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HALF CAUCHY GENERALIZED RAYLEIGH DISTRIBUTION: THEORY AND APPLICATION

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Abstract: We develop a new three parameter distribution using the half-Cauchy family of distributions, called the half-Cauchy generalized Rayleigh distribution. Some statistical properties and characteristics of the proposed distribution are provided and obtained, including the explicit expressions for the survival function, median, hazard function, mode, moments, mean deviation, order statistics, cumulative hazard function, quantiles, and the measures of dispersion based on quartiles and octiles. For the parameter estimation of the proposed model, three widely used estimation techniques namely, maximum likelihood estimators (MLE), Cramer-Von-Mises (CVM), and least-square estimation (LSE) methods are applied. Two real datasets are used for the illustration, and the goodness-of-fit test is run. It is discovered that the proposed model fits the real data very well and is more flexible than some well-known models under consideration.

Keywords and Phrases: Generalized Rayleigh distribution, Failure rate average, Half-Cauchy distribution, Order statistics, Moment.

2020 Mathematics Subject Classification: 62F15, 65C05.

1. Introduction

There are numerous continuous probability distributions found in probability theory and applied statistics literature. Some application areas, like environmental, actuarial, and medical sciences, economics, life sciences, demography, finance, insurance, etc., where the majority of the classical probability distributions have been extensively employed for modeling real datasets for many years. However, in many applied fields like finance, survival analysis, insurance, etc., there is an obvious need for customized forms of the more flexible models for modeling real datasets that can deal with a high degree of kurtosis and skewness. In this study, we have considered the half-Cauchy distribution that results from the Cauchy distribution by truncating it at the origin, thereby allowing only non-negative observations to be made. Shaw (1995) has used the half-Cauchy distribution with a heavy tail as an option to model spreading distances since it can forecast more recurrent long-distance spreading events. In addition, the half-Cauchy distribution is also used by (Paradis et al., 2002). Let X be a non negative random variable that follows the half-Cauchy distribution and its cumulative distribution function (CDF) is

$$G(x;\theta) = \frac{2}{\pi} tan^{-1}\left(\frac{x}{\theta}\right), \; x > 0, \theta > 0. \tag{1.1}$$

and its corresponding probability density function (PDF) is

$$g(x;\theta) = \frac{2}{\pi}\left(\frac{\theta}{\theta^2 + x^2}\right), \; x > 0, \theta > 0. \tag{1.2}$$

For many years, different researchers have used the half-Cauchy distribution as a parent distribution. The modification of the half-Cauchy distribution was introduced by (Cordeiro & Lemonte, 2011a) using a beta-generalized family named the beta-half-Cauchy distribution. Another extension of the half-Cauchy distribution was obtained by applying the Marshall-Olkin transformation and studying the autoregressive procedure of first order by (Jacob & Jayakumar, 2012). Polson and Scott (2012) have also used the half-Cauchy distribution for the Bayesian analysis as the prior for a universal scale parameter. Ghosh (2014) has introduced another extension of the half-Cauchy distribution named the Kumaraswamy-half-Cauchy distribution. The gamma half-Cauchy model has developed by (Alzaatreh et al., 2016). Cordeiro et al. (2017) have developed the family of distributions using half-Cauchy distribution as a generalized odd half-Cauchy generating family of distributions. Recently (Chaudhary et al., 2022) have introduced the half-Cauchy generalized exponential distribution by considering the generalized exponential as a baseline distribution.

As a result, we are involved in the creation of a new distribution based on the half-Cauchy family of distributions. In this paper, we introduced a new distribution that uses the generalized Rayleigh (GR) distribution as a baseline distribution, as defined by (Surles & Padgett, 2001); for more information, (see Kundu & Raqab, 2005). Chaudhary and Kumar (2020) have introduced an extension of Rayleigh distribution named the Logistic–Rayleigh distribution. Similarly (Joshi & Kumar, 2021) have presented the Poisson GR distribution. The CDF and PDF of the GR distribution are respectively as

$$G(x;\alpha,\lambda)=\left\{1-\exp(-\lambda x^2)\right\}^{\alpha} \ ; \alpha,\lambda>0, x>0. \tag{1.3}$$

and

$$g(x;\alpha,\lambda)=2\alpha\lambda^2 x e^{-\lambda x^2}\left[1-e^{-\lambda x^2}\right]^{\alpha-1}; \ \alpha,\lambda>0, \ x>0. \tag{1.4}$$

The main objective of this work is to create a versatile model by inserting only one extra shape parameter to the GR distribution to get a good fit to real datasets.

The rest of the sections of the article are composed as follows: In Section 2, the half-Cauchy generalized Rayleigh distribution is introduced, and we also derive some explicit expressions such as the reliability function, the PDF, the hazard function, the CDF, and the cumulative hazard function. In section 3, we discuss some statistical properties such as quantile function median, mode, mean past lifetime function, moments, mean deviation, order statistics, quantiles, skewness measures based on quartiles, and kurtosis measures based on octiles. In Section 4, the estimation of the model parameters of the proposed distribution is carried out using the three widely used estimation techniques namely maximum likelihood estimator (MLE), Cramer-Von-Mises (CVM), and least-square (LSE) methods. The application of the proposed model is discussed in Section 5. Some concluding explanations are entered in Section 6.

2. The Half-Cauchy Generalized Rayleigh (HCGR) Distribution

To develop the new family of distributions, we have used the generating family of distributions defined by (Ristic & Balakrishnan, 2012) whose CDF can be obtained as

$$F\left(x\right)=1-\int_{0}^{-\ln[G(x)]} r\left(t\right)\ dt; \tag{2.1}$$

here $r(t)$ is the PDF of any distribution and $G(x)$ is the CDF of any baseline distribution. The generating family of half-Cauchy distributions whose CDF can be calculated by considering the PDF of half-Cauchy distribution expressed in Equation (1.2) as $r(t)$ then we get

$$F\left(x\right)=1-\int_{0}^{-\ln[G(x)]} \frac{2}{\pi}\frac{\theta}{\theta^2+t^2}\ dt$$

$$= 1 - \frac{2}{\pi} \arctan \left\{ -\frac{1}{\theta} \ln \left[G\left(x\right) \right] \right\}. \tag{2.2}$$

The PDF of half-Cauchy family corresponding to Equation (2.2) can be articulated as

$$f(x) = \frac{2}{\pi} \frac{g(x)}{\theta G(x)} \left[1 + \left\{ -\frac{1}{\theta} \log G(x) \right\}^2 \right]^{-1}. \tag{2.3}$$

Now substituting Equations (1.3) and (1.4) in Equations (2.2) and (2.3), we obtain the CDF and PDF of HCGR distribution which are expressed as follows

$$F(x) = 1 - \frac{2}{\pi} \arctan \left\{ -\frac{\alpha}{\theta} \log \left\{ 1 - e^{-(\lambda x)^2} \right\} \right\};\ \alpha, \lambda, \theta > \ 0,\ x > 0. \tag{2.4}$$

$$f(x) = \frac{4\alpha\lambda^2}{\pi\theta} x e^{-(\lambda x)^2} \left[1 - e^{-(\lambda x)^2} \right]^{-1} \left[1 + \left\{ -\frac{\alpha}{\theta} \log \left\{ 1 - e^{-(\lambda x)^2} \right\} \right\}^2 \right]^{-1} x > 0. \tag{2.5}$$

Reliability/ Survival function: The reliability/survival function of HCGR distribution is

$$\begin{aligned} R(x) &= 1 - F(x) \\ &= \frac{2}{\pi} \arctan \left\{ -\frac{\alpha}{\theta} \log \left\{ 1 - e^{-(\lambda x)^2} \right\} \right\};\ \alpha, \lambda, \theta > \ 0,\ x > 0. \end{aligned} \tag{2.6}$$

Hazard rate function (HRF): The HRF of HCGR distribution is

$$\begin{aligned} h(x) &= \frac{f(x)}{R(x)} \\ &= \frac{2\alpha\lambda^2}{\theta} x e^{-(\lambda x)^2} \left[\left\{ 1 - e^{-(\lambda x)^2} \right\} \arctan \left\{ Z(x) \right\} \right]^{-1} \left[1 + \left\{ Z(x) \right\}^2 \right]^{-1}, \end{aligned} \tag{2.7}$$

where $Z(x) = -\frac{\alpha}{\theta} \log \left\{ 1 - e^{-(\lambda x)^2} \right\}$.

