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CERTAIN COSMOLOGICAL MODELS WITH VARIATION OF HUBBLE PARAMETER

Ram Bharosha Tiwari and Sudhir Kumar Srivastava

Department of Mathematics and Statistics,
DDU Gorakhpur University,
Gorakhpur - 273009, Uttar Pradesh, INDIA

E-mail : rambharoshatiwari@gmail.com, sudhir.mathstat@ddugu.ac.in

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Abstract: The present paper deals with the FRW-Cosmological Model of universe for W_2 flat perfect fluid space time. Einstein field equations with variable cosmological constant (Λ) has been obtained for such spacetime and in order to get the complete cosmological solution the law of variation for Hubble's parameter is considered. A new class of solution have been discussed for the Einstein field equations with variable cosmological constant in which the pressure, energy density, and cosmological constant Λ are found to be decreasing function of cosmic time. The physical and kinematical properties of models are also discussed.

Keywords and Phrases: W_2 flat spacetime, FRW- spacetime, Perfect fluid spacetime.

2020 Mathematics Subject Classification: 83C05, 83C15, 83F05.

1. Introduction

The study of W_2 curvature tensor in differentiable manifolds and general relativity has attracted attention of researchers from many years.Pokhariyal and Mishra [32] have introduced this new curvature tensor and studies its properties.This (0,4) type tensor denoted by W_2, For the shake of convenience, we shall denote this tensor by W and is defined as

$$W_{hijk} = R_{hijk} + \frac{1}{(n-1)}[g_{hj}R_{ik} - g_{ij}R_{hk}] \tag{1}$$

where R_{hijk} is the Riemann curvature tensor and R_{ij} the Ricci-tensor. A spacetime for which all components of W_{hijk} vanishes at each point is said to be of W_2 flatness. Zafar Ahsan and Musavvir Ali [3] have studied the W_2 curvature tensor for the spacetimes of general relativity. FRW- space time has been studied by several researchers in different physical and geometrical aspects. FRW- space time describe non-static evolution of universe for W_2 flatness (discussed in this work). Motivated by the work of [32, 3], in this paper we have made a study of the FRW-cosmological model for W_2 flatness by considering perfect fluid as source of matter distribution.Here, we look at Robertson-Walker space time, where the source of matter distribution is a perfect fluid, and Einstein's field equations for W_2 flat space time with a variable cosmological term Λ. The cosmological models with dynamical terms Λ are gaining popularity from many years because they naturally address the issue of the cosmological constant problem.

The cosmos's current, accelerated expansion suggests that an unidentified dark energy (DE), an unusual energy with negative pressure, is in charge of our universe. The most basic DE candidate is cosmic constant's mathematical equivalent to the vacuum energy density. Since then, the cosmological constant Λ has been a motivating factor in gravitational theory. In order to create the gravitational repulsion required to support a static world, Einstein first proposed it in 1917. Since the Hubble constant was found, it has been assumed that the universe is expanding. Additionally, Friedmann [17] was able to create an expanding universe without the requirement for the phrase cosmological constant. In his equations for the gravitational field, Einstein acknowledged that a term was not necessary in any particular way. Zel'dovich [50] reignited the cosmological constant Λ debate by connecting it to the vacuum energy density brought on by quantum fluctuations. In this manner, the maintenance of the cosmological constant Λ was taking place gradually and gaining solid theoretical support. Since the empirical upper bound was so great ($\Lambda \leq 10^{-120}$ Planck units) and there was no direct astronomical evidence for before 1998, several particle physicists hypothesised that some underlying principle must compel its value to be exactly zero. Two separate teams, led by Riess et al. [40] and Perlmutter et al. [31], have made similar attempts to demonstrate the universe's expansion rate through the use of type Ia supernovae. With $\Lambda \sim 1.7X10^{-121}$ Planck units, this finding offered the first definite proof that is greater than 0. The research established today holds that the finest appropriate explanation for recent discoveries that the universe appears to be expanding and speeding up involves a type of repulsive pressure, known as DE, which operates through the cosmological constant.

