamma_{gft}(T) = \ell + 1$ if and only if G is a star and $\gamma_{gft}(T) = p$ if and only if $V = L \cup S$, where L is the set of end vertices and S is the set of support vertices of T.*

Proof. Let T be a tree having ℓ end vertices. Then by Corollary 2.7, $\ell + 1 \geq \gamma_{gft}(T)$. Suppose that $\gamma_{gft}(T) = \ell + 1$ to prove T is a star. Assume the contrary that T is not a star. Then $diam(T) \geq 3$ and T has more than one support vertex. Then by Theorem 2.6, $\gamma_{gft}(T) \neq \ell + 1$ and which is a contradiction. Conversely suppose that T is a star. Then by Theorem 2.6, $\gamma_{gft}(T) = \ell + 1$. In addition suppose that $\gamma_{gft}(T) = p$. To prove $V = L \cup S$. Assume the contrary that T has at least one vertex u (say), which is neither a support vertex nor an end vertex. If $u \in N(v) \cap N(w)$ where $v, w \in S$, then $\gamma_{gft}(T) = p - 1$, which is a contradiction. Moreover if more than one vertex of T is neither a support vertex nor an end vertex, then by Theorem 2.14, $\gamma_{gft}(T) \leq p-1$, which is a contradiction. Therefore $\gamma_{gft}(T) = p$. Conversely suppose that $V = L \cup S$, where L is the set of end vertices and S is the set of support vertices of T. Then the result follows from Theorem 2.6. This implies that $\ell + 1 \leq \gamma_{gft}(T) \leq p$.

Theorem 3.5. *For the caterpillar T, $\triangle(T) \leq \gamma_{gft}(G) \leq \ell + d - 1$, where ℓ is the number of end vertices of T and d is the diameter of T.*

Proof. Let $\triangle(T) = n$ and u be a maximum degree vertex of T. Since T is a caterpillar, at least $n-2$ neighbors of u are end vertices. Then by Observation 2.5(ii), $\triangle(T) - 1 \leq \gamma_{gft}(G)$. Moreover if $diam(T) > 2$, T has T has at least two end vertices other than the neighbors of u and so $\triangle(T) < \gamma_{gft}(G)$. In addition, if $diam(T) = 2$, T is a star and by Observation 2.5(ii), $\triangle(T) < \gamma_{gft}(G)$. On other hand for any caterpillar of order p, $p = \ell + d - 1$. If $V = L \cup S$, by Observation 2.5(ii), $\gamma_{gft}(G) = p$, where S is the set support vertices and L is the set of end vertices. This implies that $\gamma_{gft}(G) \leq \ell + d - 1$.

Theorem 3.6. *Let $G \neq C_p$ be a unicyclic graph. and S be the set of all $v \in V$ such that v is neither end vertex nor support vertex of G. Then $\gamma_{gft}(G) = p-1$ if and only if $\langle S \rangle$ is either K_1 or K_2 or P_3.*

Proof. Let G be a unicyclic graph and S be the set of adjacent vertices which are not support vertices of G. Let $\langle S_1 \rangle$ is either K_1 or K_2 or P_3. Then $1 \leq\mid S \mid\leq 3$. By given condition each vertex $u_i \in V - S$ of degree more than one is adjacent with at least one pendent vertex.

Case (i). Let $\langle S \rangle = K_1$. Then G has exactly one vertex v which is not adjacent with any end vertex of G and by Theorem 2.10, $V - \{v\}$ is subset of every geodetic fault tolerant dominating set of G. Since v is dominated by at least two vertices of $V - \{v\}$, $V - \{v\}$ is a $\gamma_{gft}-$ set of G so that $\gamma_{gft}(G) = p-1$.

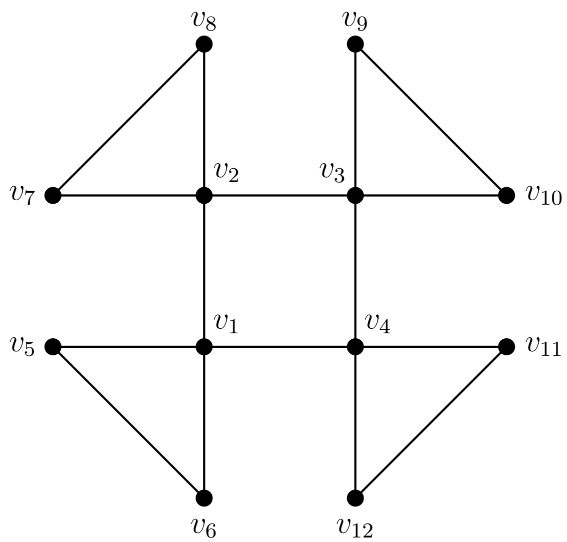
Case (ii). Let $\langle S \rangle = K_2$. Then G has exactly two adjacent vertices $S = \{u, v\}$ which are not adjacent with any end vertex of G and by Theorem 2.6, $V - \{u, v\}$ is subset of every geodetic fault tolerant dominating set of G. It is easily verified that $V - \{u\}$ or $V - \{v\}$ is a $\gamma_{gft}-$ set of G so that $\gamma_{gft}(G) = p-1$.