Reversed hazard rate function (RH): The RH of the proposed model is

$$RH = 4\alpha\lambda^2\theta^{-1} x e^{-(\lambda x)^2} \left[\left\{ 1 - e^{-(\lambda x)^2} \right\} \right]^{-1} \left[\left\{ \pi - 2\arctan \left\{ Z(x) \right\} \right\} \left\{ 1 + \left\{ Z(x) \right\}^2 \right\} \right]^{-1} \tag{2.8}$$

where $Z(x) = -\frac{\alpha}{\theta} \log \left\{ 1 - e^{-(\lambda x)^2} \right\}$. We have displayed the various shapes of PDF and HRF for different values of the parameters of HCGR distribution in Figure 1 and Figure 2 respectively. From Figure 2, the HRF can have either constant, increasing or bathtub shaped hazard rate which are necessary for reliability analysis.

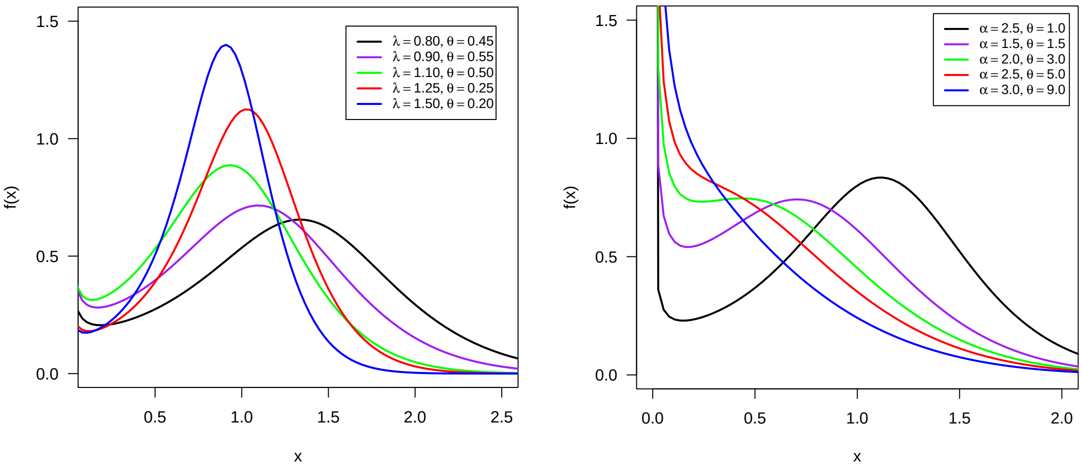

Figure 1. The plots of PDF for λ and θ keeping ($\alpha = 1$) constant (left panel) and for α and θ keeping ($\lambda = 1$) constant (right panel).

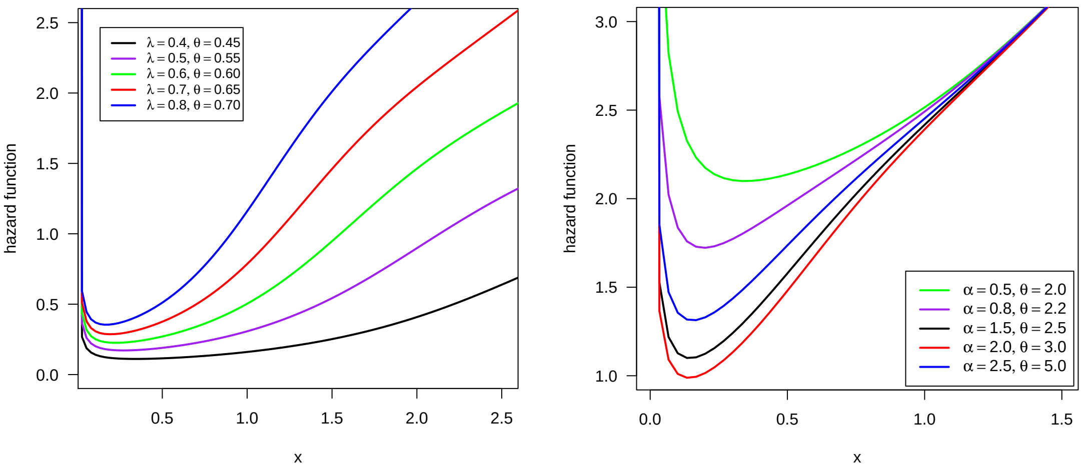

Figure 2. The plots of HRF for λ and θ keeping ($\alpha = 1$) constant (left panel) and for α and θ keeping ($\lambda = 1$) constant (right panel).

Cumulative hazard function: The cumulative hazard function of the proposed distribution is defined as

$$\begin{aligned} H(x) &= \int_{-\infty}^{x} h(t)\,dt \\ &= -\log\left[1 - F(x)\right] \\ &= -\log\left[\frac{2}{\pi}\arctan\left\{-\frac{\alpha}{\theta}\log\left\{1 - e^{-(\lambda x)^2}\right\}\right\}\right]. \end{aligned} \tag{2.9}$$

Failure rate average (FRA): Failure rate average of the random variable X can be defined for HCGR distribution as,

$$FRA(x) = \frac{H(x)}{x} = -\frac{1}{x}\log\left[\frac{2}{\pi}\arctan\left\{-\frac{\alpha}{\theta}\log\left\{1 - e^{-(\lambda x)^2}\right\}\right\}\right], \tag{2.10}$$

where $H(x)$ is the cumulative hazard function.

3. Properties of HCGR Distribution

Some statistical and mathematical properties of the HCGR distribution are briefly discussed in this section.

3.1. Quantile function of the HCGR distribution

The quantile function can be achieved by inverting the CDF defined in Equation (2.4) as

$$Q(u) = F^{-1}(u);$$

Hence we can put the quantile function for the random variable U as

$$Q(u) = \frac{1}{\lambda}\sqrt{-\ln\left(1 - t^{1/\alpha}\right)} \tag{3.1}$$

where

$$t = \exp\left\{-\theta\tan\left[\frac{(1-u)\pi}{2}\right]\right\}.$$

To generate the random numbers for HCGR we can use the following relation

$$x = \frac{1}{\lambda}\sqrt{-\ln\left(1 - t^{1/\alpha}\right)},$$

where

$$t = \exp\left\{-\theta\tan\left[\frac{(1-v)\pi}{2}\right]\right\} \ ;0 < v < 1.$$

3.2. Median of the HCGR distribution

The median of $HCGR(\alpha, \lambda, \theta)$ can be calculated by using the following relation

$$Median = \frac{1}{\lambda}\sqrt{-\ln\left(1 - e^{-\frac{\theta}{\alpha}}\right)};\ (\alpha, \lambda, \theta) > 0. \qquad (3.2)$$

3.3. Mode of HCGR distribution

The most recurring value of the probability distribution is the model value. To compute the mode $\frac{d \log f(x)}{dx} = 0$ gives $f'(x) = 0$, since $f(x) > 0$, then we can calculate the model value by solving the following equation,

$$1 + \left\{-\frac{\alpha}{\theta}\log\left(1 - e^{-(\lambda x)^2}\right)\right\}^2 + 2\left(\frac{\alpha}{\theta}\right)^2 \log\left(1 - e^{-(\lambda x)^2}\right) = 0 \qquad (3.3).$$

Analytically, it is difficult to solve since Equation (3.3) is nonlinear; hence, by using the Newton-Raphson or other appropriate algorithm, it can be solved numerically, which provides the value of the mode of the HCGR distribution.

