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PARACOMPACTNESS IN GENERALIZED TOPOLOGICAL SPACES

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Abstract: In this paper we introduce the concepts $\mathcal{G}$ - locally finite, $\sigma_{\mathcal{G}}$ - locally finite and $\mathcal{G}$ - paracompactness. Also discuss about some properties of these concepts. Here we investigate that some properties in topological spaces and generalized topological spaces (GTS) are coincides if we replace open sets by generalized open sets ($\mathcal{G}$ - open sets). Also, we provide some examples to show some results are invalid in the case of GTS.

Keywords and Phrases: $\mathcal{G}$ - locally finite, $\sigma_{\mathcal{G}}$ - locally finite, $\mathcal{G}$ - paracompactness.

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1. Introduction

Topological apace has been generalized in many ways, some of them are supra topological space [10] and generalized topological space [8]. A. S. Mashhour introduce the supra topological spaces in [10] and discuss about supra open sets, neighborhood, continuity etc in supra toplogical spaces. T.M. Al-Shami introduces and discuss about some fundamental properties of supra completly Hausdorff and completly regular spaces[1]. He also introduced supra semi compact (supra semi Lindeloff) spaces, almost supra semi compact (almost supra semi Lindeloff) spaces and mildly supra semi compact (mildly supra semi Lindeloff) spaces and extended his study into paracompactness and some wheredense properties in supra topological spaces [3, 4, 6].

Generalized topological space is an important generalization of topological spaces. $Cs'asz'ar$ and others are worked so many years on the generalized topological spaces and they develop a basic theory for them. Especially $Cs'asz'ar$ develop this theory using some basic operators. For this he consider all monotonic functions from power set of X to power set of X, where X is a nonempty set. The collection of all monotonic functions is denoted by $\Gamma(X)$. For $\gamma \in \Gamma$, let $\mathcal{G} = \{A \subset X : A \subset \gamma(A)\}$. $Cs'asz'ar$ found that $\phi \in \mathcal{G}$ and $\mathcal{G}$ is closed under arbitrary union. He named such families as generalized topology and the pair $(X, \mathcal{G})$ is called generalized topological spaces [8].

In this paper we introduce the concepts like $\mathcal{G}$ - locally finite, $\sigma_{\mathcal{G}}$ - locally finite properties in generalized topological spaces. Then introduce the concept paracompactness in generalized topological spaces. There are 5 sections. Section 2 contains the preliminary ideas used in subsequent sections. In section 3 we discuss about $\mathcal{G}$ - locally finite and $\sigma_{\mathcal{G}}$ - locally finite properties in generalized topology. In section 4 introduce the main concept paracompactness in generalized topological space and the last section is the conclusion. Here also discuss about some relation between separation axioms and paracompactness in generalized topological spaces.

2. Preliminaries

Definition 2.1. [8] *Let X be a set and exp(X) its power set. A subset $\mathcal{G}$ of exp(X) is called a generalized topology (GT) on X and (X,$\mathcal{G}$) is called a generalized topological space (GTS) if $\mathcal{G}$ has the following properties.*

1. $\phi \in \mathcal{G}$
2. *Any union of elements of $\mathcal{G}$ belongs to $\mathcal{G}$*

Here $\mathcal{G}$ is also called $\mathcal{G}$-topology on X.

Definition 2.2. [8] *A generalized topology $\mathcal{G}$ is called strong if $X \in \mathcal{G}$.*

In a generalized topological space $(X, \mathcal{G})$, define $M_{\mathcal{G}} = \cup\{U : U \in \mathcal{G}\}$.

Definition 2.3. [8] *A subset A of a GTS is called $\mathcal{G}$-open if $A \in \mathcal{G}$. A subset B is called $\mathcal{G}$-closed if (X/B) is $\mathcal{G}$-open.*

Definition 2.4. [13] *The $\mathcal{G}$-closure of A is denoted by $C_{\mathcal{G}}(A)$ is the intersection* *of all $\mathcal{G}$-closed sets containing A.*

Theorem 2.1. [12] *Let $(X, \mathcal{G})$ be a GTS and $A, B \subset X$. Then the following statements are hold.*

1. *$x \in C_{\mathcal{G}}(A)$ if and only if $x \in U \in \mathcal{G}$ implies $U \cap A \neq \phi$*
2. *If $U, V \in \mathcal{G}$ and $U \cap V = \phi$ then $C_{\mathcal{G}}(U) \cap V = \phi$ and $U \cap C_{\mathcal{G}}(V) = \phi$*

Theorem 2.2. [7] *Let A, B are subsets of a GTS $(X, \mathcal{G})$. Then the following conditions are hold.*

1. *$C_{\mathcal{G}}(A)$ is $\mathcal{G}$-closed in X. More over it is the smallest $\mathcal{G}$-closed set of X containing A.*
2. *A is $\mathcal{G}$-closed in X if and only if $C_{\mathcal{G}}(A) = A$.*
3. *$C_{\mathcal{G}}(C_{\mathcal{G}}(A)) = C_{\mathcal{G}}(A)$*
4. *$C_{\mathcal{G}}(A) \cup C_{\mathcal{G}}(B) \subset C_{\mathcal{G}}(A \cup B)$.*

Definition 2.5. [13] *Let $(X, \mathcal{G})$ be a generalized topological space. A collection $\mathcal{U}$ of subsets of X is said to be a $\mathcal{G}$ - cover of X if the union of elements of $\mathcal{U}$ equals X.*

Definition 2.6. [13] *Let $(X, \mathcal{G})$ be a generalized topological space. A $\mathcal{G}$ - subcover of a $\mathcal{G}$ - cover $\mathcal{U}$ is a subcollection μ of $\mathcal{U}$ which itself a $\mathcal{G}$ - cover. If the elements of $\mathcal{U}$ are $\mathcal{G}$ - open then we say that $\mathcal{U}$ is a $\mathcal{G}$ - open cover.*