Case (iii). Let $\langle S \rangle = P_3$. Then G has exactly three vertices $S = \{u, v, w\}$ which are not adjacent with any end vertex of G and exactly one vertex v (say) of S has degree 2. As in the previous cases $F' = V - \{u, w\}$ is subset of every geodetic fault tolerant dominating set of G. Moreover $F' - \{v\}$ is not a dominating set of G. Hence $F' \cup \{u\}$ or $F' \cup \{w\}$ is a geodetic fault tolerant dominating set of G. Thus $\gamma_{gft}(G) = p-1$.

Conversely suppose that $G \neq C_p$ be a unicyclic and $\gamma_{gft}(G) = p-1$. To prove $\langle S \rangle$ is either K_1 or K_2 or P_3. Let us assume the contrary.

Case (i). Let $|S| = 0$. Then by Theorem 2.6, $\gamma_{gft}(G) \neq p - 1$.
Case (ii). Let G has at least two non adjacent vertices $\{x, y\}$ which are not adjacent with any end vertex of G. Then x, y are adjacent with more than one vertices of $V - \{x, y\}$ so that $V - \{x, y\}$ is a geodetic fault tolerant dominating set of G. Hence $\gamma_{gft}(G) \leq p - 2$, which is a contradiction.
Case (iii). Let $| S | \geq 4$ and $S = \{u_1, u_2, ...u_m\}(m \geq 4)$. Since G is unicyclic $< S >$ is a path $P_m(m \geq 4)$. Without loss of generality assume that u_1, u_m are the end vertices of the path. Hence u_1, u_m are adjacent with at least one vertex of $V - S$. This implies that $\gamma_{gft}(< S >) \leq m - 2$ and also by Theorem 2.6, $V - S$ is a subset of every $\gamma_{gft}-$ set of G. Thus $\gamma_{gft}(G) \leq p - 2$, which is a contradiction.

Theorem A. *No cut vertex of a connected graph G belongs to any minimum geodetic set of G.*



G

Figure 3.1

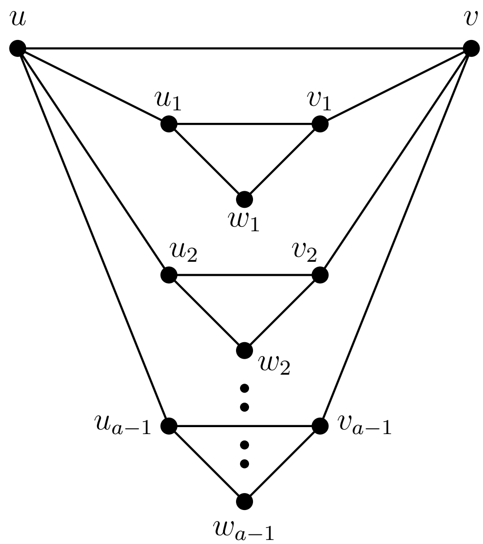
In Figure 3.1, $S = \{v_1, v_2, v_3, v_4\}$ is the set of cut vertices of G and $V - S$ is the unique geodetic fault tolerant dominating set of G
We can now present a characterization of exclusion of cut vertices.

Theorem 3.7. *Let G be connected graph of order p and G has $n(\geq 1)$ cut vertices. Then $\gamma_{gft}(G) = p - n$ if and only if $\delta(G) \geq 2$ and each components of $V - S$ are K_r for some $r \geq 2$, where S is the set of cut vertices of G.*

Proof. Assume that $\gamma_{gft}(G) = p - n$. To prove $\delta(G) \geq 2$ and each components of $V - S$ are K_r for some $r \geq 2$. Let us assume the contrary. If $\delta(G) = 1$, by Theorem 2.6, $\gamma_{gft}(G) > p - n$, which is a contradiction. On the other hand assume that at least one component G_r (say) of $V - S$ is non complete. Let $|V(G_r)| = p_r$. Then G_r has at least one vertex which is neither extreme or support vertex. This implies that $\gamma_{gft}(G_r) \leq p_r - 2$ and so $\gamma_{gft}(G) \leq p - n - 1$, which is a contradiction. Conversely suppose that $\delta(G) \geq 2$ and each components of $V - S$ are K_r for some $r \geq 2$, where S is the set of cut vertices of G. Then $V - S$ is the set of extreme vertices of G. Moreover by Observation 2.5(ii), $V - S$ is a subset of every $\gamma_{gft}-$ set of G. In addition, by Theorem A, S is not a subset of any geodetic set of G. This implies that $I[V - S] = V$. Moreover every $v \in N[v_i] \cap N[v_j]$ for all $v \in S$, where v_i, $v_j \in V - S$. Hence $V - S$ is a $\gamma_{ft}-$ set of G and so $\gamma_{ft}(G) = p - n$.