3.4. Series expansion of distribution of HCGR distribution

To expand the CDF of HCGR distribution into a simple, elegant form, we have used the following relations

$$\arctan x = \frac{\pi}{2} - \sum_{p=0}^{\infty} \frac{(-1)^p}{2p+1} x^{2p+1}\ ; x > 0$$

$$\log(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - ...$$

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + ...$$

The CDF of HCGR distribution defined in Equation (2.4) can be expressed as

$$F(x; \alpha, \lambda, \theta) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} Z_{p,q}\psi_q(-2p-1)e^{-(q-2p-1)\lambda^2 x^2}; x > 0 \qquad (3.4)$$

where $Z_{p,q} = \frac{2}{\pi}\frac{(-1)^{-(3p+2)}}{2p+1}\left(\frac{\alpha}{\theta}\right)^{-(2p+1)}$ and $\psi_q(-2p-1)$ is the coefficient of $e^{-q\lambda^2 x^2}$ in the expansion of $\left[\log(1 - e^{-\lambda^2 x^2})\right]^{-2p-1}$ for more detail (see Balakrishnan & Cohen, 1991). Now the PDF of HCGR distribution corresponding to Equation (3.4) can be obtained as

$$\begin{aligned} f(x;\alpha,\lambda,\theta) &= 2\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}(q-2p-1)Z_{p,q}\psi_q(-2p-1)\lambda^2 x e^{-(q-2p-1)\lambda^2x^2} \\ &= \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}Z^*_{p,q}x\ e^{-(q-2p-1)\lambda^2x^2}. \end{aligned} \tag{3.5}$$

where $Z^*_{p,q} = \frac{4}{\pi}\frac{(-1)^{-(3p+2)}}{2p+1}\left(\frac{\alpha}{\theta}\right)^{-(2p+1)}(2p-q+1)\psi_q(-2p-1)\lambda^2$ and $\psi_q(-2p-1)$ is defined in Equation (3.4).

3.5. Mean past Lifetime (MPL) Function

The mean past lifetime (MPL) function is also an important statistical tool which can be used in many fields like actuarial studies, survival analysis and reliability theory. The MPL is also called expected idleness time function which can be calculated using CDF and PDF defined in Equation (3.4) and Equation (3.5) as

$$\begin{aligned} K(x) &= E\left(x - X/X \leqslant x\right) = x - \frac{\int_0^x t\ f(t)dt}{F(x)} \\ &= x - \frac{1}{2}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}Z^*_{p,q}\frac{\gamma\left\{\frac{3}{2},(q-2p-1)\lambda^2x\right\}}{\left[(q-2p-1)\lambda^2\right]^{\frac{3}{2}}} \\ &\qquad \left[\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}Z_{p,q}\psi_q(-2p-1)e^{-(q-2p-1)\lambda^2x^2}\right]^{-1} \quad ; x>0, \end{aligned} \tag{3.6}$$

where $\gamma\left(a,b\right)$ is lower incomplete gamma function.

3.6. Moments and Moment Generating Function

(a) Moments

The K^{th} moment about origin of the random variable X can be defined as

$$\mu'_k = E(X^k) = \int_0^{\infty} x^k f(x)dx;\ k=1,2,..., \tag{3.7}$$

Substituting the PDF $f(x)$ defined in Equation (3.5) in Equation (3.7) we get,

$$\begin{aligned}\mu_k' &= \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} Z_{p,q}^{*} \int_0^{\infty} x^{k+1}\; e^{-(q-2p-1)\lambda^2 x^2} dx \\ &= \frac{1}{2}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} Z_{p,q}^{*} \frac{\Gamma(\frac{k+2}{2})}{[(q-2p-1)\lambda^2]^{\frac{k+2}{2}}},\end{aligned} \tag{3.8}$$

where $\Gamma(.)$ is the gamma function. The first four moments about origin of the random variable X that follows $HCGR(\alpha,\lambda,\theta)$ distribution are computed as

$$Mean = E\left(X\right) = \frac{1}{2}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} Z_{p,q}^{*} \frac{\Gamma(\frac{3}{2})}{[(q-2p-1)\lambda^2]^{\frac{3}{2}}},$$

$$E\left(X^2\right) = \frac{1}{2}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} Z_{p,q}^{*} \frac{\Gamma(2)}{[(q-2p-1)\lambda^2]^{2}},$$

$$E\left(X^3\right) = \frac{1}{2}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} Z_{p,q}^{*} \frac{\Gamma(\frac{5}{2})}{[(q-2p-1)\lambda^2]^{\frac{5}{2}}}$$

and

$$E\left(X^4\right) = \frac{1}{2}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} Z_{p,q}^{*} \frac{\Gamma(3)}{[(q-2p-1)\lambda^2]^{3}}.$$

(b) Moment Generating Function (MGF)

The MGF of X that follows HCGR distribution can be derived as

$$M_X(t) = \sum_{k=0}^{\infty}\frac{t^k}{k!}\mu_k' = \frac{1}{2}\sum_{k=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{t^k}{k!} Z_{p,q}^{*} \frac{\Gamma(\frac{k+2}{2})}{[(q-2p-1)\lambda^2]^{\frac{k+2}{2}}}, \tag{3.9}$$

where μ_k' is the K^{th} moment about origin defined in Equation (3.8).

3.7. Mean Deviation (MD)

Let X be a random variable with PDF and CDF defined in Equations (3.5) and (3.4) then MD can be defined as

(a) MD from mean is

$$\begin{aligned}MD(mean) &= \int_0^{\infty} |x-\mu|\;\; f(x)dx \\ &= 2\mu F\left(\mu\right) - 2\mu + M\left(\mu\right).\end{aligned} \tag{3.10}$$

(b) MD from median is

$$MD(median) = \int_0^\infty |x - m_d| \ f(x)dx = m_d F(m_d) - m_d - \mu + M(m_d). \tag{3.11}$$

where μ is $E(X)$, m_d is value of the median of x, and $M(m_d) = \int_{m_d}^{\infty} t \ f(t)dt$ is the first incomplete moment.

3.8. Order Statistics

Order statistics can be used in many fields of probability theory and applied statistics. Hence we present some properties of the order statistics for the proposed distribution. Let $X_1, ..., X_n$ be n independently and identically distributed random variates, each with CDF $F(x)$. If these variables are arranged in ascending order of magnitude and then written as $X_{(1)} \leqslant ... \leqslant X_{(n)}$ and the r^{th} order statistic, $X_{(r)}$; $r = 1, 2, ..., n$ having the inequality relations among them are necessarily dependent. Suppose $X_{r:n}$ represents the r^{th} order statistic and $f_{r:n}$ denote PDF of r^{th} order statistic for $X_1, ..., X_n$ be n independently and identically distributed random variables from CDF $F(x)$ and can be defined as

$$\begin{aligned} f_{r:n}(x) &= \frac{n!}{(r-1)!(n-r)!} f(x) [F(x)]^{r-1} [1 - F(x)]^{n-r} \\ &= \frac{n!}{(r-1)!(n-r)!} f(x) \sum_{j=1}^{n-r} \binom{n-r}{j} [F(x)]^{j+r-1} \\ &= Bxe^{-(\lambda x)^2} \left[1 - e^{-(\lambda x)^2}\right]^{-1} \left[1 + \{z(x_i)\}^2\right]^{-1} \sum_{j=1}^{n-r} \binom{n-r}{j} \\ &\qquad \left[1 - \frac{2}{\pi} \arctan\{z(x_i)\}\right]^{j+r-1}, \end{aligned} \tag{3.12}$$

Where $z(x_i) = -\frac{\alpha}{\theta} \log\left\{1 - e^{-(\lambda x)^2}\right\}$ and $B = \frac{4\alpha\lambda^2}{\pi\theta} \frac{n!}{(r-1)!(n-r)!}$

3.9. Skewness

The measure of skewness using quantiles can be computed as

$$S(B) = \frac{Q(3/4) + Q(1/4) - 2Q(0.5)}{Q(3/4) - Q(1/4)}. \tag{3.13}$$

3.10. Kurtosis

The measure of kurtosis using octiles was defined by (Moors, 1988) as

$$K_u(M) = \frac{Q(0.875) + Q(0.375) - Q(0.625) - Q(0.125)}{Q(3/4) - Q(1/4)}. \tag{3.14}$$

4. Parameter Estimation Methods

In this section, we explain the commonly used three methods for the estimation of unknown parameters of the $HCGR(\alpha, \lambda, \theta)$ distribution, which are listed below.

