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k-STRONG DEFENSIVE ALLIANCES IN GRAPHS

Karthik K., Chandru Hegde and B. Sooryanarayana*

Department of Mathematics,
Mangalore University,
Mangalagangothri - 574199, Karnataka, INDIA

E-mail : karthik1111820@gmail.com, chandrugh@gmail.com

*Department of Mathematics,
Dr. Ambedkar Institute of Technology,
Bengaluru - 560056, Karnataka, INDIA

E-mail : dr_bsnrao@dr-ait.org

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Abstract: In a simple connected graph $G = (V, E)$, a subset S of V is a defensive alliance if every vertex $v \in S$ has at most one more neighbour in $V - S$ than it has in S. The minimum cardinality of a defensive alliance in G is called the defensive alliance number of G, denoted by $a(G)$. A k-strong defensive alliance S is a defensive alliance in G, in which removal of any set of at most k vertices does not affect its defensive property. The k-strong defensive alliance number of G is the minimum cardinality of a k-strong defensive alliance in G, denoted by $a^k(G)$. In this paper, some properties of k-strong defensive alliances are discussed and the k-strong defensive alliance numbers of some classes of graphs are obtained.

Keywords and Phrases: Alliances, Strong Defensive Alliances, Defensive Alliance Number.

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1. Introduction

The notion of alliances are introduced by Kristiansen et al. in [9]. Let $G = (V, E)$ be a simple connected graph and $\emptyset \subset S \subseteq V$. For any $v \in V$, $N(v) =$

$\{u \in V : uv \in E\}$ is the open neighbourhood of v and $N[v] = N(v) \cup \{v\}$ is the closed neighbourhood of v. For $v \in S$, the vertices of $N(v) - S$ are the attackers of v with respect to S and those of $N[v] \cap S$ are the defenders of v with respect to S. We denote $|N[v] \cap S|$ by $Def_S(v)$ and $|N(v) - S|$ by $Att_S(v)$. The set S is said to be a defensive alliance [9] if $Att_S(v) \leq Def_S(v)\ \forall\ v \in S$. More results on various alliances are found in [1, 2, 3, 4, 6, 7, 8, 10]. For $k \geq 0$, a defensive alliance S is said to be a k-strong defensive alliance [4] if for any l with $1 \leq l \leq k$, $S - S_l$ is a defensive alliance, where S_l is any subset of S with l elements. The minimum cardinality of a k-strong defensive alliance in a graph G is the k-strong defensive alliance number of G, denoted by $a^k(G)$. A k-strong defensive alliance S in G is said to be a minimal k-strong defensive alliance if no proper subset of S is a k-strong defensive alliance in G. The maximum cardinality of a minimal k-strong defensive alliance is the upper k-strong defensive alliance number, denoted by $A^k(G)$.

A set $S \subseteq V$ is said to be a dominating set if $\underset{s\in S}{\cup} N[s] = V$. Some recent results on domination are found in [5, 11]. A k-strong defensive alliance S in a graph is said to be a k-strong global defensive alliance if S is also a dominating set. The k-strong global defensive alliance number $\gamma_{a^k}(G)$ and the upper k-strong global defensive alliance number $\gamma_{A^k}(G)$ are defined similar to $a^k(G)$ and $A^k(G)$.

The concept of k-strong defensive alliances is applicable to analyze war like situations. It can also be used to build strategies in the business field. Consider a graph model of different cities and the connection between them. Suppose a company wants to set up its manufacturing units in some of the cities so that these units can fulfill the demands of the cities where they are set up and the adjacent cities. Then a k-strong defensive alliance represents the cities where the manufacturing units can be set so that company will be able to fulfill the demands even if any k manufacturing units fail. A 0-strong defensive alliance is nothing but a defensive alliance. Throughout the article, only simple connected graphs are considered. The terms not defined here may be found in [12].

2. Properties of k-Strong Defensive Alliances

In this section, some properties of k-strong defensive alliances are discussed. For any $v \in V$, $|N(v)| = deg(v)$ is the degree of v. A vertex of degree 1 is called a pendant vertex.