Definition 2.7. [13] *If every $\mathcal{G}$ - open cover of X has a finite $\mathcal{G}$ - subcover then we say that X is $\mathcal{G}$ - compact (generalized compact).*

Definition 2.8. [7] *Let $(X, \mathcal{G})$ be a generalized topological space, $x_0 \in X$ and $N \subset X$. Then N is said to be a generalized neighbourhood ($\mathcal{G}-neighbourhood$) of x_0, if there is a $\mathcal{G}$ - open set V such that $x_0 \in V$ and $V \subset N$.*

Definition 2.9. [7] *Let $(X, \mathcal{G})$ be a generalized topological space. Let η_x be the set of all $\mathcal{G}$ -neighbourhoods of x in X. The family η_x is called the generalized neighbourhood system at x.*

Definition 2.10. [7] *Let A be a subset of a generalized topological space X and $y \in X$. Then y is said to be a generalized accumulation point ($\mathcal{G}-accumulationpoint$) of A if every $\mathcal{G}$ - open set containing y contains atleast one point of A other than y.*

Definition 2.11. [7] *Let A be a subset of a generalized topological space X, then the generalized derived set of A is the set of all generalized accumulation points of A. It is denoted by $A'^{\mathcal{G}}$.*

Theorem 2.3. [7] *Let A be a subset of a GTS X, then $C_{\mathcal{G}} = A \cup A'^{\mathcal{G}}$.*

Definition 2.12. [11] *Let $(X, \mathcal{G})$ be a generalized topological space. Then X is called a $\mathcal{G}_{T_1}$ - space, if for x_1, x_2 are two distinct points in $M_{\mathcal{G}}$, there exists $U, V \in \mathcal{G}$ such that $x_1 \in U, x_2 \notin U$ and $x_2 \in V, x_1 \notin V$.*

Definition 2.13. [11] *A generalized topological space is said to be $\mathcal{G}_{T_2}$, if x and y are two distinct points in $M_{\mathcal{G}} = \cup\{U : U \in \mathcal{G}\}$ implies there exists two disjoint $\mathcal{G}$ - open sets U and V containing x and y respectively.*

Definition 2.14. [11] *Let $(X, \mathcal{G})$ be a generalized topological space. Then X is said to be $\mathcal{G}$ - regular if for each $x \in M_{\mathcal{G}}$ and a $\mathcal{G}$ - closed set F such that $x \notin F$, there are disjoint $\mathcal{G}$ - open sets U and V such that $x \in U$ and $F \cap M_{\mathcal{G}} \subset V$. If X is $\mathcal{G}_{T_1}$ and $\mathcal{G}$ - regular then we say that X is $\mathcal{G}_{T_3}$.*

Definition 2.15. [11] *A generalized topological space X is said to be $\mathcal{G}$ - normal if for any two $\mathcal{G}$ - closed sets A and B such that $A \cap B \cap M_{\mathcal{G}} = \phi$ there exists disjoint $\mathcal{G}$ - open sets U and V such that $A \cap M_{\mathcal{G}} \subset U$ and $B \cap M_{\mathcal{G}} \subset V$. If X is $\mathcal{G}_{T_2}$ and $\mathcal{G}$ - normal we say that X is $\mathcal{G}_{T_4}$.*

For the reference of locally finite, σ - locally finite, and paracompactness. [9]

3. $\mathcal{G}$ - locally Finite and $\sigma_{\mathcal{G}}$ - locally Finite.

Definition 3.1. *Let X be a generalized topological space. Then a family $\mathcal{U}$ of subsets of X is said to be $\mathcal{G}$ - locally finite if for each $x \in M_{\mathcal{G}}$, there exists a $\mathcal{G}$ - neighbourhood hood N of x which intersect only finitely many members of $\mathcal{U}$.*

Definition 3.2. *Let X be a generalized topological space. Then a family $\mathcal{V}$ of subsets of X is said to be $\sigma_{\mathcal{G}}$ - locally finite, if it can be written as the union of countably many subfamilies each of which is $\mathcal{G}$ - locally finite.*

Example 3.1.

1. Every finite family of subsets of a generalized topological space X is $\mathcal{G}$ - locally finite.

2. Every countable family of subsets of a generalized topological space X is $\sigma_{\mathcal{G}}$ - locally finite.

Example 3.2. Let $X = \mathbb{R}$, the set of real numbers. Let $\mathcal{G}$ be the generalized topology on $\mathbb{R}$ which is generated by the set $\{\{n\} : n \in \mathbb{N}\}$.

Consider $\mathcal{U} = \{\{1,2,3,...\},\{2,3,4,...\},\{3,4,5,...\},...,\{k,k+1,k+2,...\},...\}$. It is a countable collection of subsets of $\mathbb{R}$. Note that $\mathcal{G}$ is the power set of $\mathbb{N}$. Let $x \in M_{\mathcal{G}}$, so x must be a member of N. There for $\{x\}$ is a $\mathcal{G}$ - neighbourhood of x, which intersect only finitely many members of $\mathcal{U}$. So $\mathcal{U}$ is $\mathcal{G}$ - locally finite.
Also $\mathcal{U}$ is a countable collection, so U is $\sigma_{\mathcal{G}}$ - locally finite.

Remark 3.1. *The $\mathcal{G}$ - locally finite and $\sigma_{\mathcal{G}} - locally finite$ properties are coincides with locally finite and σ - locally finite properties in ordinary topological space if we replace topological space with generalized topological space.*

Theorem 3.1. *Every $\mathcal{G}$ - locally finite collection of subsets of $\mathcal{G}$ - compact strong GTS must be finite.*

Proof. Let $(X, \mathcal{G})$ be a generalized topological space. Let $\mathcal{F} = \{F_a : a \in \Lambda\}$ be a $\mathcal{G}$ - locally finite family of subsets of X. For each point $x \in X$, choose a $\mathcal{G}$ - open neighbourhood U_x that intersect a finite number of subsets in $\mathcal{F}$. Clearly the family of subsets $\{U_x : x \in X\}$ is a $\mathcal{G}$ - open cover of X. Since X is $\mathcal{G}$ - compact, this cover has a sub collection which cover X, say $\{U_{k_n} : n = 1,2,3,...,r\}$. Since each $\{U_{k_i}, i = 1,2,3,...,n\}$ intersects only a finite number of subsets in $\mathcal{F}$.
Suppose U_{k_1} intersect $F_{k_{1_1}}, F_{k_{1_2}}, ..., F_{k_{1_{n_1}}}$
U_{k_2} intersect $F_{k_{2_1}}, F_{k_{2_2}}, ..., F_{k_{2_{n_2}}}$
.
.
.
.
U_{k_r} intersect $F_{k_{r_1}}, F_{k_{r_2}}, ..., F_{k_{r_{n_l}}}$.

