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αg-γ-REGULAR AND αg-γ-NORMAL SPACES

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Abstract: In topology, separation axioms is used to measure how close a topological space is to being metrizable. Spaces that fulfil the axioms are regarded to be nearer to being metrizable than those that do not. In this paper, we introduce new axioms namely αg-γ- regular and αg-γ-normal and analyze their properties in topological spaces. We compare αg-γ- regularity with regularity and αg-γ-normal with normality . Also we obtain the relations between the newly defined spaces and αg_γ-$T_i^{'}(i = 0, 1, 2)$ spaces.

Keywords and Phrases: Topological space, αg_γ-open sets, separation axioms, αg-γ-regular, αg-γ-normal.

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1. Introduction

In 1965, the notion of α-open sets was introduced by Njastad [15]. Levine [10] introduced the notion of generalized closed sets in topological spaces in 1970. Later in 1994, Maki et al. [11] introduced α-generalized closed sets in topological spaces.

Kasahara [9] proposed the concept of an operation on topological spaces and put forth the idea of α-closed graphs of functions in topological spaces. Jankovic [6] analyzed the functions with α-closed graphs. Following his work, Ogata [16] introduced γ-open sets using the operation γ on open sets in topological spaces. In 2009, γ-s^*-regular space was defined and several characterizations and properties of γ-s^*-regular spaces have been determined by Hussain and Ahmad [4]. In

2011, Ahmad and Hussain [1] defined and studied the properties of γ^*-regular and γ-normal spaces. Hussain and Ahmad [5] introduced γ-P-regular spaces and studied the relations between P-regularity, regularity, γ-regularity, and γ-P-regularity. Basu et al. [3] studied separation axioms with respect to an operation. Kalaivani et al. [7] studied α-γ-compact spaces, α-γ-regular spaces and α-γ-normal spaces with α-γ-open sets in topological spaces. In 2019, Ahmad et al. [2] defined γ-P_S-regular and γ-P_S-normal spaces using γ-P_S-open and γ-P_S-closed sets and discussed their characterizations. In 2022, Kalaivani et al. [8] presented the concept of $(\gamma, \gamma')^\alpha$-regular spaces and $(\gamma, \gamma')^\alpha$ -normal spaces and discussed various results on the defined spaces. Recently, Mershia Rabuni and Balamani [12, 13] introduced αg_γ-open sets using the operation γ on $\tau_{\alpha g}$ and analyzed its properties in topological spaces. Mershia Rabuni and Balamani [14] studied separation axioms via operation on αg-open sets and their characterizations.

In this paper, fundamental definitions and results required for the study of the concept are given in Section 2. In Section 3, we introduce αg-γ-regular space and we determine the relation between αg-γ-regular space and already existing separation axioms defined using αg_γ-open sets and its properties. Section 4 deals with αg-γ-normal space and its characterization theorem. In Section 5, we find the interrelation between αg - γ -regular, αg - γ-normal and $\alpha g_\gamma - T_i^{'} (i = 0, 1, 2)$.

2. Preliminaries

Throughout this paper (X, τ) signifies a topological space on which no separation axiom is assumed unless otherwise mentioned. Also closure and interior are being denoted as $cl\,(A)$ and $int\,(A)$ respectively.

Definition 2.1. *Let (X, τ) be a topological space. A subset A of (X, τ) is called*

1. *α-open* [15] *if $A \subseteq int(cl(int(A)))$. The collection of all α-open sets in X is denoted by τ_α. The complement of an α-open set is called α-closed. The intersection of all α-closed sets containing A is called α-closure of A and is denoted by $\alpha cl(A)$.*

2. *αg-closed* [11] *if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ). The complement of an αg-closed set is called αg-open. The collection of all αg-open sets in X is denoted by $\tau_{\alpha g}$.*

Definition 2.2. [12] *Let (X, τ) be a topological space. An operation γ on $\tau_{\alpha g}$ is a mapping from $\tau_{\alpha g}$ into the power set $P(X)$ of $X \ni V \subseteq \gamma(V)\ \forall\ V \in \tau_{\alpha g}$, the value of V under the operation γ is denoted by $\gamma(V)$.*

Definition 2.3. [12] *A non-empty subset A of (X, τ) with an operation γ on $\tau_{\alpha g}$*

is called an αg_γ-open set if $\forall\ x \in A$, $\exists$ an αg-open set $U \ni x \in U$ and $\gamma(U) \subseteq A$. The collection of all αg_γ-open sets in (X,τ) is denoted by $\tau_{\alpha g_\gamma}$. The complement of an αg_γ-open set is called αg_γ-closed.