Remark 2.1. *For any $S \subseteq V$, $Att_S(v) + Def_S(v) = 1 + deg(v)\ \forall\ v \in S$.*

We recall the following results for immediate references.

Theorem 2.1. [4] *Let $G = (V, E)$ be a graph of order $n > 1$. Then V is a 1-strong defensive alliance in G.*

Remark 2.3. [4] *Let $G = (V, E)$ be a graph. For any defensive alliance S in G and $v \in S$, $|S| \geq \left\lceil \frac{1+deg(v)}{2} \right\rceil$ and hence $|S| \geq \left\lceil \frac{1+\delta(G)}{2} \right\rceil$ where $\delta(G) = \min_{v \in V} \deg(v)$.*

Theorem 2.2 proves the existence of a 1-strong defensive alliance in any graph with $n > 1$. For $k > 1$, vertex set V need not be a k-strong defensive alliance; but a proper subset may be a k-strong defensive alliance. For the graph G of Figure 1, $S = \{v_1, v_2, v_3, v_4, v_5\}$ is a 2-strong defensive alliance, while the entire vertex set is not. However for $k > 1$, the existence of a k-strong defensive alliance is not assured. For $n \geq 3$, the cycle C_n does not contain k-strong defensive alliance for $k \geq 3$. The existence of a k-strong defensive alliance is assumed while discussing the results on k-strong defensive alliance number of a graph. By definition, any k-strong defensive alliance is also a $(k-1)$-strong defensive alliance.

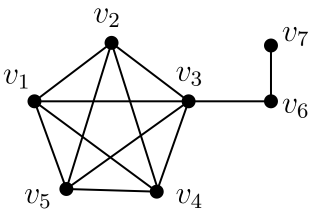

Figure 1: Graph G

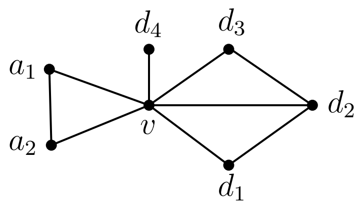

Figure 2: Graph H

Remark 2.4. *Let G be a graph and k be a positive integer.*

1. *For any $k \geq 1$, $a^{k-1}(G) \leq a^k(G) \leq A^k(G)$ and $A^{k-1}(G) \leq A^k(G)$.*

2. *Let S be a k-strong defensive alliance. Then $|S| \geq k+1$ and hence $a^k(G) \geq k+1$.*

Proposition 2.5. *If S and T are k-strong defensive alliances in a graph G, then $S \cup T$ is a k-strong defensive alliance in G.*

Lemma 2.6. *Let $k > 0$ and G be a graph of order $n \geq k+1$. A set $S \subseteq V$ with $|S| = k+1$ is a k-strong defensive alliance if and only if every vertex in S is a pendant vertex.*

Proof. Since G is connected, $deg(x) > 0 \ \forall \ x \in V$. Let $S \subseteq V$ with $|S| = k+1$. Then S is a k-strong defensive alliance if and only if for any $x \in S$, $\{x\}$ is a defensive alliance. This is true if and only if $deg(x) \leq 1$. Since G is connected, $deg(x) = 1 \ \forall \ x \in S$.

The following theorem follows by Lemma 2.6.

Theorem 2.7. *For any graph G of order at least $k+1$, $a^k(G) = k+1$ if and only* *if it has at least $k+1$ pendant vertices.*

Theorem 2.8. *For any graph G, $a^k(G) \geq k + \left\lceil \frac{1+\delta(G)}{2} \right\rceil$.*
Proof. Let S be a k-strong defensive alliance in G. By Remark 2.3, for any $v \in S$, $|S| \geq k + \left\lceil \frac{1+\deg(v)}{2} \right\rceil$. Hence $a^k(G) \geq k + \left\lceil \frac{1+\delta(G)}{2} \right\rceil$.

Corollary 2.9. *For $n \geq 3$, $a^k(K_n) = \lceil \frac{n}{2} \rceil + k \ \forall \ k \in \{0, 1, ..., \lfloor \frac{n}{2} \rfloor\}$.*
Proof. By Theorem 2.8, $a^k(K_n) \geq \lceil \frac{n}{2} \rceil + k$. Further it can be easily observed that any set of $\lceil \frac{n}{2} \rceil + k$ vertices of K_n is a k-strong defensive alliance.

