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E-SUPPLEMENT IN A RING

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Abstract: Let R be an associative ring with unity. Two new concepts namely "e-small" and "e-supplement" in R are introduced and many of its properties are discussed in this paper.

Keywords and Phrases: e-small, e-supplement, Goldie dimension.

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1. Introduction

The dualization concept of Goldie dimension was first coined by Patrick Fleury by introducing a new class of modules with finite spanning dimension in [4]. A module M is said to have finite spanning dimension if every infinite, strictly decreasing chain of sub modules is ultimately small in M i.e. for any infinite chain $N_0 \supsetneq N_1 \supsetneq N_2 \supsetneq$ ..... of sub modules of M, there exists $j \in N$ such that N_j is small in M for every $i \geqslant j$ [4].

In the study of Goldie Dimension, there is a very important role of uniform modules, essential extensions and so on. A sub module K of an R-module M is said to be essential if the intersection of K with any non-zero sub module is non zero.

It is denoted by $K \trianglelefteq M$. In this case, M is defined as essential extension of K. If in an R-module M, every sub module is essential, then M is said to be a uniform module. The dual notion of essential modules is small modules. A sub module K of a module M is said to be small if there does not exist a proper sub module L such that $K + L = M$. It is denoted by $K \ll M$. Similarly, the dual notion of uniform module is hollow module. The notion of hollow module was first coined by Fleury in [4]. A module is said to be hollow if every of its sub module is small. In [8], Miyashita calls a hollow $R-$module an $R-$sum irreducible.
A sub module N of R-module M is called a complement of a sub module L in M if it is maximal with respect to $N \cap L = 0$. It is observed that for every sub module L of M, there exists a complement. Besides, a sub module N is said to be closed if it has no proper essential extension in M. The dual notion of closed module is co-closed. It was first coined by Golan in [5]. The dual notion of the concepts of complements is supplement. A sub module N is a supplement of a sub module L if N is minimal with respect to $N + L = M$. Let N and L be two sub modules of M. Then the following two conditions are equivalent:

1. N is minimal in the set of sub module $\{K \subseteq M | L + K = M\}$.

2. $L + N = M$ and $L \cap N \ll M$.

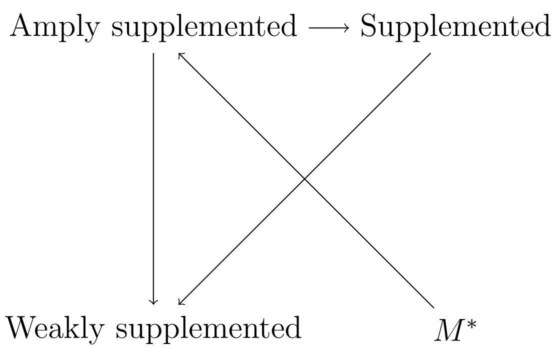
In [3], N is defined as a supplement of L in M. The detailed information about supplemented and related modules was given by Zöschinger [13], [14], [15], [16]. One major difference between complement and supplement is that complement always exists, but it is not true that every sub module has a supplement. For example in $Z-$module Z, there does not exist any proper sub module which has a supplement in Z. The notion of weak supplement was first introduced by Zöschinger in [17]. A module N is called a weak supplement of L in M iff $N + L = M$ and $N \cap L \ll M$. It is easy to verify that every supplement is weak supplement. But its converse is not true. For example, $Z_{(p)} = \{\frac{a}{b} \in Q|$p does not divide b$\}$ is weak supplement in Q but it is not a supplement in Q. The supplement and weak supplement are related to each other in the following ways-

1. N is a supplement in M.

2. N is a weak supplement in M that is closed in M.

3. N is a weak supplement in M and whenever $K \subset N$ and $K \ll M$, then $K \ll N$.

This implies that a weak supplement N in M is a supplement in M if it is coclosed in M. Besides if M is a weak supplement module, then N is a supplement in M iff it is a coclosed sub module. In [11], Varadarajan defined that a module M is said to have the property (P_1) if given any $N \subset M$, there exists a supplement K of N in M. These kind of modules are further known as supplemented modules. Similarly in [11], it was defined that a module is said to have the property (P_2) if for every pair of sub modules (K, L) with $K + L = M$, there exists supplement H of L in M which is a sub set of K. While in [8], it was termed as $R-$perfect modules and defined as for every sub module K, L of M with $K + L = M$, there exists a sub module K_0 of K which is minimal with respect to the addition $K_0 + L = M$. Now this kind of modules is known as amply supplemented modules. An amply supplemented module is always a supplemented module. But its converse is not always true. The converse is true in case M is supplemented module and its every sub module is amply supplemented. If a sub module N is a supplement in an amply supplemented module M, then N itself is amply supplemented module. A module M is said to be weakly supplemented module if its every sub module has a weak supplement. The following implications hold for a module:



In the above diagram, M^* denotes a supplemented module whose every sub module is amply supplemented.