If possible $\mathcal{F}$ is infinite, so there exist $G \in \mathcal{F}$ such that $G \neq F_{k_{i_m}}$ for each $i = 1,2,3,...,r$, $m = 1,2,3,...,l$ and for each $U_{k_i}, i = 1,2,3,....,r$ does not intersect G. So there exist $g \in G$ such that $g \notin \cup_{i=1}^{r} U_{k_i}$. But $\cup_{i=1}^{r} U_{k_i} = X$. Since $g \in G \subset X$ and $g \notin \cup_{i=1}^{r} U_{k_i}$ which is a contradiction. So $\mathcal{F}$ is infinite is wrong. Hence $\mathcal{F}$ is finite.

Remark 3.2. *In ordinary topological space the union of a locally finite collection of closed subsets of a topological space is itself closed. But in generalized topological space this result does not hold.*

That is the union of $\mathcal{G}$ - locally finite collection of $\mathcal{G}$ - closed subsets need not be $\mathcal{G}$ - closed.

Example 3.3. Let $X = \{a,b,c,d\}$ and $\mathcal{G} = \{\phi,\{c,d\},\{a,d\},\{a,c,d\}\}$. Then $\mathcal{G}$ is a generalized topology on X and $(X, \mathcal{G})$ is a generalized topological space.

Let $\mathcal{F} = \{\{a,b\},\{b,c\}\}$. Since $\mathcal{G}$ is a finite collection, it is a $\mathcal{G}$ - locally finite collection.

But $\{a, b\} \cup \{b, c\} = \{a, b, c\}$ is not $\mathcal{G}$ - closed.

Lemma 3.1. *For a subset A of a generalized topological space X, $C_{\mathcal{G}}(A) = \{y \in X$: every $\mathcal{G}$ - neighbourhood of y meets A nonvacuously. $\}$*

Proof. Let η_y be the generalized neighbourhood system of y. That is η_y be the collection of all $\mathcal{G}$ - neighbourhoods of y.

Let $B = \{y \in X : U \in \eta_y$ implies $U \cap A \neq \phi\}$.

Claim. $C_{\mathcal{G}}(A) = B$.

We have $C_{\mathcal{G}}(A) = A \cup A'^{\mathcal{G}}$. So we want to claim that $A \cup A'^{\mathcal{G}} = B$.

Let $y \in A \cup A'^{\mathcal{G}}$. So $y \in A$ or $y \in A'^{\mathcal{G}}$.

If $y \in A$, then every $\mathcal{G}$ - neighbourhood of y intersect A, because every $\mathcal{G}$ - neighbourhood of y contain y and $y \in A$. So in this case $y \in B$.

Next we consider the case $y \in A'^{\mathcal{G}}$. So y is a generalized accumulation point of A. By the definition of generalized accumulation point, we see that every $\mathcal{G}$ - neighbourhood of y meets A nonvacuously. So $y \in B$. Hence $A \cup A'^{\mathcal{G}} \subset B$.

Conversely let $y \in B$. If $y \notin A \cup A'^{\mathcal{G}}$, then $y \notin C_{\mathcal{G}}(A)$ (bcz, $C_{\mathcal{G}}(A) = A \cup A'^{\mathcal{G}}$). Since $C_{\mathcal{G}}(A)$ is the smallest $\mathcal{G}$ - closed set containing A, we see that $(X - C_{\mathcal{G}}(A))$ is a $\mathcal{G}$ - open set and so it is a $\mathcal{G}$ - neighbourhood of y which does not meet A. Which is a contradiction to the fact that $y \in B$. So $B \subset A \cup A'^{\mathcal{G}}$. Hence $B = C_{\mathcal{G}}(A)$.

Theorem 3.2. *Let $\mathcal{C}$ is a $\mathcal{G}$ - locally finite family of subsets of a generalized topological space X. Then $\{C_{\mathcal{G}}(C) : C \in \mathcal{C}\}$ is $\mathcal{G}$ - locally finite.*

Proof. Assume the contrary that $\{C_{\mathcal{G}}(C) : C \in \mathcal{C}\}$ is not $\mathcal{G}$ - locally finite. So there exist $x \in M_{\mathcal{G}}$ such that every $\mathcal{G}$ - neighbourhood of x intersect infinitely many members of $\{C_{\mathcal{G}}(C) : C \in \mathcal{C}\}$. By above lemma, $x \in C_{\mathcal{G}}(C_{\mathcal{G}}(C))$ for infinitely many $C \in \mathcal{C}$. But we have $C_{\mathcal{G}}(C_{\mathcal{G}}(C)) = C_{\mathcal{G}}(C)$, there for $x \in C_{\mathcal{G}}(C)$ for infinitely many $C \in \mathcal{C}$. Again by above lemma, every $\mathcal{G}$ - neighbourhood of x intersect infinitely many $C \in \mathcal{C}$.

Also $x \in M_{\mathcal{G}}$ and given that $\mathcal{C}$ is $\mathcal{G}$ - locally finite, so there exist a $\mathcal{G}$ - neighbourhood N_x of x which intersect only finitely many members of $\mathcal{C}$, which is a contradiction, since we just claim that every $\mathcal{G}$ - neighbourhood of x intersect infinitely many members of $\mathcal{C}$. Hence $\{C_{\mathcal{G}}(C) : C \in \mathcal{C}\}$ is $\mathcal{G}$ - locally finite.