The simplest and most popular candidate to describe this exotic component among the many options is the cosmological constant, which functions in Einstein's field equation as an source with $p = -w$ that is homogeneous and isotropic. Due to its excellent approximation of the current astronomical data, the cosmological constant holds a privileged position in the hierarchy of DE models. The applicability of a nonzero cosmological constant in relation to the observations was investigated by Krauss and Turner [23] and Dreitlein [14]. According to research by Linde [25], cosmological constant Λ is time-dependent, a function of temperature, and connected to the spontaneous breaking process. Some authors have looked into the cosmological models with decaying vacuum energy density Λ [9], [43], [37]. Present universe is homogeneous and isotropic and going on accelerating expansion confirmed by many observations [[39], [44], [15], [27], [21], [22], [23], [46], [47], [1], [30], [17]]. Many authors consider Einstein field equation with cosmological constant and find the solutions [[11], [18], [33], [35], [40], 41]. Recently Goswami et. al [19] investigated FRW dark energy cosmological model with hybrid expansion law. Dixit et. al [13] presented RHDE models in FRW Universe with two IR cut-offs with redshift parametrization. Ram Bharosha Tiwari and Sudhir Kumar Srivastva [45] investigated FRW-Cosmological Model of universe filled with dark matter for concircularly flat spacetime and Ujjal Debnath [12] investigated gravitational waves for some dark energy models in FRW Universe. Rakesh Raushan and R. Chaubey [38] investigated Dynamic evolution of FRW cosmology using variable Λ in Lyra geometry. Iver Brevik et. al [10] studied the FRW cosmological model assuming that the cosmic fluid is heterogeneous, viscous, and coupled with dark matter.

The portions of the current paper are as follows. The discussion of W_2 tensor and some basic equations for FRW- space time are presented in next section (2). In section (3) the class of solution of field equations has been discussed. To obtained an exact solution to such field equations , an attempt has been made to formulate the law of variation for Hubble's parameter. The law together with the field equations exhibits cosmological solutions in two different cases(subsection (3.1) and (3.2)).

2. Basic Equations

2.1. W_2 Tensor in FRW- space Time

The homogeneous and isotropic universe FRW - space time is defined by [49] the line element

$$ds^2 = -dt^2 + a^2(t)[(1 - kr^2)^{-1}dr^2 + r^2d\theta^2 + r^2sin^2\theta d\phi^2] \qquad (2)$$

Where a is cosmic scale factor and function of cosmic time t and k a constant known as curvature constant which have the value +1,0 or -1 for closed, flat and open universe. The Hubble's parameter and the spacial proper volume for this model are

$$V(t) = 2\pi^2 a^3, \qquad H(t) = \frac{\dot{a}(t)}{a(t)} \tag{3}$$

The vanishing of the covariant divergence of the energy momentum tensor and the Bianchi-identity for the Einstein tensor $R_{ij} - \frac{R}{2} g_{ij}$ in general relativity suggest that the cosmic term Λ is constant.In the theory of the variable Λ term, one can either add new terms to the field equation's left side to cancel the nonzero divergence of Λg_{ij} ([4, 48]) or treat Λ as a matter source and move it to the right side, as in [50], so that the energy-momentum conservation is understood as $T_k^{*ij} = 0$ where $T_{ij}^* = T_{ij} - \frac{\Lambda}{8\pi} g_{ij}$. For a particular theory, these two approaches are obviously equal [28]. Here we adopt the latter strategy and take into account the perfect fluid energy momentum tensor with Λ as

$$T_{ij}^* = (p + \rho) u_i u_j + (p - \frac{\Lambda}{8\pi}) g_{ij} \tag{4}$$

And the equation of state for perfect fluid

$$p = \omega \rho \qquad 0 \leq \omega \leq 1 \tag{5}$$

Where ρ and p are the perfect fluid energy density and pressure respectively and the four velocity vector we consider as $u_i = (1, 0, 0, 0)$.