Throughout this paper R denotes an associative ring with unity unless and otherwise stated. The concept of supplements has already been discussed. Further, the mathematician had discussed these concepts in R. An ideal of a ring R is said to be supplement of an ideal P of R if $P + Q = R$ but $P + Q' \neq R$ for any ideal $Q'(\subset Q)$ of R.Let R be a ring and P and Q are ideals with $P \subset Q$. Then P is said to be small in Q if for each ideal L $(\subseteq Q)$ of R, $P + L = Q \Rightarrow L = Q$. In this paper,an attempt is made to define and characterise the concept of supplement and small in terms of elements in R. The study of these substructures of ring elementwise leads us to the notion of dual Goldie dimension elementwise.

In the preliminary section of this paper, the concepts like e-supplement, e-small are defined in R. In this section, some basic definitions and results are discussed which are used in our work. In the next section, some results of e-small in R are proved. It is proved that if any element of an ideal P is e-small, then any element of Q ($Q \subseteq P$) is again an e-small. If P, Q are any two ideals of R and $a \in P$ is small element in Q, then a is also an e-small in R. Some results are proved to show the relation between e-supplement and e-small. It is shown that in R, if a is an e-supplement and $b \in R$ is an e-small in R with $Rb \subseteq Ra$, then b is also e-small in Ra. In theorem (3.8), it is proved that under certain conditions, if a is e-supplement of b and k is an e-small in R, then a is also an e-supplement of $b+k$ in R. In theorem (3.9), it is proved that if a is e-supplement of b in R and b is an e-supplement in R, then b is e-supplement of a in R. In theorem (3.10), it is shown that if R is e-supplemented and primitive ideal, then every principal ideal of R is direct summand of R. In theorem (3.12), it is proved that if R-module R has finite spanning dimension, then every element of R has an e-supplement.

2. Preliminaries

In this section, we have presented some basic definitions and results needed for our work.

Definition 2.1. *Let R be a ring and P is an ideal of R. An element $a \in P$ is said to be e-small of P in R if for each ideal L ($\subseteq R$) of R, $Ra + L = R \Rightarrow L = R$. The zero element of ring R is trivial e-small element. For example, consider ring $Z_{12} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$.$R6 = \{0, 6\}$ is an ideal of Z_{12}. But Z_{12} is the only ideal with respect to $R6$ which satisfies $R6 + Z_{12} = Z_{12}$ implies 6 is an e-small of Z_{12}.*

Definition 2.2. *Let R be a ring . An element $a \in R$ is said to be e-supplement of $b \in R$ if Ra is supplement of the ideal P where $b \in P = Rb$. An ideal Q is said to be e-supplemented if every element $a \in P$ is an e-supplement.*

Definition 2.3. *Let R be a ring. An element $a \in R$ is said to be e- weak supplement of $b \in R$ if Ra is a weak supplement of the ideal P where $b \in P = Rb$. An ideal Q is said to be e-weak supplemented if every element $a \in Q$ is an e-weak supplement.*

Definition 2.4. *Let P and Q are ideals of ring R, then R is said to be direct sum of P and Q denoted by $R = P \oplus Q$ if $R = P + Q$ and $P \cap Q = (0)$ [12]. In such case Q is called direct summand of R.*

Definition 2.5. (Modular law) [12] *Let P, Q, T be ideals of a ring R such that*

$P \subseteq T$. Then $(P+Q) \cap T = P + (Q \cap T)$.

Lemma 2.1. *Any small subgroup of E is contained in $J(E)$.*

Definition 2.6. *A ring R is called semi primitive or Jacobson semi simple if $J(R) = 0$. If R is sub direct product of primitive ring,then ring R is semi-primitive ([12]).*

3. Results

In this section, some of the characteristics of e-small, e-supplement are discussed.