4.1. Maximum Likelihood Estimation (MLE)

Here we present the ML estimators (MLE's) of the HCGR distribution. Let a random sample of size n be $\underset{\sim}{x} = (x_1, \ldots, x_n)$ from $HCGR(\alpha, \lambda, \theta)$ and we can express the log likelihood density function as

$$\begin{aligned} \ell(\alpha, \lambda, \theta|\underset{\sim}{x}) = n\ln(4/\pi) + n\ln\alpha + 2n\ln\lambda - n\ln\theta + \sum_{i=1}^{n}\ln x_i - \\ \lambda^2\sum_{i=1}^{n}x_i^2 - \sum_{i=1}^{n}\ln A(x_i) - \sum_{i=1}^{n}\ln\left[1 + \left\{-\frac{\alpha}{\theta}\ln A(x_i)\right\}^2\right], \end{aligned} \tag{4.1}$$

where $A(x_i) = 1 - e^{-(\lambda x_i)^2}$. After differentiating Equation (4.1) with respect to model parameters α, λ and θ, we obtain

$$\begin{aligned} \frac{\partial\ell}{\partial\alpha} &= \frac{n}{\alpha} - 2\alpha\theta^{-2}\sum_{i=1}^{n}\{\ln A(x_i)\}^2\left\{1 + \left\{-\frac{\alpha}{\theta}\ln A(x_i)\right\}^2\right\}^{-1}, \\ \frac{\partial\ell}{\partial\lambda} &= \frac{2n}{\lambda} - 2\lambda\sum_{i=1}^{n}x_i^2 - 2\lambda\sum_{i=1}^{n}\left\{x_i^2 e^{-(\lambda x_i)^2}A(x_i)^{-1}\right\} \\ &\quad -4\left(\frac{\alpha}{\theta}\right)^2\lambda\sum_{i=1}^{n}x_i^2 e^{-(\lambda x)^2}A(x_i)^{-1}\ln A(x_i)\left[1 + \left\{-\frac{\alpha}{\theta}\ln A(x_i)\right\}^2\right]^{-1}, \\ \frac{\partial\ell}{\partial\theta} &= -\frac{n}{\theta} + 2\alpha^2\theta^{-3}\sum_{i=1}^{n}\{\ln A(x_i)\}^2\left[1 + \left\{-\frac{\alpha}{\theta}\ln A(x_i)\right\}^2\right]^{-1}. \end{aligned} \tag{4.2}$$

By setting $\frac{\partial\ell}{\partial\alpha} = \frac{\partial\ell}{\partial\lambda} = \frac{\partial\ell}{\partial\theta} = 0$ and to obtain the value of α, λ and θ we have to solve Equations (4.2) numerically and we get the MLEs of the $HCGR(\alpha, \lambda, \theta)$ model. Normally, it is not possible to solve non-linear Equations (4.2) so with the aid of suitable computer software one can simply solve them. Suppose $\underline{\Theta} = (\alpha, \lambda, \theta)$ indicate the parameter vector of $HCGR(\alpha, \lambda, \theta)$ and its corresponding

MLEs of $\underline{\Theta}$ as $\widehat{\underline{\Theta}} = (\hat{\alpha}, \hat{\lambda}, \hat{\theta})$, then the asymptotic normality results in, $\left(\widehat{\underline{\Theta}} - \underline{\Theta}\right) \rightarrow N_3\left[0, \left(I\left(\underline{\Theta}\right)\right)^{-1}\right]$ where $I\left(\underline{\Theta}\right)$ is the Fisher's information matrix (FIM) given by

$$I\left(\underline{\Theta}\right) = -\begin{pmatrix} E\left(\frac{\partial^2 l}{\partial\alpha^2}\right) & E\left(\frac{\partial^2 l}{\partial\alpha\partial\lambda}\right) & E\left(\frac{\partial^2 l}{\partial\alpha\partial\theta}\right) \\ E\left(\frac{\partial^2 l}{\partial\lambda\partial\alpha}\right) & E\left(\frac{\partial^2 l}{\partial\lambda^2}\right) & E\left(\frac{\partial^2 l}{\partial\lambda\partial\theta}\right) \\ E\left(\frac{\partial^2 l}{\partial\alpha\partial\theta}\right) & E\left(\frac{\partial^2 l}{\partial\lambda\partial\theta}\right) & E\left(\frac{\partial^2 l}{\partial\theta^2}\right) \end{pmatrix}.$$

Substituting the estimated values of the parameters we can approximate an asymptotic variance. The observed FIM $O(\widehat{\underline{\Theta}})$ is utilized as an estimate of the FIM $I\left(\underline{\Theta}\right)$ as

$$O(\widehat{\underline{\Theta}}) = -\begin{pmatrix} \frac{\partial^2 l}{\partial\hat{\alpha}^2} & \frac{\partial^2 l}{\partial\hat{\alpha}\partial\hat{\lambda}} & \frac{\partial^2 l}{\partial\hat{\alpha}\partial\hat{\theta}} \\ \frac{\partial^2 l}{\partial\hat{\alpha}\partial\hat{\lambda}} & \frac{\partial^2 l}{\partial\hat{\lambda}^2} & \frac{\partial^2 l}{\partial\hat{\theta}\partial\hat{\lambda}} \\ \frac{\partial^2 l}{\partial\hat{\alpha}\partial\hat{\theta}} & \frac{\partial^2 l}{\partial\hat{\theta}\partial\hat{\lambda}} & \frac{\partial^2 l}{\partial\hat{\theta}^2} \end{pmatrix}_{|(\hat{\alpha},\hat{\lambda},\hat{\theta})} = -\Delta\left(\underline{\Theta}\right)_{|(\underline{\Theta}=\widehat{\underline{\Theta}})},$$

where Δ represent the matrix of Hessian. Using the algorithm defined by Newton-Raphson, the likelihood function can be optimized that gives the empirical information matrix. Consequently, the var-covar matrix $A = \left[-\Delta\left(\underline{\Theta}\right)_{|(\underline{\Theta}=\widehat{\underline{\Theta}})}\right]^{-1}$ is computed as

$$A = \begin{pmatrix} V(\hat{\alpha}) & \text{cov}(\hat{\alpha}, \hat{\lambda}) & \text{cov}(\hat{\alpha}, \hat{\theta}) \\ \text{cov}(\hat{\lambda}, \hat{\alpha},) & V(\hat{\lambda}) & \text{cov}(\hat{\lambda}, \hat{\theta}) \\ \text{cov}(\hat{\theta}, \hat{\alpha}) & \text{cov}(\hat{\theta}, \hat{\lambda}) & V(\hat{\theta}) \end{pmatrix}. \tag{4.3}$$

For the model parameters we can construct 100(1-Ω) % confidence intervals as,

$$\hat{\alpha} \pm Z_{\Omega/2}SD(\hat{\alpha}),\ \hat{\theta} \pm Z_{\Omega/2}SD(\hat{\theta}) \text{ and } \hat{\lambda} \pm Z_{\Omega/2}SD(\hat{\lambda}).$$

where $Z_{\Omega/2}$ is the area under standard normal curve.