Definition 2.4. [12] *An operation γ on $\tau_{\alpha g}$ is said to be αg-open if $\forall$ αg-open set U containing $x \in X$, $\exists$ an αg_γ-open set $V \ni x \in V$ and $V \subseteq \gamma(U)$.*

Definition 2.5. [12] *Let γ be an operation on $\tau_{\alpha g}$. A point $x \in X$ is said to be an αg_γ-closure point of a set A if $\gamma(U) \cap A \neq \phi$ $\forall$ αg-open set U containing x.*

Definition 2.6. [12] *Let γ be an operation on $\tau_{\alpha g}$. Then $\alpha g_\gamma Cl(A)$ is defined as the intersection of all αg_γ-closed sets containing A.*

Definition 2.7. [13] *In a topological space (X,τ), $A \subseteq X$ is an αg_γ-generalized closed (concisely αg_γ-g.closed) set if $\alpha gcl_\gamma(A) \subseteq U$ whenever $A \subseteq U$ and U is αg_γ-open in (X,τ).*

Definition 2.8. [13] *Let A be a subset of a topological space (X,τ) and γ be an operation on $\tau_{\alpha g}$. A point $x \in A$ is said to be an αg_γ-interior point of A if $\exists$ an αg-open set V of X containing $x \ni \gamma(V) \subseteq A$. $\alpha gint_\gamma(A)$ denotes the set of all such αg_γ-interior points of A.*

Definition 2.9. [13] *Let A be a subset of a topological space (X,τ) and γ be an operation on $\tau_{\alpha g}$. Then αg_γ-interior of A is the union of all αg_γ-open sets contained in A and it is denoted by $\alpha g_\gamma int(A)$.*

Definition 2.10. [13] *Let A be a subset of a topological space (X,τ) and γ be an operation on $\tau_{\alpha g}$. Then αg_γ-kernel of A is defined as the intersection of all αg_γ-open sets containing A. It is denoted by αg_γ-$ker(A)$.*

Definition 2.11. [14] *A topological space (X,τ) is called*

1. *αg_γ-T_0' if for any two distinct points $x, y \in X$, there exists an αg_γ-open set U such that either $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$.*

2. *αg_γ-T_1' if for any two distinct points $x, y \in X$, there exist αg_γ-open sets U and V containing x and y respectively such that $y \notin U$ and $x \notin V$.*

3. *αg_γ-T_2' if for any two distinct points $x, y \in X$, there exist αg_γ-open sets U and V containing x and y respectively such that $U \cap V = \phi$.*

Theorem 2.1. [12] *Let (X,τ) be a topological space and A be a subset of X and γ be an operation on $\tau_{\alpha g}$. Then for a given $x \in X$, $x \in \alpha g_\gamma Cl(A)$ if and only if $U \cap A \neq \phi$ $\forall$ αg_γ-open set U containing x.*

Proposition 2.1. *Let (X, τ) be a topological space and γ be an operation on $\tau_{\alpha g}$. Then for any subset A in X, the following statements hold:*

1. *$\alpha g_\gamma cl(A)$ is an αg_γ-closed set in X.* [12]

2. *$\alpha g_\gamma int(A)$ is an αg_γ-open set in X.* [13]

Theorem 2.2. [12] *Arbitrary union of αg_γ- open sets is αg_γ- open.*

Proposition 2.2. [13] *Every αg_γ-closed set is αg_γ-g.closed but not conversely.*
Proof. Consider an αg_γ-closed set A and an αg_γ-open set $U \ni A \subseteq U$. As A is αg_γ-closed, $A = \alpha gcl_\gamma(A)$ which implies that $\alpha gcl_\gamma(A) = A \subseteq U$. Thus A is αg_γ-g.closed.

3. αg-γ-Regular Space

Definition 3.1. *A topological space (X, τ) with an operation γ on $\tau_{\alpha g}$ is called αg-γ-regular if for each αg_γ-closed set A of X not containing x, there exist disjoint αg_γ- open sets U and V containing x and A, respectively.*

Example 3.1. Consider $X = \{a, b, c, d\}$ with $\tau = \{\phi, \{a, b\}, X\}$. Then $\tau_{\alpha g} = \{\phi, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, X\}$. Let $\gamma : \tau_{\alpha g} \to P(X)$ be an operation on $\tau_{\alpha g}$ defined by

$$\gamma(A) = \begin{cases} A & a \text{ or } b \in A \\ X & Otherwise \end{cases} \qquad \forall A \in \tau_{\alpha g}$$

Then $\tau_{\alpha g_\gamma} = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\}$. Here the space is αg-γ-regular.