The Einstein field equations of perfect fluid are

$$R_{ij} - \frac{1}{2} R g_{ij} = -8\pi T_{ij}^* \tag{6}$$

The W_2 curvature tensor for four dimensional spacetime is

$$W_{hijk} = R_{hijk} + \frac{1}{3}[g_{hj} R_{ik} - g_{ij} R_{hk}] \tag{7}$$

$$W_{ijk}^h = R_{ijk}^h + \tfrac{1}{3}[\delta_j^h R_{ik} - g_{ij} R_k^h]$$

A space time is said to be W_2-flat if its W_2-curvature tensor vanishes. For W_2-flat spacetime above equation leads to

$$R_{ijk}^h = -\tfrac{1}{3}[\delta_j^h R_{ik} - g_{ij} R_k^h]$$

which clearly shows that a W_2- flat spacetime is of constant curvature. which on contraction over h and k yields

$$R_{ij} = \tfrac{R}{4} g_{ij}$$

Thus

$$Rg_{ij} = 4R_{ij} \tag{8}$$

which indicate that W_2 - flat space-time is Einstein space.

It is also known that spaces with constant curvature play an important role in cosmology. The simplest model of the universe is that the universe is isotropic and of the same kind. This is known as the cosmological principle. Translated into the language of Riemannian geometry, this principle asserts that the three-dimensional position space is a space of constant curvature whose curvature depends on time [2]. The cosmological solution to the Einstein equation involving a three-dimensional space-like surface of constant curvature is the Friedman-Robertson-Walker metric. Therefore, we consider the FRW cosmological model for W_2-flat space-time.
Thus for W_2 flat spacetime, the Einstein field equations (6) reduce to the form

$$R_{ij} = 8\pi T^*_{ij} \tag{9}$$

which yield the following independent equations :

$$3a\ddot{a} = -(8\pi\rho + \Lambda)a^2 \tag{10}$$

$$a\ddot{a} + 2\dot{a}^2 + 2k = (8\pi p - \Lambda)a^2 \tag{11}$$

Equivalently the above equation can be written as

$$\frac{d}{dt}\left(\frac{\dot{a}}{a}\right) = ka^{-2} - 4\pi(\rho + p) \tag{12}$$

where the dot represents the derivation with respect to cosmic time t.

The Einstein field equations (10) and (11) are two equations with four unknowns a, ρ, p, and Λ. Therefore, the complete solution of the system with (5) requires another relation, which we will find in the next section by applying the new law of variation of the Hubble parameter.

3. Class of Cosmological Solutions

The solution of the field equation can be obtained by applying the law of variation of the Hubble parameter. It is interesting to note that in order to get exact solution of Einstein field equations this law was first introduced by Berman [5] in his FRW model of universe that gives a constant value for the deceleration parameter. In general, Einstein and Brans-Dicke theories are accepted for FRW metrics with constant deceleration parameters. Berman [5], Berman and Gomide [6], Maharaja and Naidu [26], Johri and Desikan [20], Pradhan and Viswakarma [34], many researchers like Rahman et al., [37], R. Kumar and S. K. Srivastava [24]

obtained a new cosmological model with constant deceleration parameters. So we also considered the same law for our models.
The law of variation of the Hubble parameter in FRW spacetime is

$$H = \alpha a^{-n} \tag{13}$$

Where $\alpha>0$ and $n \geq 0$ are constants.

The Deceleration parameter q is define by

$$q = -\frac{a\ddot{a}}{\dot{a}^2} \tag{14}$$

from (3) and (13) we obtained

$$a(t) = (n\alpha t + A_1)^{\frac{1}{n}} \qquad for \qquad n \neq 0 \tag{15}$$

$$a(t) = A_2 e^{\alpha t} \qquad for \qquad n = 0 \tag{16}$$

Here A_1 and A_2 are constant of integration.
Now using (15) and (16) into (14), we get

$$q = n - 1 \qquad or \qquad q = -1 \tag{17}$$

This indicates that the deceleration parameters for this model are constant.