4.2. Method of Least-Square Estimation (LSE)

Another well-known estimation method is the ordinary least square estimation method (Swain et al., 1988). By using this method we estimates the parameters of $HCGR(\alpha, \lambda, \theta)$ by minimizing,

$$A\left(X; \alpha, \lambda, \theta\right) = \sum_{k=1}^{n}\left[F(X_{(k)}) - \frac{k}{n+1}\right]^2. \tag{4.4}$$

Where $F(x_k)$ represents the CDF of the ordered random variables $X_1 < \ldots < X_n$ from a CDF $F(.)$. To calculate the least-square estimates $\hat{\alpha}, \hat{\lambda}$ *and* $\hat{\theta}$ of α, λ, and θ

we have to minimize Equation (4.5),

$$A\left(X\right)=\sum_{k=1}^{n}\left[1-\frac{2}{\pi}\arctan\left\{-\frac{\alpha}{\theta}\log\left\{1-e^{-\left(\lambda x_{(k)}\right)^{2}}\right\}\right\}-\frac{k}{n+1}\right]^{2} \tag{4.5}$$

w. r. to α, λ, and θ. Differentiating Equation (4.5) w. r. to α, λ, and θ we get,

$$\frac{\partial A}{\partial \alpha}=\frac{4}{\pi\theta}\sum_{k=1}^{n}\left[1-\frac{2}{\pi}\arctan\left\{-\frac{\alpha}{\theta}D(x_k)\right\}-\frac{k}{n+1}\right]D(x_k)\left[1+\left\{-\frac{\alpha}{\theta}D(x_k)\right\}^{2}\right]^{-1},$$

$$\begin{aligned}\frac{\partial A}{\partial \lambda}=&-\frac{8\alpha\lambda}{\pi\theta}\sum_{k=1}^{n}\left[1-\frac{2}{\pi}\arctan\left\{-\frac{\alpha}{\theta}D(x_k)\right\}-\frac{k}{n+1}\right](x_k)^{2}\\ &e^{-(\lambda x_k)^2}\left\{1-e^{-(\lambda x_k)^2}\right\}^{-1}\left[1+\left\{-\frac{\alpha}{\theta}D(x_k)\right\}^{2}\right]^{-1},\end{aligned}$$

$$\frac{\partial A}{\partial \theta}=\frac{4\alpha}{\pi\theta^2}\sum_{k=1}^{n}\left[1-\frac{2}{\pi}\arctan\left\{-\frac{\alpha}{\theta}D(x_k)\right\}-\frac{k}{n+1}\right]D(x_k)\left[1+\left\{-\frac{\alpha}{\theta}D(x_k)\right\}^{2}\right]^{-1},$$

where $D(x_k)=\log\left\{1-e^{-(\lambda x_k)^2}\right\}$. Likewise we can compute the weighted LS estimators by minimizing,

$$W\left(X;\alpha,\lambda,\theta\right)=\sum_{k=1}^{n}w_k\left[F(X_{(k)})-\frac{k}{n+1}\right]^{2},$$

w. r. to α, λ, and θ. Here the weights w_k are $w_k=\frac{1}{Var(X_{(k)})}=\frac{(n+2)(n+1)^2}{k(n-k+1)}$.

4.3. Cramer-Von-Mises estimation (CVME) Method

The CVME estimators for the parameters of $HCGR(\alpha,\lambda,\theta)$ distribution are calculated by optimizing the function,

$$\begin{aligned}B&=\frac{1}{12n}+\sum_{k=1}^{n}\left[F\left(x_{k:n}|\alpha,\lambda,\theta\right)-\frac{2k-1}{2n}\right]^{2};\ x\geqslant 0\\ &=\frac{1}{12n}+\sum_{k=1}^{n}\left[1-\frac{2}{\pi}\arctan\left\{-\frac{\alpha}{\theta}\log\left\{1-e^{-(\lambda x)^2}\right\}\right\}-\frac{2k-1}{2n}\right]^{2}.\end{aligned} \tag{4.7}$$

After differentiating Equation (4.7) w. r. to α, λ, and θ we have,

$$\frac{\partial B}{\partial \alpha}=\frac{4}{\pi\theta}\sum_{k=1}^{n}\left[1-\frac{2}{\pi}\arctan\left\{-\frac{\alpha}{\theta}D(x_k)\right\}-\frac{2k-1}{2n}\right]D(x_k)\left[1+\left\{-\frac{\alpha}{\theta}D(x_k)\right\}^{2}\right]^{-1},$$

$$\frac{\partial B}{\partial \lambda} = -\frac{8\alpha\lambda}{\pi\theta} \sum_{k=1}^{n} \left[1 - \frac{2}{\pi} \arctan\left\{-\frac{\alpha}{\theta} D(x_k)\right\} - \frac{2k-1}{2n}\right]$$
$$(x_k)^2 e^{-(\lambda x_k)^2} \left\{1 - e^{-(\lambda x_k)^2}\right\}^{-1} \left[1 + \left\{-\frac{\alpha}{\theta} D(x_k)\right\}^2\right]^{-1},$$

$$\frac{\partial B}{\partial \theta} = \frac{4\alpha}{\pi\theta^2} \sum_{k=1}^{n} \left[1 - \frac{2}{\pi} \arctan\left\{-\frac{\alpha}{\theta} D(x_k)\right\} - \frac{2k-1}{2n}\right] D(x_k) \left[1 + \left\{-\frac{\alpha}{\theta} D(x_k)\right\}^2\right]^{-1}.$$

Here we suppose $D(x_k) = \log\left\{1 - e^{-(\lambda x_k)^2}\right\}$. Hence we get the CVM estimators after solving $\frac{\partial B}{\partial \alpha} = 0, \frac{\partial B}{\partial \lambda} = 0$ *and* $\frac{\partial B}{\partial \theta} = 0$ simultaneously.

5. Illustration with real datasets

Here we illustrate an application of HCGR distribution by using two real data sets.

Data set 1: The data reveals the breaking stress of carbon fibres of 5 cm length (GPa) which is also employed by (Nichols & Padgett, 2006), (Cordeiro & Lemonte, 2011b) and (Oguntunde et al., 2015). The data is shown in Table 1.

Table 1. Data set I

4.42	4.7	4.9	1.57	1.61	1.87	1.89	2.03	2.03	2.05	3.33
3.39	2.12	2.35	2.41	2.43	2.48	2.5	2.53	2.55	2.55	3.65
3.7	3.75	2.56	2.59	2.67	2.73	2.74	2.79	2.81	1.61	1.69
1.84	2.82	2.85	2.87	2.88	2.93	2.95	2.96	2.97	3.09	3.11
3.15	3.15	3.19	3.22	3.22	3.27	3.28	3.31	3.31	3.56	3.6
3.39	3.68	1.8	3.11	4.2	4.38	0.39	0.85	1.08	1.25	1.47

We have calculated the MLEs directly by using `maxLik()` function available in maxLik package (see Henningsen & Toomet, 2011; Dalgaard, 2008; R Core Team, 2021) and by maximizing the likelihood function (4.1). The MLE's for α, λ, and θ are $\hat{\alpha} = 1.4585$, $\hat{\lambda} = 0.5300$ and $\hat{\theta} = 0.1655$ and their corresponding standard errors (SE) are (4.0896), (0.0350) and (0.4674) respectively. We have also calculated the variance covariance matrix defined in Equation (4.3) by utilizing MLEs and displayed in matrix (5.1)

$$A = \begin{pmatrix} 16.7250 \; 0.00098 \; 1.8956 \\ 0.00098 \; 0.0012 \text{ - } 0.0017 \\ 1.8956 \text{ - } 0.0017 \; 0.2184 \end{pmatrix}. \tag{5.1}$$

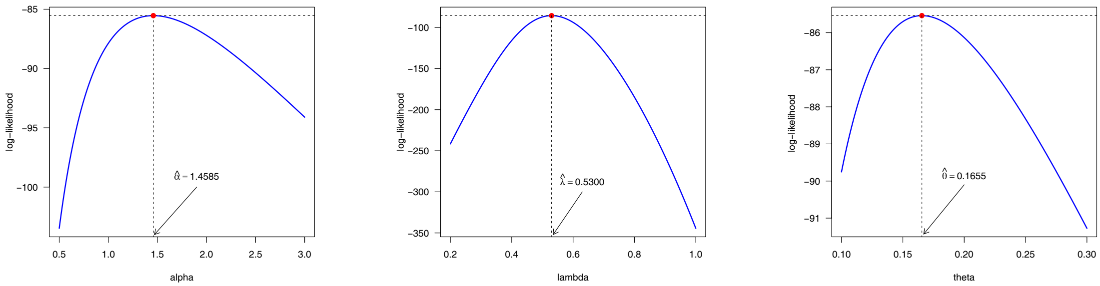

Figure 3. Profile log-likelihood functions of α, λ and θ.