Remark 1. *The concept of αg-γ - regularity and $\alpha g_\gamma - T_i'$ $(i = 0, 1, 2)$ are independent.*

Example 3.2. Consider $X = \{a, b, c, d\}$ with $\tau = \{\phi, \{a\}, \{a, b\}, X\}$. Then $\tau_{\alpha g} = P(X) \setminus \{\{c, d\}, \{b, c, d\}\}$. Let $\gamma : \tau_{\alpha g} \to P(X)$ be an operation on $\tau_{\alpha g}$ defined by

$$\gamma(A) = \begin{cases} A & A = \{a, b, c\} \text{ or } A = \{d\} \\ X & Otherwise \end{cases} \qquad \forall A \in \tau_{\alpha g}$$

Then $\tau_{\alpha g_\gamma} = \{\phi, \{d\}, \{a, b, c\}, X\}$. Here the space is αg-γ-regular, but not αg_γ-T_i' $(i = 0, 1, 2)$.

Example 3.3. Consider $X = \{a, b, c, d\}$ with $\tau = \{\phi, \{a, b\}, X\}$. Then $\tau_{\alpha g} =$

$P(X) \backslash \{\{c,d\}, \{a,c,d\}, \{b,c,d\}\}$. Let $\gamma : \tau_{\alpha g} \to P(X)$ be an operation on $\tau_{\alpha g}$ defined by

$$\gamma(A) = \begin{cases} X & A = \{b\} \text{ or } \{c\} \\ \alpha gcl(A) & Otherwise \end{cases} \qquad \forall\ A \in \tau_{\alpha g}$$

Then $\tau_{\alpha g_\gamma} = P(X) \backslash \{\{b\}, \{c\}, \{c,d\}\}$. Here the space is αg_γ- $T_i^{'}$ $(i = 0, 1, 2)$, but not αg-γ-regular.

Remark 2. *The concept of αg-γ-regularity and regularity are independent.*

Example 3.4. The topological space (X, τ) with an operation γ on $\tau_{\alpha g}$ as defined in Example 3.2 is αg-γ- regular, but not regular.

Example 3.5. Consider $X = \{a, b, c, d\}$ with the discrete topology τ. Then $\tau_{\alpha g} = P(X)$. Let $\gamma : \tau_{\alpha g} \to P(X)$ be an operation on $\tau_{\alpha g}$ defined by

$$\gamma(A) = \begin{cases} X & A = \{b\} \text{ or } \{c\} \text{ or } \{c,d\} \\ cl(A) & Otherwise \end{cases} \qquad \forall\ A \in \tau_{\alpha g}$$

Then $\tau_{\alpha g_\gamma} = P(X) \backslash \{\{b\}, \{c\}, \{c,d\}\}$. Here the space is regular but not αg-γ-regular.

Lemma 3.1. *A subset A of X is αg_γ-g.open iff $F \subseteq \alpha gint_\gamma(A)$ whenever $F \subseteq A$ and F is αg_γ-closed in X.*

Proof. Consider A to be an αg_γ-g.open set and $F \subseteq A$ where F is αg_γ-closed in X. Since $X \backslash A$ is αg_γ-g.closed and $X \backslash F$ is an αg_γ-open set containing $X \backslash A$, $\alpha gcl_\gamma(X \backslash A) \subseteq (X \backslash F)$, which implies that $X \backslash \alpha gint_\gamma(A) \subseteq (X \backslash F)$ [13]. i.e., $F \subseteq \alpha gint_\gamma(A)$.

Conversely, Let A be any subset of X such that $X \backslash A \subseteq U$ where U is αg_γ-open. Then $X \backslash U \subseteq A$ where $X \backslash U$ is αg_γ-closed. Then by the hypothesis, $X \backslash U \subseteq \alpha gint_\gamma(A)$, which implies that $X \backslash \alpha gint_\gamma(A) \subseteq U$. Then $\alpha gcl_\gamma(X \backslash A) \subseteq U$ [13]. Therefore $X \backslash A$ is αg_γ-g.closed and hence A is αg_γ-g.open.