We are now describing a new cosmological model of the universe about different values of n and their physical behaviour.

3.1. Case (i) Cosmological Model for $n \neq 0$

Using (15) into the Einstein field equation (10) and (11) and solving with (5) we get the expression for the energy density ρ, pressure p and cosmological constant Λ for the models as

$$\rho = \frac{1}{4\pi(1+\omega)}[n\alpha^2(n\alpha t + A_1)^{-2} + k(n\alpha t + A_1)^{\frac{-2}{n}}] \tag{18}$$

$$p = \frac{\omega}{4\pi(1+\omega)}[n\alpha^2(n\alpha t + A_1)^{-2} + k(n\alpha t + A_1)^{\frac{-2}{n}}] \tag{19}$$

$$\Lambda = 3\alpha^2(n\alpha t + A_1)^{-2}(n\frac{\omega-1}{\omega+1} - 1) - \frac{4k}{\omega+1}(n\alpha t + A_1)^{\frac{-2}{n}} \tag{20}$$

It is clear from the above that the equation (15) and (18)-(20) satisfy the equation (12) correspondingly and therefore this shows the exact solution of Einstein field equations. For this reason in the light of expression (15), the model for universe which the exact W_2-flat is given as

$$ds^2 = -dt^2 + (n\alpha t + A_1)^{\frac{2}{n}}[(1 - kr^2)^{-1}dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2] \qquad (21)$$

The model has expression for some other cosmological parameters.
The parameter expansion scalar Θ and spatial volume V are given as

$$\Theta(t) = u^i_{;i} = \alpha(n\alpha t + A_1)^{-1} \qquad (22)$$

$$V(t) = 2\pi^2 (n\alpha t + A_1)^{\frac{3}{n}} \qquad (23)$$

The cosmological redshift and scale factor are directly related. The most significant knowledge regarding the cosmic scale factor is discovered by looking at frequency variations in light that is emitted by far-off astronomical bodies. Such a model's cosmological redshift is $1 + z = \frac{a_{now}}{a_{then}}$ hence

$$z = (\frac{n\alpha t_1 + A_1}{n\alpha t_2 + A_2})^{\frac{1}{n}} - 1 \qquad t_1 > t_2 \qquad (24)$$

and Hubble's time is defined as the reciprocal of the Hubble constant, $1/H_0$.

$$H_0^{-1} = \alpha(n\alpha t_0 + A_1) \qquad (25)$$

where t_0 represent the current age of the universe.

3.2. Physical Behaviour of the Model

We see that at $t = \frac{-A_1}{n\alpha}$ cosmic scale factor and spatial volume vanishes and the energy density ρ, pressure p, cosmological constant Λ, the Hubble's parameters H, expansion scalar θ, becomes infinite. The universe start at $t = \frac{-A_1}{n\alpha}$ with volume $V = 0$ then expands with infinite velocity. It shows that this model has singularity at $t = \frac{-A_1}{n\alpha}$. Vanishing of scalar factor at $t = \frac{-A_1}{n\alpha}$ indicate that this singularity is of point type singularity.The energy density ρ, pressure p,cosmological constant Λ becomes proportional to t^{-2} at the value of $A_1 = 0$ and this model agree with Berman and Som [7] model. Again as we take $n = \alpha = 1$ with $A_1 = 0$, the current age of the universe becomes equal to the Hubble' time.

As time t increases cosmic scale factor and spatial volume increases and the energy density ρ, pressure p, cosmological constant Λ, the Hubble's parameters H, expansion scalar θ, decreases. Therefore, the rate of expansion slows with time. As $t \to \infty$, the cosmological parameters cosmic scale factor and spatial volume vanishes and the energy density ρ, pressure p, cosmological constant Λ, the

Hubble's parameters H, expansion scalar θ, vanishes which shows empty universe.