Theorem 2.10. *Let S be a k-strong defensive alliance and $v \in S$ with $\deg(v) > 1$. Then $Def_S(v) \geq \lfloor \frac{deg(v)}{2} \rfloor + k + 1$.*
Proof. Let $deg(v) = s$ and $N(v) \cap S = \{v_1, v_2, \ldots, v_l\}$. Then $l \geq \lfloor \frac{s}{2} \rfloor$ and $Def_S(v) = l + 1$. Suppose $l < \lfloor \frac{s}{2} \rfloor + k$. Then $1 \leq l - \lfloor \frac{s}{2} \rfloor + 1 \leq k, l$. Let S' be a set obtained by removing $l - \lfloor \frac{s}{2} \rfloor + 1$ adjacent vertices of v from S. Then $Def_{S'}(v) = \lfloor \frac{s}{2} \rfloor$. By Remark 2.1, $Att_{S'}(v) = \lceil \frac{s}{2} \rceil + 1$. Then S' is not a defensive alliance which is a contradiction to the fact that S is a k-strong defensive alliance. Thus $l \geq \lfloor \frac{s}{2} \rfloor + k$ and $Def_S(v) \geq \lfloor \frac{s}{2} \rfloor + k + 1$.

Theorem 2.11. *Let $k > 0$ and G be a graph of order $n \geq 2$. If S is a k-strong defensive alliance in G, then $deg(v) \geq 2k - 1$ for any non pendant vertex $v \in S$.*
Proof. Let $v \in S$ with $deg(v) = s > 1$. Suppose $s < 2k-1$. Then by Theorem 2.10, $|N(v) \cap S| \geq \lfloor \frac{s}{2} \rfloor + k$. Let $N(v) \cap S = \{v_1, v_2, ..., v_l\}$. Then $\lfloor \frac{s}{2} \rfloor + k \leq l \leq s \leq 2k-2$. Since $\lceil \frac{s}{2} \rceil \leq \lceil \frac{2k-2}{2} \rceil = k-1$, we get $k+1 \leq l - \lfloor \frac{s}{2} \rfloor + 1 \leq s - \lfloor \frac{s}{2} \rfloor + 1 = \lceil \frac{s}{2} \rceil + 1 \leq k$ which is a contradiction.

Corollary 2.12. *If S is a k-strong defensive alliance and $v \in S$ is any non pendant vertex, then $|N(v) \cap S| \geq 2k - 1$.*

Theorem 2.13. *Let G be a graph of order $n \geq 2$ and p be the number of pendant vertices in G. For any $k \geq p$, G contains a k-strong defensive alliance only if there exist at least $k - p + 1$ vertices of degree at least $2k - 1$.*
Proof. Let S be a k-strong defensive alliance. Then $|S| \geq k + 1$ and hence by Theorem 2.11, S has at least $k - p + 1$ non pendant vertices with degree at least $2k - 1$.

Theorem 2.14. *Let S be a minimal k-strong defensive alliance that has at most k pendant vertices. Then $\langle S \rangle$ is connected.*
Proof. Let S be a minimal k-strong defensive alliance that contains at most k pendant vertices. By (2) of Remark 2.4, S has a non pendant vertex v. Then by Theorem 2.11, $deg(v) \geq 2k - 1$. Suppose $\langle S \rangle$ is not connected. Let $\langle S_1 \rangle$ be the

component of $\langle S \rangle$ that contains v. By Corollary 2.12, $|N(v) \cap S| \geq 2k - 1$ and hence $|N(v) \cap S_1| \geq 2k - 1$. Thus $|S_1| \geq 2k \geq k + 1$. Since $Att_{S_1}(x) = Att_S(x)$ and $Def_{S_1}(x) = Def_S(x) \ \forall \ x \in S_1$, it follows that S_1 is a k-strong defensive alliance, which contradicts the minimality of S.