Lemma 3.1. *Let P and Q be ideals of R such that $P \subseteq Q$. If any element of Q is e-small in R then any element of P is also small in R.*

Proof. Let $a \in P$ and C be an ideal of R such that $Ra + C = R$. But $a \in P \subseteq Q$ is a e-small element in R. Then

$$
\begin{aligned}
Ra + C &= R \\
\Rightarrow C &= R
\end{aligned}
$$

Hence, a is small element in R.

Lemma 3.2. *Let P and Q be ideals of a ring R and $a \in P$ is small element in Q then a is small element in R.*

Proof. Let C be an ideal of R such that $Ra + C = R$

From modular law we have,

$$
\begin{aligned}
Ra + (C \cap Q) &= (Ra + C) \cap Q \\
&= Q
\end{aligned}
$$

Thus $C \cap Q$ is an ideal of R such that,

$$
\begin{aligned}
Ra + (C \cap Q) &= Q \\
\text{so, } (C \cap Q) &= Q \qquad\qquad [\because a \textit{ is small element in } R] \\
\text{Thus, } Ra + C &= C \\
\Rightarrow C &= R
\end{aligned}
$$

Hence, a is small element in R.

Theorem 3.3. *Let R be a ring and a is an e-supplement. If $b \in R$ is an e-small in R and $Rb \subseteq Ra$, then b is also an e-small in Ra.*

Proof. Since a is an e-supplement in M, therefore Ra is a supplement in M.

This implies Ra is coclosed in M. Now consider $Ra = Rb + K$, $K \subseteq Ra$. Then $M = Ra + L$, for any $L \subseteq M$ implies $M = Rb + K + L$ implies $M = K + L$ and K is co-small in Ra. But since Ra is coclosed in M implies $Ra = K$. This implies b is e-small in Ra.

The converse of the above theorem is not true in general.But instead of e-supplement, if we consider a as e-weak supplement in M, then the converse is true.Hence we have the following corollary.

Corollary 3.4. *If a is e-weak supplement in R and b is e-small in R, $Rb \subseteq Ra$ implies b is e-small in Ra, then a is e-supplement in R.*
Proof. It is easy to prove.

Theorem 3.5. *Let R be a ring. An element $a \in R$ is an e-supplement of $b \in R$ iff $Ra \cap Rb$ is small in Ra.*
Proof. Suppose $Ra \cap Rb$ is not small in Ra. Then there exists $C \subsetneq Ra$ such that $(Ra \cap Rb) + C = Ra$ Again, since Ra is a supplement of Rb in R. Then

$$\begin{aligned} R &= Ra + Rb \\ &= [Ra \cap Rb + C] + Rb \\ &= C + Rb \end{aligned}$$

which is a contradiction to the fact that a is a supplement of b in R.
Conversely, suppose $Ra \cap Rb$ is small in Ra. Consider $C(\subseteq Ra)$ is an ideal such that $Rb + C = R \Rightarrow (Ra \cap Rb) + C = Ra \Rightarrow C = Ra$. Therefore Ra is a supplement of Rb in R which implies a is an e-supplement of b in R.

Corollary 3.6. *Let R be a principal ideal ring. An element $a \in R$ is a e-supplement of $b \in R$ iff $c \in Ra \cap Rb$ is e-small in Ra where c is a generator of $Ra \cap Rb$.*
Proof. Suppose $c \in Ra \cap Rb$ is not e-small in Ra. Then there exists $C \subsetneq Ra$ such that $Rc + C = Ra$. Now since Ra is a supplement of Rb in R. Then $R = Ra + Rb \Rightarrow R = (Rc + C) + Rb \Rightarrow R = C + Rb$ which is a contradiction to the fact that a is an e-supplement of b in R.
Conversely, assume $c \in Ra \cap Rb$ is a generator of the ideal $Ra \cap Rb$ and c is an e-small in Ra. Consider $C(\subseteq Ra)$ is an ideal such that $Rb + C = R \Rightarrow (Ra \cap Rb) + C = Ra \Rightarrow Rc + C = Ra \Rightarrow C = Ra$. Therefore a is an e-supplement of b in R.

Theorem 3.7. *Let a be an e-supplement of b in $K^* = Rc$ and c is supplement element of d in R. Then a is supplement element in R.*
Proof. Given a is a supplement element of b in $K^* = Rc$. Therefore, $Ra + Rb = Rc$.