From the plot of profile log-likelihood functions of ML estimates (Figure 3) we have noticed that MLEs of α, λ, and θ are exist and unique. The MLE, LSE and CVME estimates along with log-likelihood (LL), Akaike information criterion (AIC) and Kolmogrov-Smirnov (KS) statistic with $p - value$ are presented in Table 2 and found that all three methods are almost same for the data set 1 under study.

Table 2. Estimated parameters, LL, AIC and KS statistic with p-value of MLE, LSE and CVM methods.

Estimation Method	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\theta}$	LL	AIC	KS(p-value)
MLE	1.4585	0.53	0.1655	-85.5399	177.0799	0.0515(0.9948)
LSE	1.3013	0.5263	0.1527	-85.5457	177.0914	0.0538(0.9911)
CVME	1.2612	0.5341	0.1378	-85.5467	177.0934	0.0543(0.9899)

The goodness-of-fit of any model can be evaluated graphically using the fitted PDF and HRF plots. To get further information, we may also use P-P and Q-Q plots. In general, the Q-Q plot provides lack-of-fit information about the distribution's tails, whereas the P-P plot emphasizes the model's fit at the center. Figure 5 confirms that the HCGR model fits the data set 1 very well. For the assessment of the capability and goodness-of-fit of $HCGR(\alpha, \lambda, \theta)$ model, we have selected some pre-existing distributions which are as follows,

I. Weibull Extension (WE) Model:

The PDF of the three parameters WE distribution (Tang et al., 2003) is,

$$f_{WE}(x) = \beta\lambda \left(\frac{x}{\alpha}\right)^{\beta-1} \exp\left(\frac{x}{\alpha}\right)^{\beta} \exp\left\{-\lambda\alpha \left(\exp\left(\frac{x}{\alpha}\right)^{\beta} - 1\right)\right\}; x > 0.$$

II. Power Cauchy distribution:

The PDF of power Cauchy (PC) distribution was introduced by (Rooks et al.,

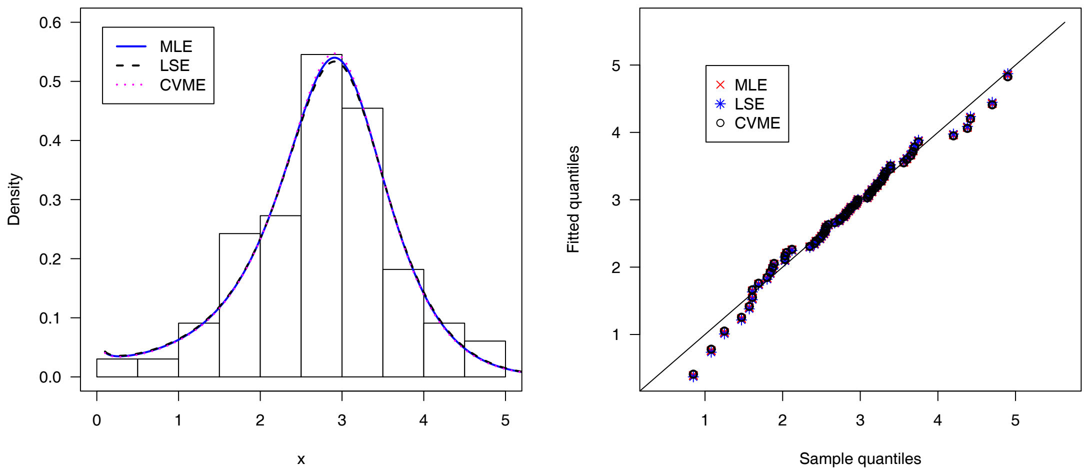

Figure 4. The Histogram and the distributions (left) and fitted quantiles verses sample quantiles (right) of ML, LS and CVM estimation methods.

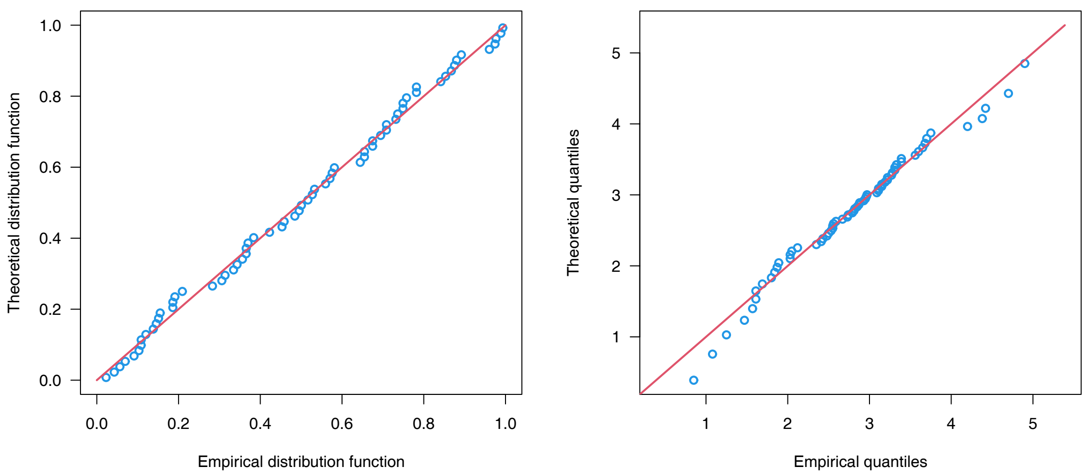

Figure 5. The P-P (left panel) and Q-Q (right panel) plots of HCGR distribution.

2010) is,

$$f_{PC}(x;\alpha,\lambda)=2\alpha\,(\pi x)^{-1}\,\left(\frac{x}{\lambda}\right)^{\alpha}\left\{1+\left(\frac{x}{\lambda}\right)^{2\alpha}\right\}^{-1};x>0.$$

III. Generalized Rayleigh (GR) model:
The PDF of GR model (Kundu & Raqab, 2005) is,

$$f_{GR}\left(\mathrm{x};\,\alpha,\,\lambda\right) = 2\alpha\lambda^2 \mathrm{x}\mathrm{e}^{-(\lambda x)^2}\left\{1-\mathrm{e}^{-(\lambda x)^2}\right\}^{\alpha-1} \qquad ; x>0.$$

IV. Generalized Exponential Extension (GEE) distribution:
The PDF of GEE given by (Lemonte, 2013) with parameters α, β and λ is,

$$f_{GEE}\left(x\right) = \alpha\beta\lambda\left(1+\lambda x\right)^{\alpha-1}\left[1-\mathrm{e}^{\{1-(1+\lambda x)^{\alpha}\}}\right]^{\beta-1}\mathrm{e}^{\{1-(1+\lambda x)^{\alpha}\}}; x>0.$$

V. Generalized Gompertz (GGZ) distribution:
The PDF of GGZ model introduced by (El-Gohary et al., 2013) is,

$$f_{GGZ}\left(x\right) = \lambda\theta e^{\alpha x}e^{-\frac{\lambda}{\alpha}\left(e^{\alpha x}-1\right)}\left[1-\exp\left(-\frac{\lambda}{\alpha}\left(e^{\alpha x}-1\right)\right)\right]^{\theta-1};\; x\geqslant 0.$$

For the test of goodness of fit of the proposed model, we have computed the fit statistics like negative log-likelihood (-LL), Akaike information criterion (AIC), Hannan-Quinn information criterion (HQIC), Corrected Akaike Information criterion (CAIC), and Bayesian information criterion (BIC) to decide the best model among the competing distributions. These statistics are calculated as.