4. Paracompactness in Generalized Topological Spaces

Definition 4.1. *A generalized topological space is called $\mathcal{G}$ - paracompact if it is $\mathcal{G}$ - regular and if every $\mathcal{G}$ - open cover of $M_{\mathcal{G}}$ has $\mathcal{G}$ - open, $\mathcal{G}$ - locally finite refinement which is also a cover of $M_{\mathcal{G}}$.*

Lemma 4.1. *Let A, B are nonempty subsets of a generalized topological space X.*

Then $A'^{\mathcal{G}} \cup B'^{\mathcal{G}} = (A \cup B)'^{\mathcal{G}}$.

Proof. Let $x \in A'^{\mathcal{G}} \cup B'^{\mathcal{G}} \implies x$ is a generalized accumulation point of A or that of B.

If x is a generalized accumulation point of A, then every $\mathcal{G}$ - neighbourhood of x intersect A. So every $\mathcal{G}$ - neighbourhood of x intersect $A \cup B$. That is x is a generalized accumulation point of $A \cup B$. Which implies $x \in (A \cup B)'^{\mathcal{G}}$.

Similarly if x is a generalized accumulation point of B we can prove $x \in (A \cup B)'^{\mathcal{G}}$.

$A'^{\mathcal{G}} \cup B'^{\mathcal{G}} \subset (A \cup B)'^{\mathcal{G}}$ ....(1).

Let $y \in (A \cup B)'^{\mathcal{G}} \implies y$ is a generalized accumulation point of $(A \cup B)$. So every $\mathcal{G}$ - neighbourhood of y intersect $A \cup B$. That is every $\mathcal{G}$ - neighbourhood of y intersect A or B or both. In any case we get $y \in A'^{\mathcal{G}} \cup B'^{\mathcal{G}}$.

$(A \cup B)'^{\mathcal{G}} \subset A'^{\mathcal{G}} \cup B'^{\mathcal{G}}$................................(2).

From (1) and (2), we get $A'^{\mathcal{G}} \cup B'^{\mathcal{G}} = (A \cup B)'^{\mathcal{G}}$.

Theorem 4.1. *Let X be a $\mathcal{G}_{T_2}$ space. Then X is $\mathcal{G}$ - paracompact if and only if it has the property that every $\mathcal{G}$ - open cover of $M_{\mathcal{G}}$ has a $\mathcal{G}$ - open, $\mathcal{G}$ - locally finite refinement which is also a cover of $M_{\mathcal{G}}$.*

Proof. Suppose X is $\mathcal{G}$ - paracompact space. So by the definition of $\mathcal{G}$ paracompactness, we get X is $\mathcal{G}$ - regular and if every $\mathcal{G}$ - open cover of $M_{\mathcal{G}}$ has $\mathcal{G}$ - open, $\mathcal{G}$ - locally finite refinement which is also a cover of $M_{\mathcal{G}}$.

Conversely suppose X is $\mathcal{G}_{T_2}$ and the given property hold. We want to prove that X is $\mathcal{G}$ - paracompact. We need only to show that X is $\mathcal{G}$ - regular.

Let $x \in X$ and C is a $\mathcal{G}$ - closed subset of X not containing x. We want to find two disjoint $\mathcal{G}$ - open sets U and V such that $x \in U$ and $C \cap M_{\mathcal{G}} \subset V$. Since C is a $\mathcal{G}$ - closed set not containing x, we get $X - C$ is a $\mathcal{G}$ - open set containing x . So $x \in X - C \subset M_{\mathcal{G}} \implies x \in M_{\mathcal{G}}$.

Let $y \in C \cap M_{\mathcal{G}} \implies y \in C$ and $y \in M_{\mathcal{G}} \implies x$ and y are two distinct elements in $M_{\mathcal{G}}$. Since X is $\mathcal{G}_{T_2}$, there exists disjoint $\mathcal{G}$ - open sets U_x and U_y such that $x \in U_x$ and $y \in U_y$. So $X - U_x$ is $\mathcal{G}$ - closed in X and $U_y \subset X - U_x$. That is $X - U_x$ is a $\mathcal{G}$ - closed set containing U_y, but $C_{\mathcal{G}}(U_y)$ is the smallest $\mathcal{G}$ - closed set containing U_y. There for $C_{\mathcal{G}}(U_y) \subset X - U_x$. So $x \notin C_{\mathcal{G}}(U_y)$ as $x \notin X - U_x$.

Let $\mathcal{U} = \{U_y : y \in C \cap M_{\mathcal{G}}\} \cup \{X - C\}$. Then $\mathcal{U}$ is a $\mathcal{G}$ - open cover of $M_{\mathcal{G}}$. Let $\mathcal{V}$ be a $\mathcal{G}$ - open, $\mathcal{G}$ - locally finite refinement of $\mathcal{U}$ which is also a cover of $M_{\mathcal{G}}$. So by the definition of refinement, every member of $\mathcal{V}$ is contained in some member of U_y or in $X - C$. In the second case it cannot intersect C. Let $\mathcal{W} = \{V \in \mathcal{V} : V \cap C \neq \phi\}$. Then every member of $\mathcal{W}$ is contained in some U_y. Let $G = \cup_{W \in \mathcal{W}} W$. Since each $W \in \mathcal{W}$ is $\mathcal{G}$ - open, we see that G is $\mathcal{G}$ - open.

Claim. $C \cap M_{\mathcal{G}} \subset G$

Let $a \in C \cap M_{\mathcal{G}} \implies a \in C$ and $a \in M_{\mathcal{G}}$. Since $a \in M_{\mathcal{G}}$ and $\mathcal{V}$ is a $\mathcal{G}$ - open cover of $M_{\mathcal{G}}$, there exists $V \in \mathcal{V}$ such that $a \in V$. Since $\mathcal{V}$ is a refinement of $\mathcal{U}$, there exists $U_y \in \mathcal{U}$ such that $a \in V \subset U_y$. Since $a \in V$ and $a \in C \implies V \cap C \neq \phi$. There for $V \in \mathcal{W} \implies a \in G$. That is $C \cap M_{\mathcal{G}} \subset G$.

