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THE FORCING CONVEX DOMINATION NUMBER OF A GRAPH

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Abstract: Let G be a connected graph and D a minimum convex domination set of G. A subset $T \subseteq D$ is called a forcing subset of D, if D is the unique minimum convex dominating set containing T. A forcing subset for D of minimum cardinality is a minimum forcing subset of D. The forcing convex domination number of D, denoted by $\gamma_{con}(D)$, is the cardinality of a minimum forcing subset of D. The forcing convex domination number of G, denoted by $f_{\gamma con}(G)$ and is defined by $f_{\gamma con}(G) =$ min $\{f_{\gamma con}(D)\}$, where the minimum is taken over all minimum convex dominating sets D in G. Some general properties satisfied by this concepts are studied. The forcing fair dominating number of certain standard graphs are determined. It is shown that for every pair a, b of integers with $0 \leq a < b$, there exists a connected graph G such that $f_{\gamma con}(G) = a$ and $\gamma_{con}(G) = b$.

Keywords and Phrases: Forcing convex domination, convex domination number, convex number.

2020 Mathematics Subject Classification: 05C69.

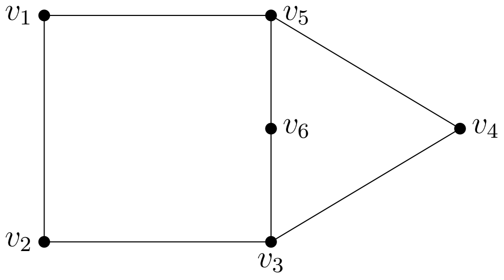
1. Introduction

By a graph $G = (V,E)$, we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by n and m respectively. For basic definitions and terminologies we refer to [4]. Two vertices u and v are said to be *adjacent* if uv is an edge of G. The *open neighbourhood of a vertex* v in a graph G is defined as the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$, while the *closed neighbourhood* of v in G is defined as $N_G[v] = N_G(v) \cup \{v\}$. For any vertex v in a graph G, the number of vertices adjacent to v is called the *degree* of v in G, denoted by $deg_G(v)$. If the degree of a vertex is 0, it is called an *isolated vertex*, while if the degree is 1, it is called an *end-vertex*. The *minimum degree* of vertices in G is defined by $\delta(G) = min\{deg(v)/v \in V(G)$. The *maximum degree* of vertices in G is defined by $\triangle(G) = max\{deg(v)/v \in V(G)$. A vertex v is called an *universal vertex* if $deg_G(v) = n - 1$. For any set S of vertices of G, the *induced subgraph* $< S >$ is the maximal subgraph of G with vertex set S. A subset $S \subseteq V(G)$ is called a *dominating set* if every vertex $v \in V(G)\backslash S$ is adjacent to a vertex $u \in S$. The *domination number*, $\gamma(G)$, of a graph G denotes the minimum cardinality of such dominating sets of G. A minimum dominating set of a graph G is hence often called as a γ-*set* of G. The domination concept was studied in [1, 4]. A vertex v of a connected graph G is said to be a *dominating vertex* of G if v belongs to every γ-set of G. If G has a unique γ-set S, then every vertex of S is a dominating vertex of G. The forcing set in a graph is a very interesting concept. Let S be a γ-set of G. A subset $T \subseteq S$ is called a *forcing subset* for S if S is the unique γ-set containing T. A forcing subset for S of minimum cardinality is a *minimum forcing subset* of S. The *forcing domination number* of S, denoted by $f_\gamma(S)$, is the cardinality of a minimum forcing subset of S. The *forcing domination number* of G, denoted by $f_\gamma(G)$, is $f_\gamma(G) = min\{f_\gamma(S)\}$, where the minimum is taken over all γ-sets in G. The forcing concept was first introduced and studied in [1]. The forcing domination number of a graph was first introduced by G. Chatrand in [3]. Further studied in [13], [14] and [15]. Many authors have studied the forcing concept with respect to several parameters like domination, geodetic, Steiner, hull, detour, monophonic, etc. In this paper we study the forcing concept with respect convex domination. A convex dominating set, abbreviated *con*-set in G. A set S of vertices in a graph G is *convex* if $I(S) = S$. Certainly, $V(G)$ is convex. For a nontrivial connected graph G, the convexity number $con(G)$ was defined in [4] as the maximum cardinality of a proper convex set of G, that is, $con(G)$ $= max\{|S| : S \quad is \quad a \quad convex \quad set \quad of \quad G \quad and \quad S = V(G)$. A convex set S in G with $|S| = con(G)$ is called a maximum convex set. A nontrivial connected graph G of order n with $con(G) = k$ is called a (k, n) graph. The convexity number was also studied in [6] and [8]. The *join* of two graphs G and H is a graph formed from disjoint copies of G and H by connecting every vertex of G to every vertex of H. The *corona* $G_1 \odot G_2$ of two graphs G_1 and G_2 is defined as the graph obtained by taking one copy of G_1 and n_1 copies of G_2 , and then join the i^{th} vertex of G_1 with an edge to every vertex in the i^{th} copy of G_2. The *cross product* $a \times b$ is defined as a vector c that is orthogonal to both a and b , with a direction given by the right hand rule and a magnitude equal to the area of the parallelogram that the vector span.