$$\begin{aligned} AIC &= -2l(\hat{\theta})+2k, & CAIC &= \frac{2\left(k+1\right)k}{n-k-1}+AIC. \\ BIC &= -2l(\hat{\theta})+k\log\left(n\right), & HQIC &= -2l(\hat{\theta})+2k\log\left[\log\left(n\right)\right] \end{aligned}.$$

Here, n represents the size of the sample, and k is the number of parameters. Table 3 shows that the HCGR model fits well compared to the five other models taken for the study, and the result is also supported by the graphical illustration (Figure 6). Further, in order to test the fits attained by the distribution, we have computed the Kolmogrov-Smirnov (KS), the Cramer-Von Mises (A^2) and the Anderson-Darling (W) statistics. These statistics are computed as

$$A^2 = \frac{1}{12n}+\sum_{i=1}^{n}\left[\frac{(2i-1)}{2n}-d_i\right]^2, \qquad KS = \max_{1\leqslant i\leqslant n}\left(d_i-\frac{i-1}{n},\frac{i}{n}-d_i\right),$$

$$W = -n-\frac{1}{n}\sum_{i=1}^{n}\left(2i-1\right)\left[\ln d_i+\ln\left(1-d_{n+1-i}\right)\right].$$

where $d_{\mathrm{i}} = CDF\left(x_{\mathrm{i}}\right)$; the x_i's are the ordered samples.

Table 3. Some model selection statistics

Model	LL	AIC	BIC	CAIC	HQIC
HCGR	-85.5399	177.0799	183.6489	177.467	179.6756
GGZ	-85.6858	177.3716	183.9406	177.7587	179.9673
WE	-86.0643	178.1286	184.6976	178.5157	180.7243
GEE	-87.2704	180.5408	187.1098	180.9279	183.1365
GR	-88.6368	181.2735	185.6528	181.464	183.004
PC	-90.5126	185.0252	189.4045	185.2157	186.7557

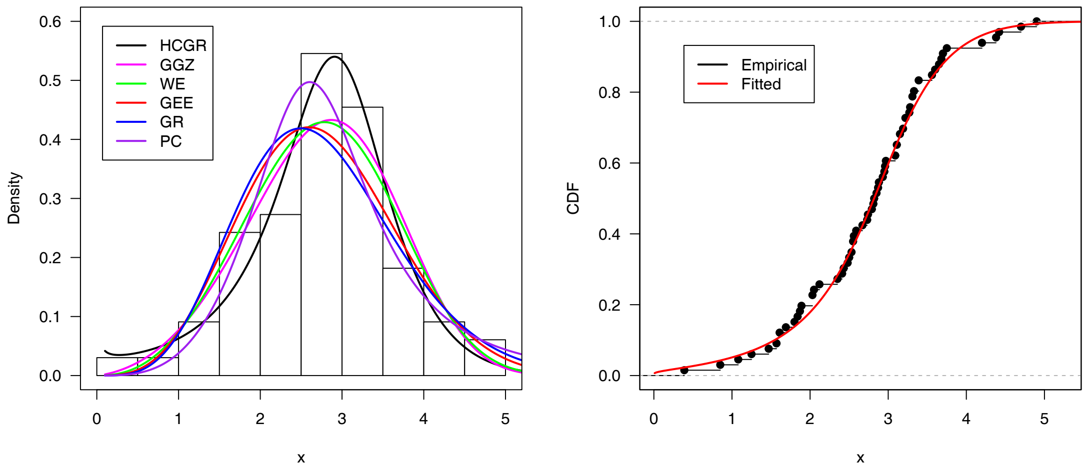

Figure 6. The PDF of fitted models and the histogram (left panel) and Empirical against estimated CDF (right panel).

The PDF of fitted distributions and the histogram and Empirical vs. fitted distribution function of GGZ, WE, PC, GEE, and GR distributions are shown in Figure 6. To select the best model, we have calculated KS, W, and A^2 statistics and presented them in Table 4. From Table 4, the HCGR distribution has the smallest test statistic value and greatest p-value; therefore, it performs better as compared to the other candidate distributions.

Table 4. Various statistics for goodness-of-fit along with p-value

Model	***KS(p-value)***	***W(p-value)***	A^2***(p-value)***
HCGR	0.0515(0.9948)	0.0267(0.9864)	0.2474(0.9717)
GGZ	0.0833(0.7498)	0.0715(0.7443)	0.4457(0.8020)
WE	0.0828(0.7562)	0.0850(0.6652)	0.4941(0.7522)
GEE	0.1096(0.4065)	0.1530(0.3812)	0.7816(0.4940)
GR	0.1205(0.2935)	0.1947(0.2784)	1.0044(0.3547)
PC	0.0963(0.5731)	0.1246(0.4782)	1.0733(0.3207)

Data set 2: The data set taken under study represents the strengths of 15 mm glass fibres (Smith & Naylor, 1987), measured at the National Physical Lab, UK. Unfortunately, the measurement units are not specified in the article. The data set is presented in Table 5.

Table 5. Data set II

0.55	0.93	1.25	1.36	1.49	1.52	1.58	1.61	1.64	1.68	1.73
1.53	1.59	1.61	1.66	1.68	1.76	1.82	2.01	0.77	1.11	1.28
1.76	1.84	2.24	0.81	1.13	1.29	1.48	1.5	1.55	1.61	1.62
1.48	1.51	1.55	1.61	1.63	1.67	1.7	1.78	1.89	1.04	1.27
1.6	1.62	1.66	1.69	1.84	0.84	1.24	1.3	0.74	1.54	1.77
1.81	2	1.81	2	1.81	2	1.81	2			

We have obtained the MLEs of HCGR distribution as α, λ, and θ are $\hat{\alpha} = 15.6378$, $\hat{\lambda} = 1.30865$ and $\hat{\theta} = 0.24028$ and log-likelihood -12.7267 and their corresponding standard errors (SE) are (6.9685), (0.07942) and (0.17035) respectively. Similarly we have calculated the estimate matrix of variance-covariance defined in (4.3) using MLEs as

$$B = \begin{pmatrix} 48.5600 \; 0.0015 \; 0.7436 \\ 0.0015 \; 0.0063 \text{ - } 0.0010 \\ 0.7436 \text{ - } 0.0010 \; 0.0290 \end{pmatrix}.$$

In Figure 7, the graphical views of profile log-likelihood are displayed using ML estimates of the parameters α, λ, and θ. We have noticed that MLEs of α, λ, and θ are unique.

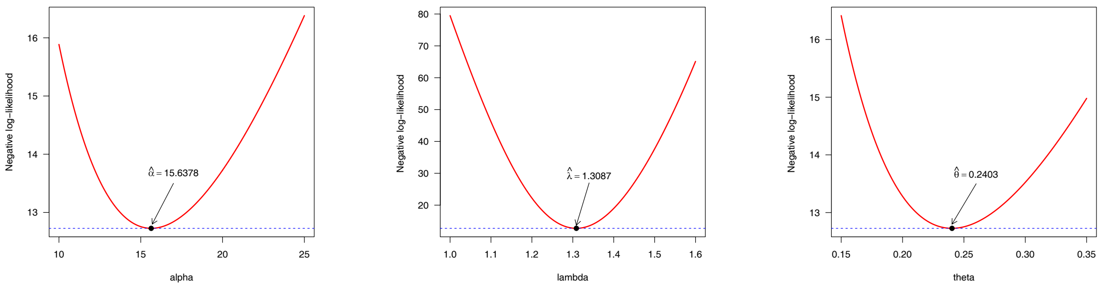

Figure 7. The plots of profile log-likelihood functions of ML estimates of α, λ and θ.

