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ON THE NUMBER OF FUZZY SUBGROUPS AND FUZZY NORMAL SUBGROUPS OF S_2, S_3 AND A_4

Dhiraj Kumar and Manoranjan Kumar Singh

Department of Mathematics,
Magadh University,
Bodh-Gaya - 824234, Bihar, INDIA

E-mail : dhirajraj1982@gmail.com, drmksingh_ gaya@yahoo.com

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Abstract: Counting fuzzy subgroups of a finite group is a fundamental problem of fuzzy group theory. Many researchers have made significant contributions to the rapid growth of this topic in recent years. The number of fuzzy subgroups of any group is infinite without the aid of equivalence relation. Some authors have used the equivalence relation of fuzzy sets to study the equivalence of fuzzy subgroups ([5], [6], [16]). The problem of counting the number of distinct fuzzy subgroups of a finite group is relative to the choice of the equivalence relation. The number of fuzzy subgroups of a particular group varies from one equivalence relation to the other. The equivalence relation applied in our computation can be seen in the existing literature. Sulaiman and Abd Ghafur [10] define an equivalence relation for counting fuzzy subgroups of group G. We have used this relation to find fuzzy subgroups and fuzzy normal subgroups of S_2, S_3 and A_4. Lattice subgroup diagrams were used in our computation.

Keywords and Phrases: Fuzzy Subgroups, Fuzzy Normal Subgroups, Equivalence Relation, Chain, Subgroup Lattice, Symmetric Group, Alternating Group.

2020 Mathematics Subject Classification: 20N25, 06D72, 20E15, 22F05.

1. Introduction

The idea of fuzzy set was first propounded by Lotfi A. Zadeh in 1965. Since the inception of the conception of a fuzzy set, which laid the foundations of fuzzy set theory (FST), the literature on FST and its application has been proliferating rapidly and are widely scattered over many disciplines. In 1971, Azriel Rosenfeld initiated fuzzy sets in the realm of group theory and formulated the notion of fuzzy subgroup of a group. Since then, the study of different fuzzy algebraic structures was started. Many authors use the equivalence relation of fuzzy sets to study the number of fuzzy subgroups of group G. In this regard, R. Sulaiman and Abd Ghafur define an equivalence relation. In this paper, we will use this relation to count the number of fuzzy subgroups and fuzzy normal subgroups of $S_2,\ S_3$ and A_4.

This paper is constructed into four sections.In section one introduction is given. Section two provides some preliminary results, definitions, and an overview of the equivalence defined by Sulaiman and Abd Ghafur. In section three, we have the effects of finding the number of fuzzy subgroups and fuzzy normal subgroups of $S_2,\ S_3$ and A_4. Finally, some conclusions and further research directions are given in section four.

2. Preliminaries

Definition 2.1. Zadeh [15] *Let X be a nonempty set. A fuzzy subset of X is a function μ from X into* $[0, 1]$

$$\mu : X \to [0, 1]$$

Definition 2.2. Rosenfeld [7] *Let G be a group. A fuzzy subset of G is said to be a fuzzy subgroup of G if*

1. $\mu(xy) \geq min\{\mu(x), \mu(y)\}, \forall x,\ y \in G,$

2. $\mu(x^{-1}) \geq \mu(x), \forall x \in G$

Definition 2.3. *Sulaiman and Abd Ghafur: A fuzzy subset μ of G is a fuzzy subgroup of G if there is a sequence of subgroups $P_1 < P_2 < ..... < P_n = G$ in subgroup lattice of G such that μ can be written as*

$$\mu(x) = \begin{cases} \theta_1, & x \in P_1, \\ \theta_2, & x \in P_2 \backslash P_1, \\ \vdots \\ \theta_n, & x \in P_n \backslash P_{n-1}. \end{cases}$$

Definition 2.4. *R. Sulaiman, Let α, β be fuzzy subgroup of G of the form*

$$\alpha(x) = \begin{cases} \theta_1, & x \in P_1, \\ \theta_2, & x \in P_2 \backslash P_1, \\ \vdots & \\ \theta_n, & x \in P_n \backslash P_{n-1}. \end{cases}$$

$$\beta(x) = \begin{cases} \delta_1, & x \in M_1, \\ \delta_2, & x \in M_2 \backslash M_1, \\ \vdots & \\ \delta_n, & x \in M_n \backslash M_{n-1}. \end{cases}$$