Claim. $x \notin C_{\mathcal{G}}(G)$.

If possible $x \in C_{\mathcal{G}}(G) \implies x \in C_{\mathcal{G}}(\cup_{W \in \mathcal{W}} W) = (\cup_{W \in \mathcal{W}} W) \cup (\cup_{W \in \mathcal{W}} W)'^{\mathcal{G}}$. So either $x \in (\cup_{W \in \mathcal{W}} W)$ or $x \in (\cup_{W \in \mathcal{W}} W)'^{\mathcal{G}}$. That is either $x \in W$ for some $W \in \mathcal{W}$ or x is a generalized accumulation point of W for some $W \in \mathcal{W}$ (From above lemma). Note that $x \in U_x$ and each U_y does not intersect U_x, we see that $x \notin V, forall V \in \mathcal{V}$. So $x \notin W$, for all $W \in \mathcal{W}$. So x must be a generalized accumulation point of W, for some $W \in \mathcal{W}$. So every $\mathcal{G}$ - neighbourhood of x intersect W. But this is impossible, because the $\mathcal{G}$ - open set U_x containing x, which does not intersect W.

There for $x \notin C_{\mathcal{G}}(G) \implies x \in X - C_{\mathcal{G}}(G)$. Also note that $X - C_{\mathcal{G}}(G)$ is $\mathcal{G}$ - open. So we take $U = X - C_{\mathcal{G}}(G)$ and $V = G$. Hence $x \in U$ and $C \cap M_{\mathcal{G}} \subset V$ and $U \cap V = \phi$.

Lemma 4.2. *Let X is $\mathcal{G}$ - regular space then for any $x \in X$ and any $\mathcal{G}$ - open set G containing x there exists a $\mathcal{G}$ - open set H containing x such that $C_{\mathcal{G}}(H) \subset G \cup M_{\mathcal{G}}^c$.*

Proof. Suppose X is a $\mathcal{G}$ - regular space. Let $x \in X$ and G be any $\mathcal{G}$ - open subset of X containing x.

So $X - G$ is a $\mathcal{G}$ - closed set not containing x. So there exists disjoint $\mathcal{G}$ - open sets U and V such that $x \in U$ and $(X - G) \cap M_{\mathcal{G}} \subset V$. $\implies (G^c \cap M_{\mathcal{G}})^c \supset V^c$ (taking complements on both sides) . By De Morgan's law $(X - V) \subset (G \cup M_{\mathcal{G}}^c)$

Since $U \subset (X - V)$ and $(X - V)$ is $\mathcal{G}$ - closed, we get $C_{\mathcal{G}}(A) \subset (X - V)$. Hence $C_{\mathcal{G}}(U) \subset G \cup M_{\mathcal{G}}^c$. Take $H = U$, we see that $x \in H$ and $C_{\mathcal{G}}(H) \subset G \cup M_{\mathcal{G}}^c$.

Theorem 4.2. *Every $\mathcal{G}$ - paracompact space is $\mathcal{G}$ - normal.*

Proof. Let X is a $\mathcal{G}$ - paracompact space. Let A and B are two $\mathcal{G}$ - closed subsets of X such that $A \cap B \cap M_{\mathcal{G}} = \phi$. Since X is $\mathcal{G}$ - paracompact, it is $\mathcal{G}$ - regular and every $\mathcal{G}$ - open cover of $M_{\mathcal{G}}$ has $\mathcal{G}$ - open , $\mathcal{G}$ - locally finite refinement which is also a cover of $M_{\mathcal{G}}$.

Since X is $\mathcal{G}$ - regular, for each $x \in A \cap M_{\mathcal{G}}$ and $x \notin B \cap M_{\mathcal{G}}$, there are disjoint $\mathcal{G}$ - open sets U_x and V_x such that $x \in U_x$ and $B \cap M_{\mathcal{G}} \subset V_x$. Since X is $\mathcal{G}$ - regular and using above lemma, there exists $\mathcal{G}$ - open set H_x such that $C_{\mathcal{G}}(H_x) \subset U_x \cup M_{\mathcal{G}}^c, x \in H_x$.

Let $\mathcal{U} = \{H_x : x \in A \cap M_{\mathcal{G}}\} \cup \{X - A\}$. Then $\mathcal{U}$ be a $\mathcal{G}$ - open cover of $M_{\mathcal{G}}$. Since X is $\mathcal{G}$ - paracompact , by the definition of $\mathcal{G}$ - paracompactness $\mathcal{U}$ has a $\mathcal{G}$ - open, $\mathcal{G}$ - locally finite refinement which is also a cover of $M_{\mathcal{G}}$.

Let $\mathcal{V}$ be a $\mathcal{G}$ - open, $\mathcal{G}$ - locally finite refinement of $\mathcal{U}$ which is also a cover of $M_{\mathcal{G}}$. So by the definition of refinement, every member of $\mathcal{V}$ is contained in some member of $\mathcal{U}$. Let $\mathcal{W}$ be $\{V \in \mathcal{V} : V \cap A \neq \phi\}$. Now let $G = \cup_{W \in \mathcal{W}} W$. Then G is $\mathcal{G}$ - open.

Claim. $A \cap M_{\mathcal{G}} \subset G$.

Let $z \in A \cap M_{\mathcal{G}} \implies z \in H_z$, where $H_z \in \mathcal{U}$. So $z \in W$, for some $W \in \mathcal{W}$. Hence $z \in G$.

Claim. $B \cap M_{\mathcal{G}} \subset (X - C_{\mathcal{G}}(G))$.

Let $z \in B \cap M_{\mathcal{G}} \implies z \in B$ and $z \in M_{\mathcal{G}}$. To prove that $z \notin C_{\mathcal{G}}(G)$.