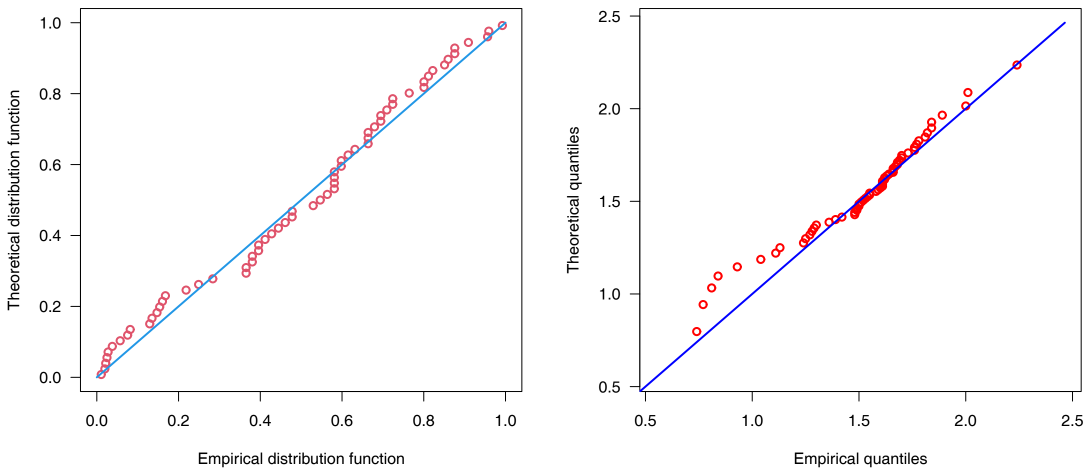

Figure 8. The P-P (left panel) and Q-Q (right panel) plots of HCGR distribution.

In Table 6, the MLE, LSE, and CVME estimates of parameters of the proposed model are presented with log-likelihood (LL), Akaike information criterion (AIC), Kolmogrov-Smirnov (KS) statistics, and a p-value (in parentheses). It is found that the MLE method performs well as compared to the LSE and CVME methods for the second data set.

From Tables 7 and 8, we observed that the HCGR model yields the least value of the test statistics and the highest p-value, this indicates that the suggested model can be considered best among the GGZ, WE, PC, GEE, and GR models. Also, we

Table 6. Estimated parameters, and some goodness-of-fit statistic of MLE, LSE, and CVM methods.

Estimation Method	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\theta}$	LL	AIC	KS(p-value)
MLE	15.6378	1.3087	0.2403	-12.7267	31.4535	0.0797(0.8181)
LSE	2.0618	1.3261	0.0281	-12.7511	31.5022	0.0766(0.8537)
CVME	2.0828	1.3467	0.0247	-12.8414	31.6828	0.0832(0.7757)

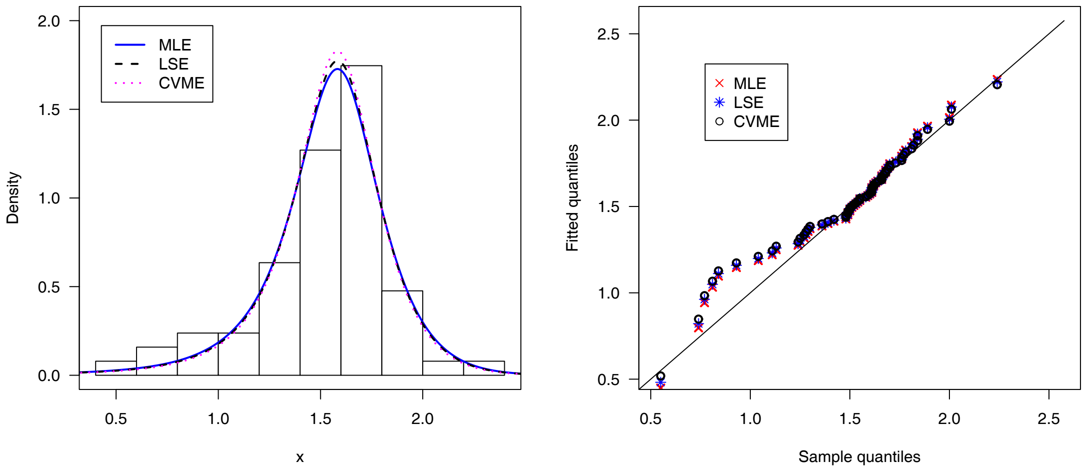

Figure 9. The Histogram and the distributions (left) and fitted quantiles verses sample quantiles (right) of ML, LS and CVM estimation methods.

Table 7. Some model selection statistics

Model	LL	AIC	BIC	CAIC	HQIC
HCGR	-12.7267	31.4535	37.8829	31.8603	33.9822
GGZ	-14.1452	34.2904	40.7198	34.6972	36.8191
WE	-15.1885	36.3771	42.8065	36.7839	38.9058
PC	-20.6521	45.3041	49.5904	45.5041	46.9899
GEE	-21.3634	48.7268	55.1562	49.1336	51.2555
GR	-23.9288	51.8575	56.1438	52.0575	53.5433

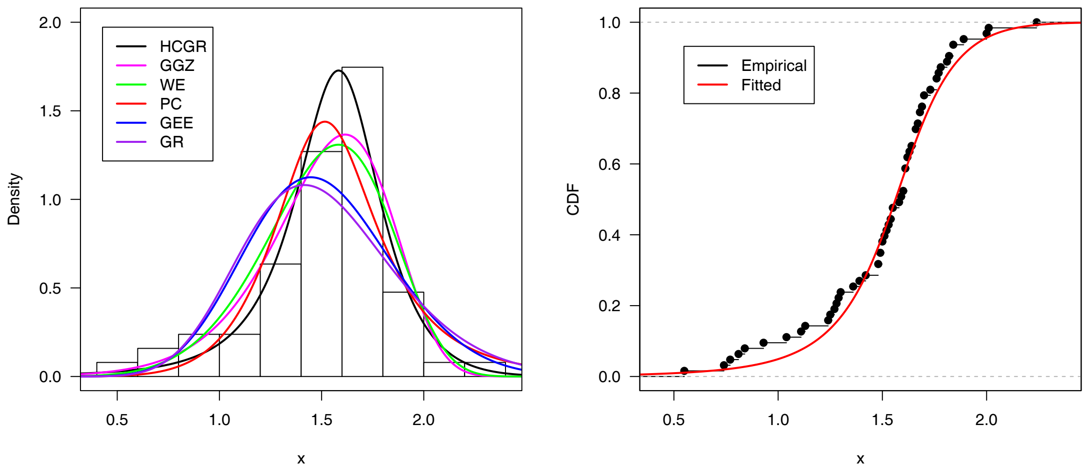

Figure 10. The PDF of fitted models and the histogram (left panel) and Empirical with estimated CDF (right panel).

have displayed the graphs of the fitted PDF and CDF plots in Figure 10 for the second data set. From these graphical illustrations, we conclude that the HCGR distribution performs better as compared to other candidate models.

Table 8. Various statistics for goodness-of-fit along with p-value

Model	***KS(p-value)***	***W(p-value)***	A^2***(p-value)***
HCGR	0.0797(0.8181)	0.0810(0.6876)	0.6453(0.6056)
GGZ	0.1318(0.2240)	0.1564(0.3714)	0.8864(0.4223)
WE	0.1520(0.1087)	0.2155(0.2395)	1.2242(0.2584)
PC	0.1235(0.2915)	0.2369(0.2061)	2.1344(0.0778)
GEE	0.2108(0.0074)	0.5269(0.0337)	2.7529(0.0368)
GR	0.2151(0.0059)	0.5832(0.0243)	3.1292(0.0237)

6. Conclusion

In this paper, we suggest a generalization of the half-Cauchy model and name it the half-Cauchy generalized Rayleigh (HCGR) distribution. For this study, some statistical and mathematical properties of the proposed distribution, such as the median, mode, mean past lifetime function, moments and its generating function, mean deviation, order statistics, quantiles, and the measures of skewness based on quartiles and kurtosis based on octiles, are provided. The Cramer-Von

Mises (CVM), maximum likelihood estimation (MLE), and least squares estimation (LSE) methods are used to estimate model parameters. To illustrate the goodness-of-fit and potentiality of the proposed distribution, we have considered two real datasets, and it is found that the HCGR distribution can attain a better fit as compared to some existing distributions. As a result, it is expected that the HCGR distribution will be used in applied statistics and survival analysis.

Acknowledgments

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