2. The Forcing convex domination number of a graph

Definition 2.1. *Let G be a connected graph and D a minimum convex domination set of G. A subset $T \subseteq D$ is called a forcing subset of D, if D is the unique minimum convex dominating set containing T. A forcing subset for D of minimum cardinality is a minimum forcing subset of D. The forcing convex domination number of D, denoted by $\gamma_{con}(D)$, is the cardinality of a minimum forcing subset of D. The forcing convex domination number of G, denoted by $f_{\gamma con}(G)$ and is defined by $f_{\gamma con}(G) = min\{f_{\gamma con}(D)\}$, where the minimum is taken over all minimum convex dominating sets D in G.*

Example 2.2. For the graph G given in Figure 2.1, $S_1 = \{v_1, v_2, v_3\}$,$S_2 = \{v_1, v_5, v_6\}$, $S_3 = \{v_2, v_3, v_4\}$ and $S_4 = \{v_1, v_4, v_5\}$ are the only four γ_{con}-sets of G such that $f_{\gamma con}(S_1) = 2$, $f_{\gamma con}(S_2) = 1$, $f_{\gamma con}(S_3) = 2$ and $f_{\gamma con}(S_4) = 2$ so that $f_{\gamma con}(G) = 1$.



G

Fig 2.1

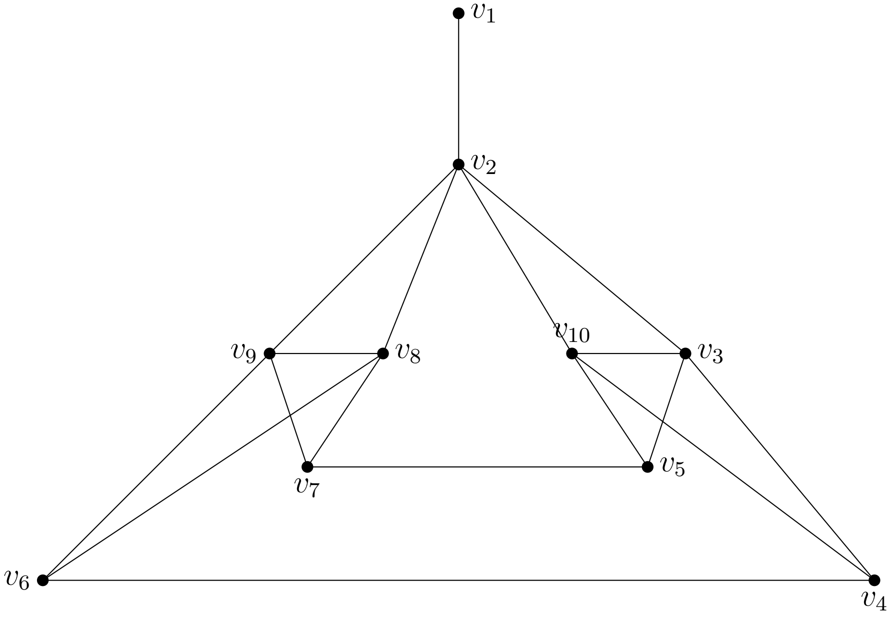
The following theorem follows from the definition of the forcing convex domination number of a graph.