Remark 3. *In a space (X, τ) with an αg-open operation on $\tau_{\alpha g}$, A subset A of X is αg_γ-g.open iff $F \subseteq \alpha g_\gamma int(A)$ whenever $F \subseteq A$ and F is αg_γ-closed in X.*

Theorem 3.1. *The following are equivalent for a topological space (X, τ) with an operation on $\tau_{\alpha g}$:*

1. *X is αg-γ-regular.*
2. *For each $x \in X$ and each αg_γ-open set U containing x, there exists an αg_γ-open set V containing x such that $\alpha g_\gamma cl(V) \subseteq U$.*

3. *For each $x \in X$ and each αg_γ-closed set C not containing x, there exists an αg_γ-open set V containing x such that $\alpha g_\gamma cl\,(V) \cap C = \phi$.*

4. *For each αg_γ-closed set A of X, $\cap \left\{\alpha g_\gamma cl\,(U) : A \subseteq U,\ U \in \tau_{\alpha g_\gamma}\right\} = A$.*

5. *For each $A \subseteq X$ and each αg_γ-open set U with the condition $A \cap U \neq \phi$, there exists an αg_γ-open set V such that $A \cap V \neq \phi$ and $\alpha g_\gamma cl(V) \subseteq U$.*

6. *For each non-empty subset A of X and each αg_γ-closed subset C of X with $A \cap C = \phi$, there exist αg_γ-open sets U and V such that $A \cap U \neq \phi$, $C \subseteq V$ and $U \cap V = \phi$.*

Proof. (1) $\Longrightarrow$ (2) Assume X to be αg-γ-regular. Consider an αg_γ-open set U containing x. Then the αg_γ-closed set $X\backslash U$ does not contain x. By αg-γ-regularity of X, there exist αg_γ- open sets V and W containing x and $X\backslash U$, such that $V \cap W = \phi$. Then $V \subseteq X\backslash W$. Hence $x \in V \subseteq \alpha g_\gamma cl\,(V) \subseteq \alpha g_\gamma cl\,(X\backslash W) = X\backslash W \subseteq U$.

(2) $\Longrightarrow$ (3) Let x be any point in X and C be any αg_γ-closed set of X such that $x \notin C$. Then $X\backslash C$ is an αg_γ-open set containing x. By (2) there exists an αg_γ-open set V containing x such that $\alpha g_\gamma cl(V) \subseteq X\backslash C$. This implies that $\alpha g_\gamma cl\,(V) \cap C = \phi$.

(3) $\Longrightarrow$ (4) Let x be any point in X and A be any αg_γ-closed set of X. Suppose that $x \in X\backslash A$. Then by (3), for each $x \in X$ and each αg_γ-closed set A not containing x, there exists an αg_γ-open set V containing x such that $\alpha g_\gamma cl\,(V) \cap A = \phi$, which implies $A \subseteq X\backslash \alpha g_\gamma cl(V)$. Let $U = X\backslash \alpha g_\gamma cl(V)$ and U is αg_γ-open in X. Then $U \cap V = \phi$. Hence by Theorem 2.1, $x \notin \alpha g_\gamma cl(U)$. Therefore $A \supseteq \cap \left\{\alpha g_\gamma cl\,(U) : A \subseteq U,\ U \in \tau_{\alpha g_\gamma}\right\}$. Obviously, we have $A \subseteq \cap \left\{\alpha g_\gamma cl\,(U) : A \subseteq U, U \in \tau_{\alpha g_\gamma}\right\}$. Hence $A = \cap \left\{\alpha g_\gamma cl\,(U) : A \subseteq U, U \in \tau_{\alpha g_\gamma}\right\}$.

(4) $\Longrightarrow$ (5) Consider any subset A of X and an αg_γ-open set U of X such that $A \cap U \neq \phi$. Let $x \in A \cap U$. Then $x \notin X\backslash U$. Since $X\backslash U$ is αg_γ-closed, by (4), there exists an αg_γ-open set O such that $X\backslash U \subseteq O$ and $x \notin \alpha g_\gamma cl\,(O)$. Let $V = X\backslash \alpha g_\gamma cl\,(O)$. Since, $V \subseteq U$ and V is an αg_γ-open set containing x, $A \cap V \neq \phi$. Now $V \subseteq X\backslash O$, implies $\alpha g_\gamma cl\,(V) \subseteq X\backslash O \subseteq U$.

(5) $\Longrightarrow$ (6) Consider any non-empty subset A of X and an αg_γ-closed set C of X such that $A \cap C = \phi$. Then $X\backslash C$ is an αg_γ-open set and $A \cap X\backslash C \neq \phi$. Then by (5), there exists an αg_γ-open set U such that $A \cap U \neq \phi$ and $\alpha g_\gamma cl(U) \subseteq X\backslash C$. Let $V = X\backslash \alpha g_\gamma cl(U)$. Then $C \subseteq V$ and $U \cap V = \phi$.