In view of Theorems,we have the following realization results.

Theorem 3.8. *For each positive integers $2 < a \leq b \leq a + b - 2$, there exist a connected graph G such that $\gamma(G) = a$, $\gamma_g(G) = b$ and $\gamma_{gft}(G) = a + b - 2$.*



G

Figure 3.2

Proof. Case (i). $a = b$. Let $P : u, v$ be a path on two vertices and let P_i : $u_i, v_i (1 \leq i \leq a-2)$ be copy of path on two vertices. Add new vertices $w_i (1 \leq i \leq a)$ and introducing new edges $u_i w_i (1 \leq i \leq a-2)$ and $v_i w_i (1 \leq i \leq a-2)$. The graph

G can be obtained by introducing new edges $uu_i(1 \le i \le a)$ and $vv_i(1 \le i \le a-2)$. The graph G is given figure 3.2. First we show that $\gamma(G) = \gamma_g(G)$. Let $Z = \{w_1, w_2, ..., w_{a-2}\}$. Since $\{u, v\}$ dominates $u_i(1 \le i \le a)$ and $v_i(1 \le i \le a)$, $\{u, v\}$ is a subset of every dominating set of G. Let $D = \{u, v\} \cup \{w_1, w_2, ..., w_{a-2}\}$. Then it is easily verified that D is a minimum dominating set of G so that $\gamma(G) = a$. By Theorem 1.1, Z is a subset of every geodetic dominating set of G. Moreover $I[D] = V$. Hence D is a geodetic dominating set of G so that $\gamma(G) = \gamma_g(G)$. Let $F = D \cup \{u_1, u_2, ..., u_{a-2}\}$. Moreover $F - \{v\}$ for all $v \in F$ is geodetic dominating set of G so that $\gamma_{gft}(G) = 2a - 2$

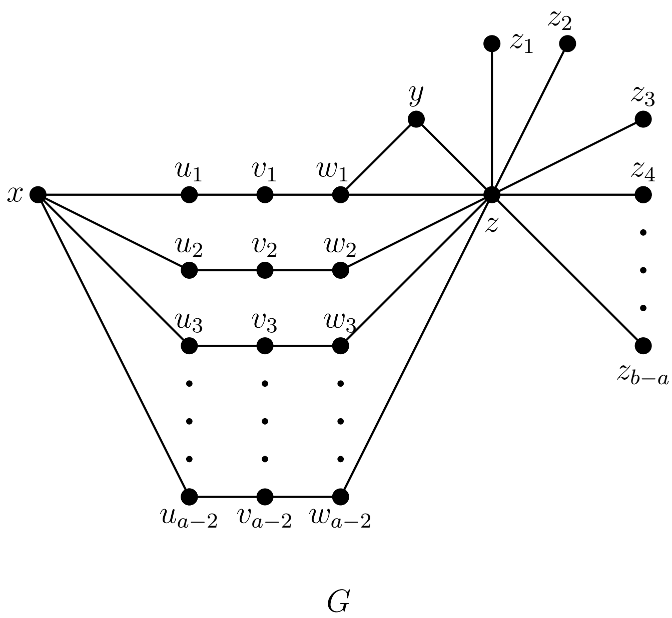


Figure 3.3

Case (ii). $a < b$. Let $P_i : u_i, v_i w_i (1 \le i \le a-2)$ be copy of path on three vertices. Let H be the graph obtained from P_i by adding new vertices x and y. Introducing the new edges xu_i and $v_i y(1 \le i \le a-2)$. The graph G can be obtained from H by adding new vertices $z_1, z_2 ... z_{b-a}$ and introducing new edges $yz_i(1 \le i \le b-a)$. The graph G is given figure 3.3. First we prove that $\gamma(G) = a$. Let $D_1 = \{v_1, v_2, ..., v_{a-2}\}$. Then D is not a dominating set of G. Let $D = D_1 \cup \{x, y\}$ Then it is clear that D is a minimum dominating set of G so that $\gamma(G) = a$. Next to prove that $\gamma_g(G) = b$. Let $Z = \{z_1, z_2 ... z_{b-a}\}$ be the set of end vertices of G. Then by Theorem 1.1, Z is a subset of every γ_g- set of G so that $|\gamma_g(G)| \ge b - a$. Let $S = D \cup Z$. Then it is clear that S is a minimum geodetic domination set of