Theorem 2.3. *For a connected graph G, $0 \leq f_{\gamma con}(G) \leq \gamma_{con}(G)$.*

Observation 2.4. *Let G be a connected graph, then*
(i)$f_{\gamma con}(G) = 0$ if and only if G has a unique minimum convex dominating set.
(ii)$f_{\gamma con}(G) = 1$ if and only if G has at least two minimum convex dominating sets one of which is a unique minimum convex dominating set containing one of its elements, and
(iii) $f_{\gamma con}(G) = \gamma_{con}(G)$ if and only if no minimum convex dominating set of G is the unique minimum convex dominating set containing any of its proper subsets.

Definition 2.5. *A vertex v of a graph G is said to be convex dominating vertex if v belongs to every minimum convex dominating set of G.*

Example 2.6. For the graph G given in Figure 2.2, $S_1 = \{v_2, v_3, v_4\}$, $S_2 = \{v_2, v_8, v_{10}\}$, $S_3 = \{v_2, v_9, v_{10}\}$, $S_4 = \{v_1, v_3, v_8\}$ are the only four γ_{con}- sets of G, so that v_2 is the only convex dominating vertex of G.



G

Fig 2.2

Observation 2.7. *Let G be a connected graph and let $\Im$ be the set of relative complements of the minimum forcing subsets in their respective minimum convex dominating sets in G. Then $\cap_{F\in\Im}F$ is the set of convex dominating vertices of G.*

Corollary 2.8. *Let G be a connected graph and S a minimum convex dominating set of G. Then no convex dominating vertex of G belongs to any minimum forcing set of S.*

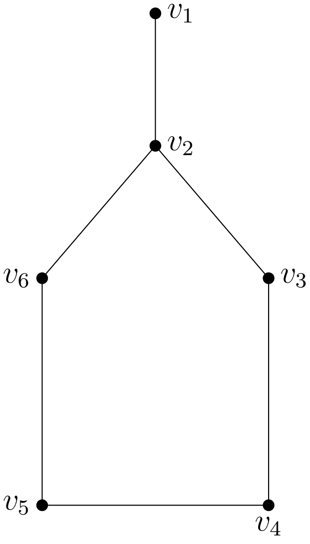
The following observation is clear from the definition of forcing convex domination number and the convex dominating vertex of a graph.

Observation 2.9. *Let G be a connected graph and W be the set of all convex dominating vertices of G. Then $f_{\gamma_{con}}(G) \leq \gamma_{con}(G) - |W|$.*

Example 2.10. *The bound in observation 2.9 is sharp. For the graph G given in Figure 2.2, $D_1 = \{v_2, v_3, v_9\}$, $D_2 = \{v_2, v_8, v_{10}\}$, $D_3 = \{v_2, v_9, v_{10}\}$ and $D_4 =$*

$\{v_2, v_3, v_8\}$ *are the only four* γ_{con}*- sets of* G*, such that* $f_{\gamma con}(D_1) = f_{\gamma con}(D_2) = f_{\gamma con}(D_3) = f_{\gamma con}(D_3) = 2$ *so that* $f_{\gamma con}(G) = 2$ *and* $\gamma_{con}(G) = 3$*. Also* $W = \{v_2\}$ *is the set of convex dominating vertices of* G *so that* $f_{\gamma con}(G) = \gamma_{con}(G) - |W|$*. Also the bound in Observation 2.9 is strict.*

Remark 2.11. *For the graph* G *given in Figure 2.3,* $S_1 = \{v_2, v_3, v_4\}$*,* $S_2 = \{v_2, v_3, v_6\}$*,* $S_3 = \{v_2, v_5, v_6\}$*. Here* $W = \{v_2\}$ *is the set of convex dominating vertices of* G *so that* $\gamma_{con}(G) = 2$ *and* $f_{\gamma con}(G) = 1$*. Therefore* $f_{\gamma con}(G) = \gamma_{con}(G) - |W|$*.*



G

Fig 2.3

Theorem 2.12. *For the complete graph* $G = K_n$ $(n \geq 3)$*,* $f_{\gamma con}(G) = 2$*.*

Proof. Let $v_1, v_2, ..., v_n$ be the vertex set of K_n. Then $D_i = \{v_i, v_j\}$, $(1 \leq i \leq n)$ is a γ_{con}- set of G. Since D_i $(1 \leq i \leq n)$ is not unique, it follows that $f_{\gamma con}(G) = 2$.