(6) $\Longrightarrow$ (1) Consider any point x in X and an αg_γ-closed set C of X such that $x \notin C$. Then $\{x\} \cap C = \phi$. Then by (6), there exist αg_γ-open sets U and V such

that $\{x\} \cap U \neq \phi$, $C \subseteq V$ and $U \cap V = \phi$. Consequently, $x \in U$. Therefore X is αg-γ-regular.

Theorem 3.2. *The following are equivalent for a topological space (X, τ) with an αg-open operation on $\tau_{\alpha g}$:*

1. *X is αg-γ-regular.*
2. *For each αg_γ-closed set C and each $x \notin C$, there exist an αg_γ-open set U and an $\alpha g_\gamma.g$- open set V such that $x \in U$, $C \subseteq V$ and $U \cap V = \phi$.*
3. *For each $A \subseteq X$ and each αg_γ-closed set C with $A \cap C = \phi$, there exist an αg_γ- open set U and an $\alpha g_\gamma.g$- open set V such that $A \cap U \neq \phi$, $C \subseteq V$ and $U \cap V = \phi$.*

Proof. (1) $\Longrightarrow$ (2) Let $x \in X$ and C be an αg_γ-closed set not containing x. Since X is αg-γ-regular, for each x in X and each αg_γ-closed subset C of X not containing x, there exist disjoint αg_γ-open sets U and V such that $\{x\} \cap U \neq \phi$, $C \subseteq V$. Since every αg_γ-open set is $\alpha g_\gamma.g$-open, V is $\alpha g_\gamma.g$-open. Therefore V is an $\alpha g_\gamma.g$-open set such that $x \in U$, $C \subseteq V$ and $U \cap V = \phi$.

(2) $\Longrightarrow$ (3) Consider any subset A of X and an αg_γ-closed set C of X such that $A \cap C = \phi$. Let $x \in A$. Then $x \notin C$. Therefore by (2), there exist an αg_γ-open set U and an $\alpha g_\gamma.g$- open set V such that $x \in U$, $C \subseteq V$ and $U \cap V = \phi$. Therefore $A \cap U \neq \phi$.

(3) $\Longrightarrow$ (1) Consider any point x in X and an αg_γ-closed set C of X such that $x \notin C$. Then $\{x\} \cap C = \phi$. Then by (3), there exist an αg_γ- open set U and an $\alpha g_\gamma.g$- open set V such that $\{x\} \cap U \neq \phi$, $C \subseteq V$ and $U \cap V = \phi$. Consequently, we have $x \in U$. Set $W = \alpha g_\gamma int(V)$. Since V is $\alpha g_\gamma.g$- open, $C \subseteq \alpha g_\gamma int\,(V) = W$, by Remark 3 and $U \cap W = \phi$. Therefore X is αg-γ-regular.

Theorem 3.3. *A topological space (X, τ) with an αg-open operation on $\tau_{\alpha g}$ is αg-γ-regular iff for each αg_γ-closed set A of X, $A = \cap\{\alpha g_\gamma cl\,(V) : A \subseteq V,\ V$ is $\alpha g_\gamma.g$-open$\}$.*

Proof. Consider X to be αg-γ-regular and A be an αg_γ-closed set. By Theorem 3.1, for each αg_γ-closed set A of X, $\cap \left\{\alpha g_\gamma cl\,(U) : A \subseteq U,\ U \in \tau_{\alpha g_\gamma}\right\} = A$. Since every αg_γ-open set is $\alpha g_\gamma.g$-open, $\cap\{\alpha g_\gamma cl\,(V) : A \subseteq V,\ V$ is $\alpha g_\gamma.g$- open$\} = A$. Conversely, Assume any point x in X and an αg_γ-closed set A of X such that $x \notin A$. Then by hypothesis, there exists an $\alpha g_\gamma.g$-open set V such that $A \subseteq V$ and $x \in X \backslash \alpha g_\gamma cl\,(V)$. By Remark 3, $A \subseteq \alpha g_\gamma int(V)$. Let $U = \alpha g_\gamma int(V)$ and $W = X \backslash \alpha g_\gamma cl\,(V)$. Then both U and W are αg_γ-open set containing A and x

respectively and $U \cap W = \phi$. Hence X is αg-γ-regular.