3.3. Case (ii) Cosmological Model for $n = 0$

Substituting (14) into the Einstein field equations (10) and (11) and solving with (5) yields energy density ρ, pressure p and cosmological constant Λ term in the model

$$\rho = \frac{ke^{-2\alpha t}}{A_2^2 4\pi(1+\omega)} \tag{26}$$

$$p = \frac{k\omega e^{-2\alpha t}}{A_2^2 4\pi(1+\omega)} \tag{27}$$

$$\Lambda = -3[\alpha^2 + \frac{2ke^{-2\alpha t}}{3A_2^2 4\pi(1+\omega)}] \tag{28}$$

Since the solutions (15) and (18)-(20) equally satisfy the equation (12), they represent exact solutions to the Einstein field equations. Therefore, in light of the equation (15), the exact W_2 flat model of the universe.

$$ds^2 = -dt^2 + (A_2^2 e^{2\alpha t}[(1-kr^2)^{-1}dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2] \tag{29}$$

Here we found formulas for other cosmic parameters of the model.
The expansion term Θ and spatial volume of universe V of the model are

$$\Theta(t) = 3\alpha \tag{30}$$

$$V(t) = 2\pi^2 A_2^3 e^{3\alpha t} \tag{31}$$

The scale factor and the cosmological redshift are closely related. The studies of frequency shifts in the light emitted by distant celestial bodies provide the most significant knowledge regarding the cosmic scale factor. For a model like this, the cosmic redshift is $1 + z = \frac{a_{now}}{a_{then}}$ hence

$$z = e^{\alpha(t_1 - t_2)} - 1 \qquad t_1 > t_2 \tag{32}$$

and also the Hubble's time which is defined as the reciprocal of the Hubble constant, $1/H_0$.

$$H_0^{-1} = \alpha^{-1} \tag{33}$$

3.4. Physical Behaviour of the Model

At $t = 0$ the cosmological parameters,the cosmic scale factor a, spatial volume V, energy density ρ, pressure p, are constant. Therefore, the universe begins to evolve with a constant volume and expands exponentially. The model is free from initial singularity. It is interesting to note that the universe exhibits uniform expansion because the Hubble parameter and the expansion scalar are constant throughout the universe's evolution. As t increases the cosmic scale factor, spatial volume, the energy density, pressure, increase exponentially.

Universe become empty for the value of curvature index $k = 0$ as energy density and pressure becomes zero. As $t \to \infty$, the cosmic scalar factor and spatial volumes grew infinitely, and ρ, p, and Λ approaches to zero. This indicates that the late universe is dominated by vacuum energy, suggesting an accelerated expansion of the universe which is also satisfied with recent data.

4. Concluding Remarks

In this paper we have studied the FRW cosmological model of W_2 flat spacetime filled with perfect fluid and variable cosmological constant Λ, using variations in Hubble's parameter law that give constant values for deceleration parameter. This article is divided into two parts. The first part examines the universe model of $n \neq 0$. This model exhibit expansion of the power law of the universe, where as at $t = \frac{-A_1}{n\alpha}$ the model has a point singularity and the energy density, pressure, and Λ terms diverge at the initial singularity. The universe begins to evolve from zero volume at $t = \frac{-A_1}{n\alpha}$ with an infinite expansion rate, and as t increases the cosmological parameters the cosmic scale factor and spatial volume increase whereas the expansion scalar decreases.

In the second case $n = 0$ the universe at initial epoch has constant energy density, pressure and cosmological constant and start expanding exponentially with increase of time.

The universe becomes empty when energy density and pressure tends to zero as $t \to \infty$ in both cases. It is interesting to see that our model becomes similar to [24] although some cosmological parameters are different as we consider different spacetime. As our models shows that the universe becomes empty when energy density and pressure tends to zero as $t \to \infty$ in both cases this quite indicate that some modification required according to Weyl's postulates and geometric properties of different curvature tensors defined by Pokhriyal and Mishra [32] and its application in relativity and cosmology.

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