If possible $z \in C_{\mathcal{G}}(G)$. Using theorem 1.4.13, we have $C_{\mathcal{G}}(G) = G \cup G'^{\mathcal{G}}$. There for $z \in G \cup G'^{\mathcal{G}} \implies z \in G$ or $z \in G'^{\mathcal{G}}$.

Case(i). $z \in G$

If $z \in G \implies z \in (\cup_{W \in \mathcal{W}} W)$.

$\implies z \in W$ for some $W \in \mathcal{W}$.

$\implies z \in V$ for some $V \in \mathcal{V}$ and $V \cap A \neq \phi$.

Since every member of $\mathcal{V}$ is contained in some member of $\mathcal{U}$. So there exists $U \in \mathcal{U}$ such that $V \subset U$. This U is of the form $H_x, x \in A \cap M_{\mathcal{G}}$ or is $(X - A)$. Since $(X - A)$ does not intersect A, so U is of the form $H_x, x \in A \cap M_{\mathcal{G}}$

Let U is of the form of H_x, then $z \in H_x$ as $z \in V \subset U$. So by the definition of $H_x, x \in A \cap M_{\mathcal{G}}$, there exists disjoint $\mathcal{G}$ - open sets such that $x \in U_x$ and $B \cap M_{\mathcal{G}} \subset V_x$. Also $C_{\mathcal{G}}(H_X) \subset U_x \cup M_{\mathcal{G}}^c$. So $z \in V \subset H_x \subset C_{\mathcal{G}}(H_x) \subset U_x \cup M_{\mathcal{G}}^c$. Since $z \in M_{\mathcal{G}}$, we get $z \in U_x$. Also $z \in B \cap M_{\mathcal{G}} \implies z \in V_x$. Which is a contradiction to the fact that $U_x \cap V_x = \phi$. There for $z \notin G$.

Case(ii). $z \in G'^{\mathcal{G}}$**.**

If $z \in G'^{\mathcal{G}}$. By theorem 3.4.2 we have $G'^{\mathcal{G}} = (\cup_{W \in \mathcal{G}} W)'^{\mathcal{G}} = \cup_{W \in \mathcal{W}} (W'^{\mathcal{G}})$. There for $z \in \cup_{W \in \mathcal{W}} (W'^{\mathcal{G}}) \implies z \in W'^{\mathcal{G}}$, for some $W \in \mathcal{G} \implies z$ is a $\mathcal{G}$ - accumulation point of W. By the definition of $\mathcal{W}, W = V, V \in \mathcal{V}$ such that $V \cap A \neq \phi$. Since $\mathcal{V}$ is a refinement of $\mathcal{U}$, by the definition of refinement, there exist $U \in \mathcal{U}$ such that $V \subset U$. By the definition of $\mathcal{U}, U$ is of the form $H_x, x \in A$. That is there exists $H_x \in \mathcal{U}$, such that $W \subset H_x, x \in A$. So there exist disjoint $\mathcal{G}$ - open sets U_x and V_x such that $x \in U_x, B \cap M_{\mathcal{G}} \subset V_x$ and $C_{\mathcal{G}}(H_x) \subset U_x \cup (M_{\mathcal{G}})^c$. Since $W \subset H_x$ and $H_x \subset C_{\mathcal{G}}(H_x) \implies W \subset C_{\mathcal{G}}(H_x) \subset U_x \cup M_{\mathcal{G}}^c$.

$\implies W \subset U_x$ as $W \subset M_{\mathcal{G}}$.

Also $z \in B \cap M_{\mathcal{G}}$ and $B \cap M_{\mathcal{G}} \subset V_x \implies z \in V_x$. Since V_x is $\mathcal{G}$ - open and $z \in V_x$, it is a $\mathcal{G}$ - neighbourhood of z. Since z is a $\mathcal{G}$ - accumulation point of W, we see that $V_x \cap W \neq \phi$.

We get $W \subset U_x$ and $W \cap V_x \neq \phi$, we get $U_x \cap V_x \neq \phi$. Which is a contradiction. There for $B \cap M_{\mathcal{G}} \subset (X - C_{\mathcal{G}}(G))$.

That is there are two $\mathcal{G}$ - open sets G and $(X - C_{\mathcal{G}}(G))$ such that $A \cap M_{\mathcal{G}} \subset G$ and $B \cap M_{\mathcal{G}}$. Hence X is $\mathcal{G}$ - normal.

Lemma 4.3. *Let X is a $\mathcal{G}$ - normal space. Then for any $\mathcal{G}$ - closed set C and any $\mathcal{G}$ - open set G containing C, there exist $\mathcal{G}$ - open set H such that $C \cap M_{\mathcal{G}} \subset H$ and $C_{\mathcal{G}}(H) \subset G \cup M_{\mathcal{G}}^c$.*

Proof. Suppose X is a $\mathcal{G}$ - normal space. Let C be a $\mathcal{G}$ - closed subset of X and G be a $\mathcal{G}$ - open subset of X containing C. Then $X - G$ is a $\mathcal{G}$ - closed subset of X not containing C. That is C and $(X - G)$ are two disjoint $\mathcal{G}$ - closed subset of X. Since X is $\mathcal{G}$ - normal, by the definition of $\mathcal{G}$ - normality, there are disjoint $\mathcal{G}$ - open sets U and V such that $C \cap M_{\mathcal{G}} \subset U$ and $(X - G) \cap M_{\mathcal{G}} \subset V$. Since $U \cap V = \phi$, we see that $U \cap (X - V)$ and $X - V$ is a $\mathcal{G}$ - closed subset of X . But we know that $C_{\mathcal{G}}(U)$ is the smallest $\mathcal{G}$ - closed set containing U. Therefor we get $C_{\mathcal{G}}(U)$ is the smallest $\mathcal{G}$ - closed set containing U. So $C_{\mathcal{G}}(U) \subset (X - V)$.

Also $(X - G) \cap M_{\mathcal{G}} \subset V \implies ((X - G) \cap M_{\mathcal{G}})^c \supset (X - V)$.