Theorem 3.4. *A topological space (X, τ) with an operation γ on $\tau_{\alpha g}$ is αg-γ-regular if and only if for each αg_{γ}-closed set C of X and $x \notin C$ with $\alpha g_{\gamma} ker\,(C) \neq \alpha g_{\gamma} ker\ \{x\}$, there exist αg_{γ}-closed sets F and G such that $\alpha g_{\gamma} ker(C) \subseteq F$, $\alpha g_{\gamma} ker(C) \cap G = \phi$ and $\alpha g_{\gamma} ker\{x\} \subseteq G$, $\alpha g_{\gamma} ker\{x\} \cap F = \phi$ and $F \cup G = X$.*
Proof. Let C be an αg_{γ}-closed set, $x \notin C$. Then by hypothesis, there exist αg_{γ}-open sets U and V such that $C \subseteq U$, $x \in V$ and $U \cap V = \phi$. Now $X \backslash U = G$ (say) and $X \backslash V = F$ (say) are αg_{γ}-closed sets such that $G \cup F = X$. This implies $\alpha g_{\gamma} ker(C) \subseteq U \subseteq F$ and $\alpha g_{\gamma} ker\,(C) \cap G = \phi$. Similarly, $\alpha g_{\gamma} ker\ \{x\} \subseteq V \subseteq G$, $\alpha g_{\gamma} ker\ \{x\} \cap F = \phi$ and $F \cup G = X$.

Conversely, Let C be an αg_{γ}-closed set of X and $x \notin C$ such that $\alpha g_{\gamma} ker\,(C) \neq \alpha g_{\gamma} ker\ \{x\}$. Then by hypothesis, $X \backslash F$ and $X \backslash G$ are αg_{γ}-open sets such that $(X \backslash F) \cap (X \backslash G) = \phi$ and $\alpha g_{\gamma} ker(C) \cap G = \phi \Rightarrow C \cap G = \phi \Rightarrow C \subseteq X \backslash G$. Similarly, $\alpha g_{\gamma} ker\{x\} \cap F = \phi \Rightarrow \{x\} \cap F = \phi \Rightarrow x \in X \backslash F$. Therefore (X, τ) is an αg-γ-regular space.

4. αg-γ - Normal Space

Definition 4.1. *A topological space (X, τ) with an operation γ on $\tau_{\alpha g}$ is called αg-γ- normal if for any pair of disjoint αg_{γ}-closed sets A and B of X, there exist disjoint αg_{γ}- open sets U and V containing A and B respectively.*

Example 4.1. Consider $X = \{a, b, c\}$ with $\tau = \{\phi, \{a\}, X\}$. Then $\tau_{\alpha g} = P\,(X) \backslash \{b, c\}$. Let $\gamma : \tau_{\alpha g} \to P\,(X)$ be an operation on $\tau_{\alpha g}$ defined by

$$\gamma\,(A) = \begin{cases} A & A \text{ is singleton} \\ cl(A) & otherwise \end{cases} \qquad \forall\ A \in \tau_{\alpha g}$$

Here the space X is αg-γ -normal.

Remark 4. *The concept of αg-γ- normal and αg_{γ}-$T_i^{'}$ $(i = 0, 1, 2)$ are independent.*

Example 4.2. The topological space (X, τ) with an operation γ on $\tau_{\alpha g}$ as defined in Example 3.2 is αg-γ-normal but not αg_{γ}-$T_i^{'}$ $(i = 0, 1, 2)$. The topological space (X, τ) with an operation γ on $\tau_{\alpha g}$ as defined in Example 3.3 is αg_{γ}-$T_i^{'}$ $(i = 0, 1, 2)$ but not αg-γ-normal.

Remark 5. *The concept of αg-γ-normality and normality are independent.*

Example 4.3. The topological space (X, τ) with an operation γ on $\tau_{\alpha g}$ as defined in Example 3.2 is αg-γ- normal, but not normal. The topological space (X, τ) with an operation γ on $\tau_{\alpha g}$ as defined in Example 3.5 is normal, but not αg-γ-normal.

Theorem 4.1. *The following are equivalent for a topological space (X, τ) with an αg-open operation γ on $\tau_{\alpha g}$:*