Again, c is supplement of d in R. This implies

$$\begin{aligned} Rc + Rd &= R \\ \Rightarrow Ra + (Rb + Rc) &= R \end{aligned}$$

Let there exists some $K \subseteq Ra$ such that

$$\begin{aligned} K + (Rb + Rd) &= R \\ \Rightarrow (K + Rb) + Rd &= R \end{aligned}$$

Since, $K \subseteq Ra$ and Rc is minimal in $Rc + Rd = R$. Therefore, $K + Rb = Rc = K^*$ which is a contradiction to the fact that a is supplement element of b in K^*. Therefore a is a supplement element in R.

Theorem 3.8. *If a is e-supplement of b and k is an e-small in R, then a is an e-supplement $b + k$ in R provided $R(a + b)$ is a maximal ideal in R.*
Proof. Let $X \subseteq Ra$ such that $R(b + k) + X = R$ implies $R(b) + R(k) + X = M$ implies $R(b) + X = M$ since b is e-small in R. This implies a is e-supplement in R.

Theorem 3.9. *If a is e-supplement of b in R and b is itself an e-supplement in R, then b is e-supplement of a in R.*
Proof. Since a is e-supplement of b in R i.e. $Ra + Rb = M$ and $Ra \cap Rb \ll R$. Now b is e-supplement in R. Therefore $Ra \cap Rb \ll Rb$ implies b is e-supplement of a in R.

Theorem 3.10. *Let R be e-supplemented and semi-primitive. Then every principal ideal of R is a direct summand of R.*
Proof. Let a be any element of R. Consider a is supplement element of $b \in R$. Therefore, there exist $P = Rb \subseteq R$ such that Ra is a supplement of Rb in R. Therefore, $Ra + Rb = R$ and $Ra \cap Rb \subseteq_s R$. Since $Ra \cap P \subseteq J(R) = 0 \Rightarrow Ra \cap P = 0$ Thus Ra is direct summand of R.

Corollary 3.11. *If R is e-supplemented and sub direct product of primitive rings, then every principal ideal of R is a direct summand of R.*
Proof. It is easy to follow.

Theorem 3.12. *Let R be a ring R module R has finite spanning dimension, then every element of R has an e-supplement.*
Proof. Let a be an element of R and Ra is module generated by a
There are two cases:
Case I. If Ra is small, then R is supplement of Ra in R.
Case II. If Ra is not small, there exists $S \subseteq R$ such that $Ra + S = R$ it is minimal

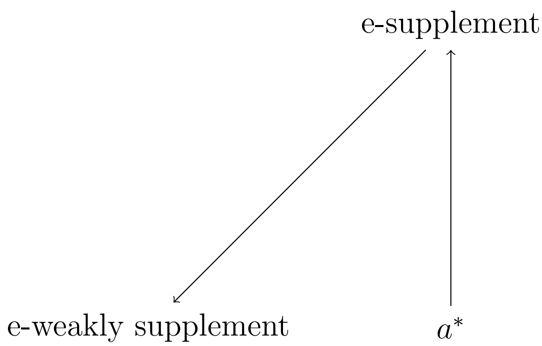
w.r.t $Ra + S = R$ and b be an element of S then a is supplement in R.
If Ra is not supplement of S in R then there exist b such that $S + Rb = R$.
Continuing in this process we have $Ra \supseteq Rb. \ldots .$
decreasing sequence of non small submodule.
Since, it has finite spanning dimension, so there exists $c \in R$ such that Rc is supplement of S.
Therefore c is an e-supplement of S.

4. Some Observations

Some observations from the above discussion are given below :

1. Every e-supplement is e- weakly supplement.
2. If $eS(R)$ denotes the e-supplement in R and $eW(R)$ denotes all e-weak supplement in R, then clearly $eW(R) \subseteq eS(R)$.
3. For every e-supplement in R, there exists an ideal in ring R which is coclosed in R.
4. For every element in ring R, it is not necessary that it has an e-supplement.
5. If R is supplemented, then R is also an e-supplemented.

Clearly, we have the following implication in diagram form:



In the above diagram, a^* denotes an e-weak supplement of b in R and b is e-small in R,$Rb \subseteq Ra$ implies b is e-small in Ra.

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