$\implies (X - V) \subset G \cup M_{\mathcal{G}}^c$ (by De Morgan's law).

But $C_{\mathcal{G}}(U) \subset X - V \subset G \cup M_{\mathcal{G}}^c \implies C_{\mathcal{G}}(U) \subset G \cup M_{\mathcal{G}}^c$. So we take $H = U$. Then we get $C \cap M_{\mathcal{G}} \subset H$ and $C_{\mathcal{G}}(H) \subset G \cup M_{\mathcal{G}}^c$.

Theorem 4.3. *Let $\mathcal{U}$ be the $\mathcal{G}$ - open, $\mathcal{G}$ - locally finite cover of a $\mathcal{G}$ - normal space X. Then for each $U \in \mathcal{U}$ there exists a $\mathcal{G}$ - open set $G(U)$ such that $C_{\mathcal{G}}(G(U)) \subset U \cup M_{\mathcal{G}}^c$ and the family $\{G(U) : U \in \mathcal{U}\}$ covers X.*

Proof. Let $(X, \mathcal{G})$ be a generalized topological space. Also given that X is a $\mathcal{G}$ - normal space and let $\mathcal{U}$ be be a $\mathcal{G}$ - open, $\mathcal{G}$ - locally finite cover of X. Since $\mathcal{U}$ is a $\mathcal{G}$ - open cover of X, we see that $M_{\mathcal{G}} = X$.

We want to find a function $G : \mathcal{U} \to \mathcal{G}$ with the property that for each $U \in \mathcal{U}, G(U)$ is $\mathcal{G}$ - open, $C_{\mathcal{G}}(G(U)) \subset U \cup M_{\mathcal{G}}^c$ and the family $\{G(U) : U \in \mathcal{U}\}$ covers X. Call such a function G as a total shrinking function for $\mathcal{U}$.

Next define a partial shrinking function for $\mathcal{U}$ to be a function $F : \mathcal{V} \to \mathcal{G}$, where $\mathcal{V} \subset \mathcal{U}, C_{\mathcal{G}}(F(V)) \subset V \cup M_{\mathcal{G}}^c$, for all $V \in \mathcal{V}$ and the family $(\mathcal{U} - \mathcal{V}) \cup \{F(V) : V \in \mathcal{V}\}$ is a cover of X. Let $\mathcal{F}$ be a family of all partial shrinking functions for $\mathcal{U}$. Note that $\mathcal{F}$ is nonempty because the function with empty domain belong to it.
Define an order $' \leq '$ in $\mathcal{F}$ as follows. Given two partial shrinking functions F, H for $\mathcal{F}$, we say that $F \leq H$ if

1. domain of F is contained in that of H.

2. $F(V) = H(V)$ for all $V \in$ domain of F.

Claim. $' \leq '$ is a partial order on $\mathcal{F}$.

1. *Reflexivity :*
   Let $F \in \mathcal{F}$, then clearly $F \leq F$.

2. *Antisymmetric:*
   Let $F, H \in \mathcal{F}$ such that $F \leq H$ and $H \leq F$. To prove that $F = H$.
   $F \leq H \implies$ the domain of F is contained in the domain of H.
   $H \leq F \implies$ the domain of H is contained in the domain F.
   There for domain $F =$ domain of H.
   Let domain of $F =$ domain of $H = \mathcal{V}$. Then
   $F \leq H$ and $H \leq F \implies F(V) = H(V)$, for all $V \in \mathcal{V}$. Hence $F = H$.

3. *Transityvity:*
   Let F, G, H are three members in $\mathcal{F}$ such that $F \leq G$ and $G \leq H$. To prove that $F \leq H$.
   $F \leq G \implies$ domain of F is contained in the domain of G and
   $G \leq H \implies$ domain of G is contained in the domain of H. Hence domain of F is contained in the domain of H.
   Let $\mathcal{D}(F), \mathcal{D}(G), \mathcal{D}(H)$ are denotes the domain of F, G and H respectively. Then
   $F \leq G \implies F(V) = G(V)$ for all $V \in \mathcal{D}(F)$.
   $G \leq H \implies G(V) = H(V)$ for all $V \in \mathcal{D}(G)$.
   Since $\mathcal{D}(F) \subset \mathcal{D}(G) \subset \mathcal{D}(H) \implies \mathcal{D}(F) \subset \mathcal{D}(H)$ and hence $F(V) = H(V)$ for all $V \in \mathcal{F}$.
   Therefor $F \leq H$. Hence $' \leq '$ is a partial order on $\mathcal{F}$.

Claim. $\mathcal{F}$ has a maximal element.

Let $\{F_i : i \in I\}$ is a chain in $\mathcal{F}$. We want to find an upper bound for this chain. For this construct a partial shrinking function F from this chain as follows:

Let $\mathcal{D}(F)$ be the domain of F and let $\mathcal{D}(F) = \cup_{i \in I} \mathcal{D}(F_i)$, where $\mathcal{F}_\rangle$ is the domain of F_i. Now if $V \in \mathcal{D}(F)$, so $V \in \mathcal{D}(F_i)$ for some $i \in I$ and set $F(V) = F_i(V)$. This is well defined because if $V \in \mathcal{D}(F_i) \cap \mathcal{D}(F_j)$ for $i \neq j$ in I, then $V \in \mathcal{D}(F_i)$ and $\mathcal{D}(F_j)$. If $F_i \neq F_j$, then we may assume that $F_i \leq F_j$. So $\mathcal{D}(F_i) \subset \mathcal{D}(F_j) \implies \mathcal{D}(F_i) \cap \mathcal{D}(F_j) = \mathcal{D}(F_i)$. Also $F_i = F_j$ on $\mathcal{D}(F_i) \implies F_i(V) = F_j(V)$ as $V \in \mathcal{D}(F_i)$.
Claim. $C_{\mathcal{G}}(F(V)) \subset V \cup M_{\mathcal{G}}^c, V \in \mathcal{D}(F)$.