Theorem 2.13. *For the complete bipartite graph* $G = K_{m,n}$ $(2 \leq m \leq n)$*,* $f_{\gamma con}(G) = 2$*.*

Proof. Let $X = \{x_1, x_2, ..., x_m\}$ and $Y = \{y_1, y_2, ..., y_n\}$ be the two bipartite sets of G. Let $D_{ij} = \{x_i, y_j\}$, $1 \leq i \leq m$ and $1 \leq j \leq n$. Then D_{ij} is the γ_{con}-set of G so that $\gamma_{con}(G) = 2$, Since $m, n \geq 2$, D_{ij} is unique and so $f_{\gamma con}(D_{ij}) \geq 1$ $\forall i, j$. By Theorem $1 \leq f_{\gamma con}(D_{ij}) \leq 2$ for all i, j. We prove that $f_{\gamma con}(D_{ij}) = 2$ for all i and j.

Suppose that $f_{\gamma con}(D_{ij}) = 1$, for some i, j. Then D_{ij} is a unique γ_{con}-set containing x_i and y_j, which is a contradiction to G is a complete bipartite set of G. Therefore $f_{\gamma con}(D_{ij}) = 2\ \forall i, j$. Hence it follows that $f_{\gamma con}(G) = 2$.

Theorem 2.14. *For the path $G = P_n$ $(n \geq 2)$, $f_{\gamma con}(G) = 0$.*
Proof. Let $V(P_n) = \{v_2, v_3, ..., v_n\}$. Then $S = \{v_2, v_3, ..., v_{n-1}\}$ is the unique γ_{con}-set of G so that $f_{\gamma con}(G) = 0$.

Theorem 2.15. *For a cycle graph $G = C_n$ $(n \geq 4)$,*

$$f_{\gamma con}(G) = \begin{cases} 2 & \text{if } n{=}4,5 \\ 0 & \text{if } n \geq 6 \end{cases}$$

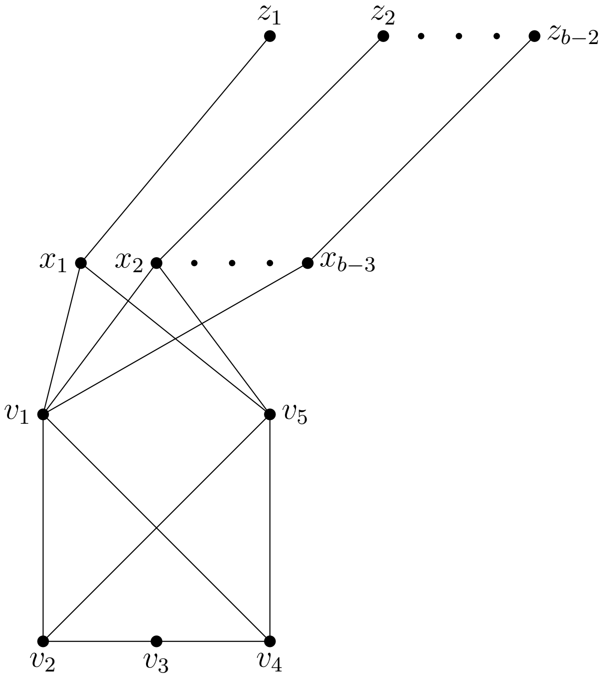
Proof. Let $V(C_n) = \{v_1, v_2, ..., v_n, v_1\}$.
Case (i):
when $n = 4$. Let $S_1 = \{v_1, v_2\}$, $S_2 = \{v_2, v_3\}$, $S_3 = \{v_3, v_4\}$ and $S_4 = \{v_1, v_4\}$ are the only two γ_{con}-sets of G such that $f_{\gamma con}(S_1) = f_{\gamma con}(S_2) = f_{\gamma con}(S_3) = f_{\gamma con}(S_4) = 2$ so that $f_{\gamma con}(G) = 2$.
Case (ii):
when $n = 5$. Then $S_1 = \{v_1, v_2, v_3\}$, $S_2 = \{v_2, v_3, v_4\}$, $S_3 = \{v_3, v_4, v_5\}$, $S_4 = \{v_1, v_4, v_5\}$ and $S_5 = \{v_1, v_2, v_5\}$ are the five γ_{con}-sets of G such that $f_{\gamma con}(S_1) = f_{\gamma con}(S_2) = f_{\gamma con}(S_3) = f_{\gamma con}(S_4) = f_{\gamma con}(S_5) = 2$ so that $f_{\gamma con}(G) = 2$.
Case (iii):
Let $n \geq 6$. Let $S = V(G)$ is the unique γ_{con}-set of G so that $f_{\gamma con}(G) = 0$.