Corollary 2.15. *If S is a minimal k-strong defensive alliance which is not connected, then $|S| = k + 1$. The converse holds for all $k > 1$.*
Proof. By Theorem 2.14, S has at least $k + 1$ pendant vertices. Minimality of S proves that $|S| = k + 1$. Conversely, suppose $|S| = k + 1$ where $k > 1$. Then by Lemma 2.6, all the vertices of S are pendant vertices and hence $\langle S \rangle$ is not connected.

Converse part of Corollary 2.15 fails in path P_2 when $k = 1$. For the graph H of Figure 2, $S = \{v, d_1, d_2, d_3, d_4\}$ is a minimal 1-strong defensive alliance with 1 pendant vertex, which shows that there exists a k-strong defensive alliance with at most k pendant vertices. This leads to the following.

Corollary 2.16. *Let S be a minimal k-strong defensive alliance with at most k pendant vertices. If v is a pendant vertex in S and u is the adjacent vertex of v, then $u \in S$.*

3. Bounds for k and $a^k(G)$

Let S be a minimal k-strong defensive alliance. Then there is a subset $\{v_1, v_2, \ldots, v_{k+1}\}$ of S such that $S - \{v_1, v_2, \ldots, v_{k+1}\}$ is either empty or not a defensive alliance.

Theorem 3.1. *Let G be a graph of order $n \geq 2k + 1$. If there is no vertex of degree $2k$ or $2k - 1$, then $a^k(G) \leq n - 1$.*
Proof. Suppose $a^k(G) = n$. Then V is a minimal k-strong defensive alliance in G. Then there exists a set $\{v_1, v_2, \ldots, v_{k+1}\} \subseteq V$ such that $S = V - \{v_1, v_2, \ldots, v_{k+1}\}$ is not a defensive alliance. Then there exists $z \in S$ such that $Att_S(z) > Def_S(z)$. For $1 \leq i \leq k + 1$, each $S_i = S \cup \{v_i\}$ is a defensive alliance, which shows that $zv_i \in E$ for each i. Hence $Def_S(z) < Att_S(z) = k+1$ and $Def_{S_i}(z) \geq Att_{S_i}(z) = k$. Thus $Def_S(z) = k$ or $k - 1$. Then by Remark 2.1, it follows that $deg(z) = 2k$ or $2k - 1$, which completes the proof.

Theorem 3.2. *For any graph G, $a^k(G) = k + 1$ or $a^k(G) \geq 2k$.*
Proof. Suppose $a^k(G) \neq k+1$. By Theorem 2.7, G has at most k pendant vertices. Let S be any minimal k-strong defensive alliance in G. Then by (2) of Remark 2.4, S has a non pendant vertex v. By Corollary 2.12, $|N(v) \cap S| \geq 2k - 1$. Thus $|S| \geq 2k$.

Theorem 3.3. *For any graph G and $k > 1$, $a^k(G) = 2k$ if and only if $G = K_{2k}$.*

Proof. By Corollary 2.9, $a^k(K_{2k}) = 2k$. Conversely, let $a^k(G) = 2k$ and S be a minimal k-strong defensive alliance in G with $|S| = 2k$. Let l be the number of pendant vertices and m be that of non pendant vertices in S respectively. Let $Att_S(x) = t_x \ \forall \ x \in S$.

Claim-1. $l = 0$.

Suppose $l > 0$. By Corollary 2.12, every non pendant vertex in S is adjacent to every other vertex in S. Thus $m \leq 1$. Since S is a minimal k-strong defensive alliance, $l \leq k$. Therefore $|S| = l + m \leq k+1 < 2k$ which is a contradiction. Hence the claim holds.

Claim-2. $t_x = 0 \ \forall \ x \in S$.

Suppose $t_x \geq 1$ for some $x \in S$. By Corollary 2.12, $|N(x) \cap S| \geq k$. Let S' be a set obtained by removing any k vertices of $N(x) \cap S$ from S. Then $Att_{S'}(x) = t_x + k > k = Def_{S'}(x)$ which is a contradiction to the fact that S is a k-strong defensive alliance. Hence the claim holds.

By Claim-1 and Claim-2, $\langle S \rangle = G$ and $m = 2k$. Thus by Corollary 2.12, $\langle S \rangle = K_{2k}$.