Let $V \in \mathcal{D}(F)$, so $V \in \mathcal{D}(F_i)$ for some $i \in I$. Also by the definition F, we have $F(V) = F_i(V)$ for some $i \in I$. Since for each $i \in I$, F_i is a partial shrinking on $\mathcal{U}$, by the definition of partial shrinking $C_{\mathcal{G}}(F_i(V)) \subset V \cup M_{\mathcal{G}}^c \implies C_{\mathcal{G}}(F(V)) \subset V \cup M_{\mathcal{G}}^c$ for all $V \in \mathcal{D}(F)$.

In order to show that F is a partial shrinking for $\mathcal{U}$, it only remains to show that the family $(\mathcal{U}-\mathcal{D}(F))\cup\{F(V):V\in\mathcal{D}(F)\}$ is a cover of X.

Let $x\in X$, since $M_{\mathcal{G}}=X \implies x\in M_{\mathcal{G}}$. Given that $\mathcal{U}$ is locally finite , then by the definition of locally finite property, there is a $\mathcal{G}$ - neighbourhood N of x which intersect only finitely many members of $\mathcal{U}$. So x contain only finitely many members of $\mathcal{U}$ Suppose that $U_1, U_2, ..., U_n$ are the only members in $\mathcal{U}$ contain x. If at least one of the U_i is $(\mathcal{U}-\mathcal{D}(F))$ we are done. Otherwise , since $\mathcal{F}$ is the collection of all partial shrinking for $\mathcal{U}$, there exist $i_1, i_2, i_3, ..., i_n$ in I, such that $U_r\in\mathcal{D}(F_{i_r})$ for $r=1,2,3,....,n$. Since $\{F_i : i\in I\}$ is a chain , without loss of generality we may assume that $F_{i_1}\leq F_{i_2}\leq ....\leq F_{i_n}$. By the definition of $'\leq'$, we get $\mathcal{D}(F_{i_1})\subset\mathcal{D}(F_{1_2})\subset ....\subset\mathcal{D}(F_{i_n})$. Since $U_1\in\mathcal{D}(F_{i_1}), U_2\in\mathcal{D}(F_{i_2}), ..., U_n\in\mathcal{D}(F_{i_n})$, we get $U_r\in\mathcal{D}(F_{i_n})$ for all $r=1,2,...,n$. Since these are the only members of $\mathcal{U}$ containing x and by the definition of F_{i_n}, we see that $(\mathcal{U}-\mathcal{D}(F_{i_n}))\cup\{F_{i_n}(V):V\in\mathcal{D}(F_{i_n})\}$ is a cover of X. So there is a $V\in\mathcal{D}(F_{i_n})$ such that $x\in V$. But we have $\mathcal{D}(F)=\cup_{i\in I}\mathcal{D}(F_i)$ and $F=F_i$ on $\mathcal{D}(F_i)$. There for $F(V)=F_{i_n}(V)$. Thus we get $\mathcal{U}-\mathcal{D}(F)\cup\{F(V):V\in\mathcal{D}(F)\}$ is a cover of X. So F is a partial shrinking function for $\mathcal{U}$ and by its construction it is an upper bound for the chain $\{F_i : i\in I\}$.

Thus every chain in $\mathcal{F}$ has an upper bound in $\mathcal{F}$. By Zorn's lemma, we have $\mathcal{F}$ has a maximal element. That is we get a partial shrinking function, say G for $\mathcal{U}$ which is maximal with respect to the ordering $'\leq'$.

To conclude the proof, we assert that G is a total shrinking for $\mathcal{U}$. That is to show that the domain of $G, \mathcal{D}(G)=\mathcal{U}$. Clearly $\mathcal{D}(G)\subset\mathcal{U}$ as G is a partial shrinking function on $\mathcal{U}$. So we want to prove that $\mathcal{U}\subset\mathcal{D}(G)$

If not there exist $U\in\mathcal{U}$ such that $U\notin\mathcal{D}(G)$. That is $U\in\mathcal{U}-\mathcal{D}(G)$. Let Wbe the union of the sets $\cup\{G(V):V\in\mathcal{D}(G)\}$ and the set $\cup(\mathcal{U}-\mathcal{D}(G)-U)$. Note that in the definition of W is a collection some $\mathcal{G}$ - open sets because the domain of G is a sub collection of $\mathcal{U}$ and the second collection is also a sub collection of $\mathcal{U}$. So their union must be $\mathcal{G}$ - open. Hence W is a $\mathcal{G}$ - open subset of X. There for $X-W$ is $\mathcal{G}$ - closed. Since $(\mathcal{U}-\mathcal{D}(G))\cup\{G(V):V\in\mathcal{D}(G)\}$ is a cover of X, we get $X-W\subset U$. Now by $\mathcal{G}$ - normality and using above lemma, we see that there exist a $\mathcal{G}$ - open set Q such that $(X-W)\cap M_{\mathcal{G}}\subset Q$ and $C_{\mathcal{G}}(Q)\subset U\cup M^c_{\mathcal{G}}$.

Define $H:\mathcal{D}(G)\cup\{U\}\to\mathcal{G}$ by $H(V)=G(V)$ for $V\in\mathcal{D}(G)$ and $H(U)=Q$. Since G is a partial shrinking function on $\mathcal{U}$, we get H is a partial shrinking function for $\mathcal{U}$. Also note that H is strictly greater than G. Which contradict the maximality of G. Thus $\mathcal{D}(G)=\mathcal{U}$.

5. Conclusion

Recently many scholars are working in generalized topological spaces and number of articles are published in various journals. Here we found that some results in ordinary topological spaces which are related to paracompactness are valid in generalized topological spaces whenever it is a strong GTS. But some results in topological spaces never hold in GTS for example the union of locally finite collection of closed subsets of a topological space is closed , in the case of GTS the union of $\mathcal{G}$ - locally finite collection of $\mathcal{G}$ - closed sets need not be $\mathcal{G}$ - closed. T. M. Al shami present the concept of sum of the ordered spaces using pairwise disjoint topological ordered spaces [8]. We plan in an upcoming paper to introduce this concept in GTS.

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