Theorem 2.16. *Let G be a connected graph of order $n \geq 3$, with at least two universal vertices. Then $f_{\gamma con}(G) = 1$.*
Proof. Let x $(1 \leq i \leq n)$ be a universal vertex of G. It is easily observed that $S_i = \{x_i\}$ $(1 \leq i \leq n)$ is a convex dominating vertex of G such that $f_{\gamma con}(S_i) = 1$ for $1 \leq i \leq n$ so that $f_{\gamma con}(G) = 1$.

Corollary 2.17. *(i) For the complete graph $G = K_n$ $(n \geq 2)$, $f_{\gamma con}(G) = 1$.*
(ii) For the Fan graph $G = F_n = K_1 + P_{n-1}$ $(n \geq 3)$, $f_{\gamma con}(G) = 0$.
(iii) For the graph $G = K_n - \{e\}$ $(n \geq 4)$, $f_{\gamma con}(G) = 1$.
(iv) For the graph $G = K_1 + (m_1K_1 + m_2K_2 + ... + m_rK_r)$, where $m_1 + m_2 + ... + m_r \geq 2$, $f_{\gamma con}(G) = 0$.

Theorem 2.18. *For every pair a, b of integers with $0 \leq a \leq b$, there exists a connected graph G such that $f_{\gamma con}(G) = a$ and $\gamma_{con}(G) = b$.*
Proof. For $a = 0$, $b \geq 2$, let $G = P_{b+2}$ Then by a Theorem 2.14 , $f_{\gamma con}(G) = 0$ and $\gamma_{con}(G) = b$. For $a = 1$, $b \geq 2$. For the graph G, given in Figure 2.4, $S_1 = \{x_1, x_2, ..., x_{b-3}, v_1, v_5\} \cup \{v_4\}$ and $S_2 = \{x_1, x_2, ..., x_{b-3}, v_1, v_5\} \cup \{v_2\}$ are the only two γ_{con}-set of G so that $f_{\gamma con}(G) = 1$ and $\gamma_{con}(G) = b$. So let $a \geq 2$ and $b \geq 3$.

Let $Q_i : w_i, y_i, z_i$ $(1 \leq i \leq a)$ be a copy of K_4. Let $P_i : u_i, v_i$ $(1 \leq i \leq b-a)$ be a copy on two vertices. Let G be the graph obtained from Q_i $(1 \leq i \leq a)$ and P_i $(1 \leq i \leq b-a)$ by joining each v_i $(1 \leq i \leq b-a)$ with each w_j and y_j $(1 \leq j \leq a)$. The graph G is shown in Figure 2.5.