Then we say that α and β are equivalent and write $\alpha \sim \beta$ if

$$(i)\ m = n\ (ii) P_i = M_i\ \forall\ i \in \{1, 2, \ldots, m\}.$$

3. Results

3.1. Constructing fuzzy subgroup of S_2

$$S_2 = \{I, (1, 2)\}$$

We have two subgroups of S_2, those are $\{I\}$, $\{I, (12)\}$. Subgroup lattice diagram of S_2 is as shown below

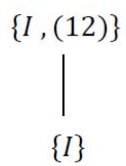

Figure 1

We notice the diagram and calculate how many subgroups there are in S_2. If $Q_1(\mu) = S_2$, S_2 has 1 fuzzy subgroup, namely

$$\mu_1(x) = \theta_1\ \forall\ x \in S_2$$

If $Q_1(\mu) = I$, we have one fuzzy subgroup of S_2

$$i.e. \mu_2(x) = \begin{cases} \theta_1, & x \in \{I\}, \\ \theta_2, & x \in S_2 \backslash \{I\}, . \end{cases}$$

S_2 has $1 + 1 = 2$ fuzzy subgroups.

3.2. Counting fuzzy normal subgroups of S_2

We infer that every subgroup of S_2 is normal. So S_2 has 2 fuzzy normal subgroups.

3.3. Constructing fuzzy subgroups of S_3

$$S_3 = \{I,\ (1\ 2),\ (1\ 3),\ (2\ 3),\ (1\ 2\ 3),\ (1\ 3\ 2)\}$$

We have six subgroups of S_3 , namely $K_1 = \{I\},\ K_2 = \{I, (1\ 2)\},\ K_3 = \{I, (2\ 3)\},$ $K_4 = \{I, (1\ 3)\},\ M = \{I, (1\ 2\ 3),\ (1\ 3\ 2)\}$ *and* S_3.
The diagram of subgroup lattice of S_3 is as depicted below.

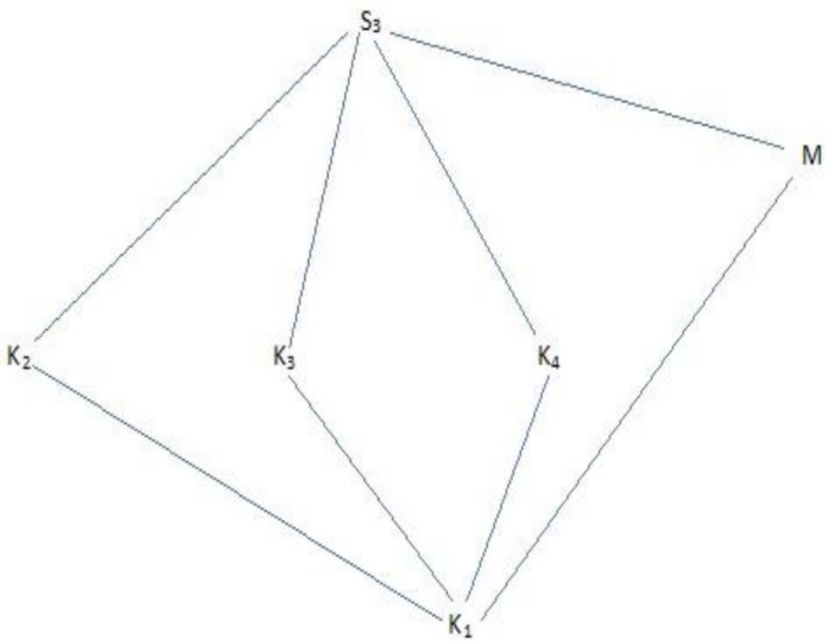

Figure 2

By observing the diagram we obtain the number of fuzzy subgroups in S_3.

If $Q_1(\mu) = S_3,\ S_3$ has 1 fuzzy subgroup, namely

$$\mu_1(x) = \theta_1\ \forall\ x \in K_2.$$

If $Q_1(\mu) = K_2$ there are 1 fuzzy subgroup of S_3 with $Q_1(\mu) = K_2$ i.e.

$$\mu_2(x) = \begin{cases} \theta_1, & x \in K_2, \\ \theta_2, & x \in S_3 \backslash K_2. \end{cases}$$

Also there is 1 fuzzy subgroup each for $Q_1(\mu) = K_3,\ Q_1(\mu) = K_4,\ Q_1(\mu) = M$. For $Q_1(\mu) = K_1 = \{I\}$ we have 5 fuzzy subgroups. S_3 has $2 \times (1+1+1+1+1) = 10$ fuzzy subgroups.