1. *X is αg-γ- normal.*
2. *For any pair of disjoint αg_γ-closed sets A and B of X, there exist disjoint αg_γ-g.open sets U and V such that $A \subseteq U$ and $B \subseteq V$.*
3. *For each αg_γ-closed set A and any αg_γ-open set B containing A, there exists an αg_γ-g.open set U such that $A \subseteq U \subseteq \alpha g_\gamma cl(U) \subseteq B$.*
4. *For each αg_γ-closed set A and any αg_γ-g.open set B containing A, there exists an αg_γ-g.open set U such that $A \subseteq U \subseteq \alpha g_\gamma cl(U) \subseteq \alpha g_\gamma int(B)$.*
5. *For each αg_γ-closed set A and any αg_γ-g.open set B containing A, there exists an αg_γ-open set U such that $A \subseteq U \subseteq \alpha g_\gamma cl(U) \subseteq \alpha g_\gamma int(B)$.*
6. *For each αg_γ-g.closed set A and any αg_γ-open set B containing A, there exists an αg_γ-open set U such that $\alpha g_\gamma cl(A) \subseteq U \subseteq \alpha g_\gamma cl(U) \subseteq B$.*
7. *For each αg_γ-g.closed set A and any αg_γ-open set B containing A, there exists an αg_γ-g.open set U such that $\alpha g_\gamma cl(A) \subseteq U \subseteq \alpha g_\gamma cl(U) \subseteq B$.*

Proof. (1) $\Longrightarrow$ (2) It is evident from the fact that every αg_γ-open set is αg_γ-g.open set.

(2) $\Longrightarrow$ (3) Consider an αg_γ-closed set A and any αg_γ-open set B such that $A \subseteq B$. Now, $(X \backslash B)$ is an αg_γ-closed set such that $A \cap (X \backslash B) = \phi$. Then by (2), there exist disjoint αg_γ-g.open sets U and V such that $A \subseteq U$ and $(X \backslash B) \subseteq V$. By Remark 3, $(X \backslash B) \subseteq \alpha g_\gamma int(V)$. Now, $U \cap \alpha g_\gamma int(V) = \phi$ consequently implies that $\alpha g_\gamma cl(U) \cap \alpha g_\gamma int(V) = \phi$. Then $\alpha g_\gamma cl(U) \subseteq X \backslash \alpha g_\gamma int(V) \subseteq B$. Therefore $A \subseteq U \subseteq \alpha g_\gamma cl(U) \subseteq B$.

(3) $\Longrightarrow$ (4) Consider any αg_γ-closed set A of X and an αg_γ-g.open set B containing A. Since $\alpha g_\gamma int(B)$ is αg_γ-open and by (3), there exists an αg_γ-g.open set U such that $A \subseteq U \subseteq \alpha g_\gamma cl(U) \subseteq \alpha g_\gamma int(B)$.

(4) $\Longrightarrow$ (5) Consider any αg_γ-closed set A of X and an αg_γ-g.open set B containing A. Then by (4), there exists an αg_γ-g.open set U such that $A \subseteq U \subseteq \alpha g_\gamma cl(U) \subseteq \alpha g_\gamma int(B)$. Now since $A \subseteq U$ and U is αg_γ-g.open, $A \subseteq \alpha g_\gamma int(U)$, by Remark 3. As $\alpha g_\gamma cl(\alpha g_\gamma int(U)) \subseteq \alpha g_\gamma cl(U)$, $A \subseteq \alpha g_\gamma int(U) \subseteq \alpha g_\gamma cl(\alpha g_\gamma int(U)) \subseteq \alpha g_\gamma cl(U) \subseteq \alpha g_\gamma int(B)$.

(5) $\Longrightarrow$ (6) Consider any αg_γ-g.closed set A of X and an αg_γ-open set B containing A. Then $\alpha g_\gamma cl(A) \subseteq B$. Since every αg_γ-open set is αg_γ-g.open, B is

αg_γ-g.open and $\alpha g_\gamma cl(A)$ is αg_γ-closed, by (5), there exists an αg_γ-open set U such that $\alpha g_\gamma cl\,(A) \subseteq U \subseteq \alpha g_\gamma cl\,(U) \subseteq \alpha g_\gamma int\,(B) = B$.

(6) $\Longrightarrow$ (7) Obvious by the fact that every αg_γ-open set is αg_γ-g.open set.

(7) $\Longrightarrow$ (1) Consider any two disjoint αg_γ-closed sets A and B. Then $A \subseteq X\backslash B$, A is $\alpha g_\gamma.g$-closed and $X\backslash B$ is αg_γ-open. Then by (7), there exists an $\alpha g_\gamma.g$-open set U such that $\alpha g_\gamma cl(A) \subseteq U \subseteq \alpha g_\gamma cl(U) \subseteq X\backslash B \Rightarrow B \subseteq X\backslash \alpha g_\gamma cl(U)$. Since A is αg_γ-closed and by Remark 3, $A \subseteq \alpha g_\gamma int(U)$. Let $V = \alpha g_\gamma int(U)$ and $W = X\backslash \alpha g_\gamma cl(U)$. Therefore, V and W are disjoint αg_γ-open sets containing A and B respectively. Hence X is αg-γ-normal.