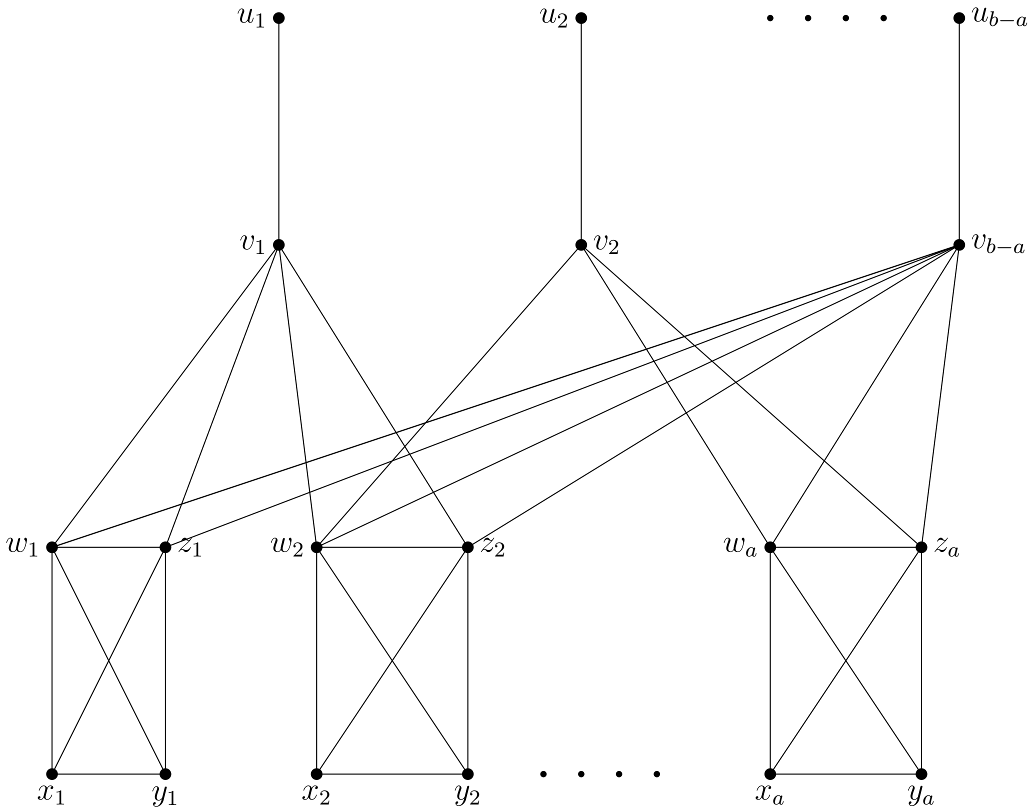


G

Fig 2.4

First we prove that $\gamma_{con}(G) = b$. Let $z = \{v_1, v_2, ..., v_{b-a}\}$ be the set of all cut vertices of G. By Theorem z is a subset of every convex dominating set of G. Let $H_i = \{w_i, z_i\}$ $(1 \leq i \leq a)$. Then every convex dominating set of G contains exactly one vertex from each H_i $(1 \leq i \leq a)$ and so $\gamma_{con}(G) \geq b - a + a = b$. Let $S = X \cup \{x, z_1, z_2, ..., z_a\}$. Then S is a convex dominating set of G so that $\gamma_{con}(G) = b$.

Next we prove that $f_{\gamma con}(G) = a$. By Theorem, $f_{\gamma con}(G) \leq \gamma_{con}(G) - |Z|$. Since X is a subset of every convex dominating set of G and every γ_{con}-set of G contains exactly one vertex from each H_i $(1 \leq i \leq a)$, every $\gamma_{con}(G)$-set is of the form $S = Z \cup \{c_1, c_2, ..., c_a\}$, where $c_i \in H_i$ $(1 \leq i \leq a)$. Let T be any proper subset of S with $|T| < a$. Then for some i, there exists H_i such that $H_i \cap T = \phi$, which is a contradiction. Therefore $f_{\gamma con}(G) = a$.



G

Fig 2.5

Theorem 2.19. *Let G and H be two non trivial connected graphs. Then $f_{\gamma con}(H \circ K) = 0$.*

Proof. Since $S = V(H)$ is the unique γ_{con}-set of G, $f_{\gamma con}(H \circ K) = 0$.

Theorem 2.20. *For the graph $G = \overline{K}_2 + \overline{K}_{n-2}$, $f_{\gamma con}(G) = 1$ $(n \geq 4)$.*

Proof. Since $V(\overline{K}_2) = \{x, y\}$ and $V(\overline{K}_{n-2}) = \{v_1, v_2, ..., v_{n-2}\}$. Then $S_i = \{x, v_i\}$ $(1 \leq i \leq n-2)$ and $S_j = \{y, v_j\}$ $(1 \leq j \leq n-2)$ are the γ_{con}-sets of G such that $f_{\gamma con}(S_i) = 1$ and $f_{\gamma con}(S_j) = 1$ so that $f_{\gamma con}(G) = 1$.

Theorem 2.21. *Let H and K be two connected graphs of order n_1 and n_2 respectively. Then $f_{\gamma con}(H + K) = 1$.*

Proof. Since $V(H) = \{v_1, v_2, ..., v_{n_1}\}$ and $V(K) = \{u_1, u_2, ..., u_{n_2}\}$. Then $S_i = \{v_i\}$ $(1 \leq i \leq n_2)$ and $S_j = \{u_j\}$ $(1 \leq j \leq n_2)$ are the only γ_{con}-sets of G such that $f_{\gamma con}(S_i) = 1$ $(1 \leq i \leq n_1)$ and $f_{\gamma con}(S_j) = 1$ $(1 \leq i \leq n_2)$ so that $f_{\gamma con}(H + K) = 1$.

3. Conclusion

In future study, we compare the results regarding the forcing convex domination number to other forcing domination concepts in graphs.

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