3.4. Constructing fuzzy normal subgroups of Symmetric Group S_3

$$S_3 = \{I,\ (1\ 2),\ (1\ 3),\ (2\ 3),\ (1\ 2\ 3),\ (1\ 3\ 2)$$

We have three normal subgroups of S_3, those are $K_1 = \{I\},\ M = \{I,\ (1\,2\,3),\ (1\,3\,2)\}$ and S_3 itself. The diagram of normal subgroup lattice of S_3 is as shown below:

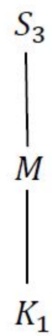

Figure 3

Now we determine the number of fuzzy normal subgroups in S_3. If $Q_1(\mu) = S_3,\ S_3$ has 1 fuzzy normal subgroup, namely

$$\mu_1(x) = \theta_1\ \forall\ x \in S_3.$$

If $Q_1(\mu) = M$, there is 1 fuzzy normal subgroup of S_3 i.e.

$$\mu_2(x) = \begin{cases} \theta_1, & x \in M, \\ \theta_2, & x \in S_3 \backslash M. \end{cases}$$

If $Q_1(\mu) = K_1 = \{I\}$, we have 2 fuzzy normal subgroups So S_3 has $2 \times (1+1) = 4$ fuzzy normal subgroups.

3.5. Constructing fuzzy subgroups of Alternating Group A_4

$A_4 = \{I,\ (1\,2)(3\,4),\ (1\,3)(2\,4),\ (1\,4)(2\,3),\ (1\,2\,3),\ (1\,3\,2),\ (1\,2\,4),\ (1\,4\,2),\ (1\,3\,4),\ (1\,4\,3),\ (2\,3\,4),\ (2\,4\,3)\}$

We have 10 subgroups of A_4, those are $K_1 = \{I\}$, $K_2 = \{I,\ (1\ 2)(3\ 4)\}$, $K_3 = \{I,\ (1\ 3)(2\ 4)\}$, $K_4 = \{I,\ (1\ 4)(2\ 3)\}$, $L_5 = \{I,\ (1\ 2\ 3)(1\ 3\ 2)\}$, $L_6 = \{I,\ (1\ 2\ 4)(1\ 4\ 2)\}$, $L_7 = \{I,\ (1\ 3\ 4)(1\ 4\ 3)\}$, $L_8 = \{I,\ (2\ 3\ 4)(2\ 4\ 3)\}$, $M_9 = \{I, (1\ 2)(3\ 4),\ (1\ 3)(2\ 4),\ (1\ 4)(2\ 3)\}$ and A_4 itself.