Theorem 4.2. *The following are equivalent for a topological space (X, τ) with an operation γ on $\tau_{\alpha g}$:*

1. *X is αg-γ- normal.*
2. *For each αg_γ-closed set A and each αg_γ-open set U containing A, there exists an αg_γ-open set V containing A such that $\alpha g_\gamma cl(V) \subseteq U$.*
3. *For each pair of disjoint αg_γ-closed sets A and B of X, there exists an αg_γ-open set V containing A such that $\alpha g_\gamma cl\,(V) \cap B = \phi$.*

Proof. (1) $\Longrightarrow$ (2) Consider an αg_γ-closed set A and any αg_γ-open set U containing A. Then $X\backslash U$ is an αg_γ-closed set and it does not contain A. By the hypothesis, there exist αg_γ-open sets V and W containing A and $X\backslash U$ respectively, such that $V \cap W = \phi$. This implies $V \subseteq X\backslash W$. Hence $V \subseteq \alpha g_\gamma cl\,(V) \subseteq \alpha g_\gamma cl\,(X\backslash W) = X\backslash W \subseteq U$.

(2) $\Longrightarrow$ (3) Consider any pair of disjoint αg_γ-closed sets A and B of X. Then $X\backslash B$ is an αg_γ-open set containing A. By (2), there exists an αg_γ-open set V containing A such that $\alpha g_\gamma cl(V) \subseteq X\backslash B$. This implies that $\alpha g_\gamma cl\,(V) \cap B = \phi$.

(3) $\Longrightarrow$ (1) Consider any two disjoint αg_γ-closed sets A and B of X. Then by (3), there exists an αg_γ-open set V containing A such that $\alpha g_\gamma cl\,(V) \cap B = \phi$. Consequently, we have $X\backslash \alpha g_\gamma cl\,(V)$ to be an αg_γ-open set that contains B and $(X\backslash \alpha g_\gamma cl\,(V)) \cap V = \phi$. Therefore X is αg-γ- normal.

5. Interrelation between αg-γ-regular, αg-γ-normal and $\alpha g_\gamma - T_i^{'}(i = 0, 1, 2)$

Proposition 5.1. *If (X, τ) is both αg_γ-$T_1^{'}$ and αg-γ-normal then (X, τ) is αg-γ-regular.*

Proof. Let x be any point in X and A be any αg_γ-closed set such that $x \notin A$. Since X is αg_γ-$T_1^{'}$, every singleton is αg_γ-closed [14]. Then $\{x\}$ and A are disjoint αg_γ-closed sets of X. Since X is αg-γ-normal, there exist disjoint αg_γ-open sets

U and V such that $x \in \{x\} \subseteq U$ and $A \subseteq V$ respectively. Therefore X is αg-γ-regular.

Remark 6. *Converse of Proposition 5.1 is not true.*

Example 5.1. The topological space (X, τ) with an operation γ on $\tau_{\alpha g}$ as defined in Example 3.2 is both αg-γ-regular and αg-γ-normal, but not αg_γ-$T_1^{'}$.

Proposition 5.2. *If (X, τ) is both αg_γ-$T_1^{'}$ and αg-γ-regular, then (X, τ) is αg_γ-$T_2^{'}$.*
Proof. Consider any pair of distinct points x, y in X. Since X is αg_γ-$T_1^{'}$, every singleton is αg_γ-closed [14]. Then $\{y\}$ is an αg_γ-closed set of X which does not contain x. As X is αg-γ-regular, there exist disjoint αg_γ-open sets U and V such that $x \in U$ and $y \in \{y\} \subseteq V$ respectively. Therefore, (X, τ) is αg_γ-$T_2^{'}$.

Proposition 5.3. *If (X, τ) is both αg_γ-$T_1^{'}$ and αg-γ-normal, then (X, τ) is αg_γ-$T_2^{'}$.*
Proof. Obvious by Proposition 5.1 and Proposition 5.2.

6. Conclusion

In this manuscript, we have introduced αg-γ-regular and αg-γ-normal spaces. we have studied the interrelations and characteristics of the defined spaces with corresponding counter examples. In future, continuous maps using operation approach on αg open sets and their characteristics in the defined spaces can be analysed. Also, αg-γ-compact spaces can be defined and its properties can be determined. Generalised versions of Urysohn lemma and Tietze extension theorem using αg-γ-normal space can be studied in near future. The produced results can also be extended to bi-operations via αg-open sets in topological spaces.

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