G so that $\gamma_g(G) = b$. Next to prove that $\gamma_{gft}(G) = a + b - 2$. By Theorem 1.1 and Theorem 2.6, S is a subset of every geodetic fault tolerant dominating set of G. Let $F = S \cup \{u_1, u_2, ..., u_{a-2}\}$. Then it is easily verified that F is a geodetic fault tolerant dominating set of G so that $\gamma_{gft}(G) = a + b - 2$.

4. Conclusion

We have discussed geodetic fault tolerant dominating set and geodetic fault tolerant domination number of graphs, which has applications in location theory and networking. This concept is extending to monophonic and edge geodetic.

Acknowledgments

We thank the anonymous referees for their useful suggestions.

References

[1] Abdollahzadeh Ahangar H., Samodivkin V. and Yero IG, Independent transversal dominating sets in graphs: complexity and structural properties, Filomat, 30 (2), (2016), 293-303.

[2] Abdollahzadeh Ahangar H., Samodivkin V., The total geodetic number of a graph, Utilitas Mathematica, 100 (2016), 253-268.

[3] Abdollahzadeh Ahangar H., Kosari S., Sheikholeslami S. M. and Volkmann L., Graphs with large geodetic number, Filomat, 29 (6), (2015), 1361-1368.

[4] Abdollahzadeh Ahangar H., Samodivkin V., Sheikholeslami S. M. and Khodkar A., The restrained geodetic number of a graph, Bulletin of the Malaysian Mathematical Sciences Society, 38 (3), (2015), 1143-1155.

[5] Arul Paul Sudhahar P., Ajitha A. and Subramanian A., The total edge geodetic domination number of a graph, South East Asian J. of Mathematics and Mathematical Sciences, 13 (1), (2017), 19-26.

[6] Buckley F. and Harary F., Distance in Graph, Addison-Wesley, Redwood City, CA, 1990.

[7] Chellathurai S. R. and Vijaya S. P., The geodetic domination number for the product of graphs, Math. Comput. Modelling, 3 (4), (2014), 19-30.

[8] Harary F., Loukakis E. and Tsouros C., The geodetic number of a graph, Math. Comput. Modelling, 17 (11), (1993), 89-95.

[9] John J. and Sujin Flower V., On the Forcing Domination and the Forcing Total Domination Numbers of a Graph, Graphs and Combinatorics, 38 (5), (2022), 117-129.

[10] Kokilambal G., Antipodal domination number, South East Asian J. of Mathematics and Mathematical Sciences, 18 (3), (2022), 339-346.

[11] Mustapha Chellali and Haynes T. W., A note on the double domination number in trees, AKCE International Journal of Graphs and Combinatorics, 3 (2), (2006), 147-150.

[12] Nader M. H. and Hajian Majid, A new lower bound on the double domination number of a graph, Discrete Applied Mathematics, 254 (2019), 280-282.

[13] Nader Jafari Rad and Hajibaba M., On domination, 2-domination, and Italian domination numbers, Utilitas Mathematica, 111 (111), (2019), 271-280.

[14] Nader Jafari Rad and Volkmann Lutz, A note on the independent domination number in graphs, Discrete Applied Mathematics, 161 (18), (2013), 3087-3089.

[15] Santhakumaran A. P. and John J., Edge Geodetic Number of a Graph, Journal of Discrete Mathematical Sciences and Cryptography, 10 (3), (2007), 415-432.

[16] Shailaja S. Shirkol, Pavitra P. Kumbargoudra and Meenal M. Kaliwal, Double roman domination number of middle graph, South East Asian J. of Mathematics and Mathematical Sciences, 18 (3), (2022), 369-380.

[17] Stalin D. and John J., Edge Geodetic Dominations in Graphs, International journal of Pure and Applied Mathematics, 116 (22), (2017), 31-40.

[18] Stalin D. and John J., Upper Edge Geodetic Dominations in Graphs, Journal of Advanced research in Dynamical and Control System, 9 (9), (2018), 201-205.

[19] Stalin D. and John J., Forcing Edge Geodetic Dominations in Graphs, Journal of Advanced research in Dynamical and Control System, 10 (4), (2018), 172-177.

[20] Van Rooij J. M. M. and Bodlaender H. I., Exact algorithms for dominating set, Discrete Applied Mathematics, 159 (2011), 2147-2164.