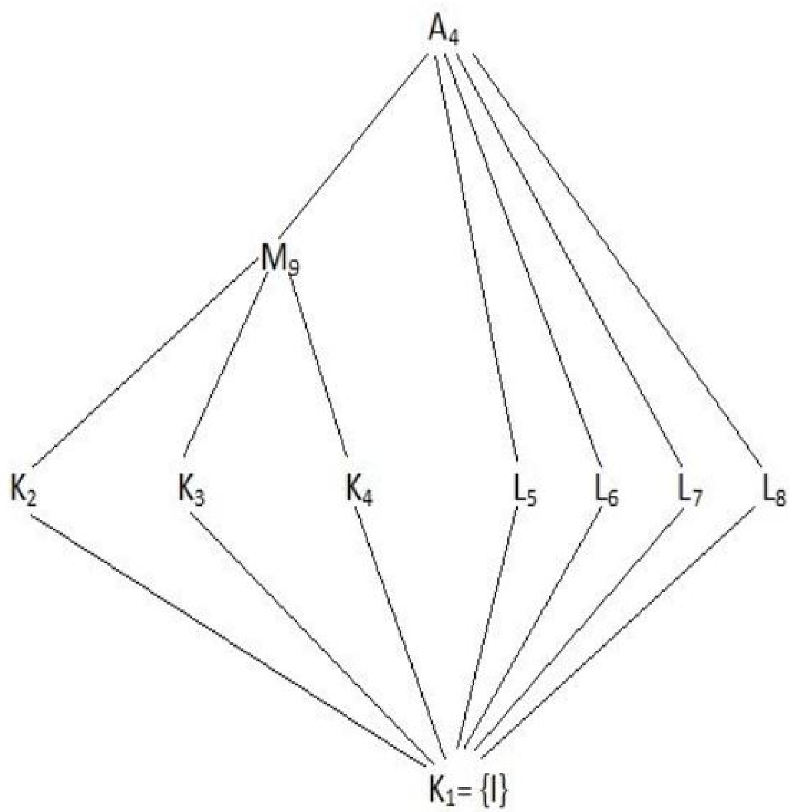

Figure 4

The diagram of subgroup lattice of A_4 is depicted above. By observing the diagram we compute the no. of fuzzy subgroups in A_4. If $Q_1(\mu) = A_4,\ A_4$ has 1 fuzzy subgroup, namely

$$\mu_1(x) = \theta_1\ \forall x \in A_4.$$

If $Q_1(\mu) = M_9$, there is 1 fuzzy subgroup of A_4 i.e.

$$\mu_2(x) = \begin{cases} \theta_1, & x \in M_9, \\ \theta_2, & x \in A_4 \backslash M_9. \end{cases}$$

If $Q_1(\mu) = K_2$, there are 2 chains , namely $K_2 < A_4,\ K_2 < M_9 < A_4$. Hence A_4 has 2 fuzzy subgroups with $Q_1 = K_2$, namely

$$\mu_3(x) = \begin{cases} \theta_1, & x \in K_2, \\ \theta_2, & x \in A_4 \backslash K_2, \end{cases}$$

and

$$\mu_4(x) = \begin{cases} \theta_1, & x \in K_2, \\ \theta_2, & x \in M_9 \backslash K_2, \\ \theta_3, & x \in A_4 \backslash M_9. \end{cases}$$

Also we have 2 F.S. for $Q_1(\mu) = K_3,\ Q_1(\mu) = K_4$. If $Q_1(\mu) = L_5,\ A_4$ has 1 fuzzy subgroup, namely

$$\mu_9(x) = \begin{cases} \theta_1, & x \in L_5, \\ \theta_2, & x \in A_4 \backslash L_5. \end{cases}$$

Also there is 1 fuzzy subgroup each for $Q_1(\mu) = L_6,\ Q_1(\mu) = L_7,\ Q_1(\mu) = L_8$ For $Q_1(\mu) = \{I\}$, we have $1 + 1 + 2 \times 3 + 1 + 1 + 1 + 1 = 12$ fuzzy subgroups. So total number of fuzzy subgroups $= 2 \times 12 = 24$.

3.6. Constructing fuzzy normal subgroups of A_4

We have three normal subgroups of A_4, those are
$K_1 = \{I\},\ K_2 = V = \{I, (12)(34), (13)(24), (14)(23)\}$ and A_4 itself.
The diagram of normal subgroup lattice of A_4 is as shown below.

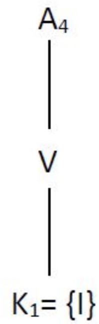

Figure 5

Now we calculate the no. of fuzzy normal subgroups in A_4. If $Q_1(\mu) = A_4$, there is 1 fuzzy normal subgroup of A_4 namely

$$\mu_1(x) = \theta_1 \ \forall x \in A_4$$

If $Q_1(\mu) = V$, there is 1 fuzzy normal subgroup of A_4 with $Q_1(\mu) = V$ i.e.

$$\mu_2(x) = \begin{cases} \theta_1, & x \in V, \\ \theta_2, & x \in A_4 \backslash V. \end{cases}$$

For $Q_1(\mu) = \{I\}$ we have two fuzzy normal subgroups of A_4. A_4 has $2\times(1+1) = 4$ fuzzy normal subgroups.

4. Conclusion

In this paper, we have found the number of Fuzzy subgroups and Fuzzy normal Subgroups of S_2, S_3 and A_4. We conclude this paper by giving two open problems about the above results.

Problem 1: Determine the number of fuzzy subgroups of $D_8 \times C_{2^n}$, $n \geq 1$ by the equivalence relation given by Sulaiman and Abd Ghafur.

Problem 2: Solve the problem of classifying Fuzzy subgroups of $C_4 \times S_n$, $n \leq 5$ by the same equivalence relation.

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