The following theorem gives a bound for k.

Theorem 3.4. *Let $G = (V, E)$ be a graph with l pendant vertices. For $k \geq l$, if G has a k-strong defensive alliance, then $k \leq \lfloor \frac{\triangle(G)+1}{2} \rfloor$, where $\triangle(G) = \max_{v \in V} \deg(v)$.*

Proof. Let S be any k-strong defensive alliance in G. Since G has at most k pendant vertices, by (2) of Remark 2.4, S has a non pendant vertex v. By Theorem 2.11, $2k - 1 \leq deg(v)$. Thus $k \leq \lfloor \frac{deg(v)+1}{2} \rfloor$ and hence $k \leq \lfloor \frac{\triangle(G)+1}{2} \rfloor$.

Theorem 3.5. *Let $k > 0$ and G be a graph of order n with $\delta(G) \geq 2k+1$. Then $a^k(G) \leq n - \lfloor \frac{\delta(G)-(2k-1)}{2} \rfloor$.*

Proof. Let $v \in V$ with $deg(v) = \delta(G)$ and $v_1, v_2, ..., v_{\lfloor \frac{\delta(G)-1}{2} \rfloor - k} \in N(v)$. Let $S = V - \{v, v_1, ..., v_{\lfloor \frac{\delta(G)-1}{2} \rfloor - k}\}$. Then $Att_S(x) \leq \lfloor \frac{\delta(G)-1}{2} \rfloor - k + 1$ for each $x \in S$. By Remark 2.1, $Def_S(x) \geq 1 + \delta(G) - \lfloor \frac{\delta(G)-1}{2} \rfloor + k - 1 = \lceil \frac{\delta(G)-1}{2} \rceil + k + 1$. Thus S is a defensive alliance with $|S| \geq k+1$. Let S' be a set obtained by removing any l vertices from S, where $l \leq k$. Then for any $z \in S'$, $Att_{S'}(z) \leq \lfloor \frac{\delta(G)-1}{2} \rfloor - k + 1 + k = \lfloor \frac{\delta(G)-1}{2} \rfloor + 1$. By Remark 2.1, $Def_{S'}(z) \geq \lceil \frac{\delta(G)-1}{2} \rceil + 1$. Therefore S' is a defensive alliance. Thus S is a k-strong defensive alliance and $a^k(G) \leq |S| = n - \lfloor \frac{\delta(G)-(2k-1)}{2} \rfloor$.

For the complete graph K_n, equality holds in Theorem 3.5 when $k = \lfloor \frac{n}{2} \rfloor$.

4. k-Strong Defensive Alliances in Some Classes of Graphs

The join $G_1 \vee G_2$ of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2 \cup \{uv : u \in V_1, v \in V_2\}$. The graph $K_1 \vee C_n$ is referred as wheel on n vertices, denoted by $W_{1,n}$. The vertex corresponding to K_1 is the central vertex and the vertices corresponding to C_n are the rim vertices of $W_{1,n}$. Some results on 1-strong defensive alliance number of $W_{1,n}, P_n, C_n$ are found in [4]. By Theorem 3.4, C_n and P_n do not have k-strong defensive alliances for $k > 1$.

Theorem 4.1. *For any $n \geq 3$, $a^2(W_{1,n}) = n+1$ and $W_{1,n}$ has no k-strong defensive alliance for $k \geq 3$.*

Proof. Let V be vertex set of $W_{1,n}$. Then V is a 2-strong defensive alliance. For any 2-strong defensive alliance S, note that at least one rim vertex lies in S. Let $v \in S$ be a rim vertex. Then by Theorem 2.10, $N(v) \subseteq S$. Hence it follows that $S = V$. Thus $a^2(W_{1,n}) = n + 1$. Let $k \geq 3$. Since V is the only 2-strong defensive alliance in $W_{1,n}$, there can not exist any k-strong defensive alliance other than V. Further V is also not a k-strong defensive alliance, which completes the proof.

Theorem 4.2. *For $m, n > 2$ and $0 \leq k \leq min\{\lceil \frac{m+2}{2} \rceil, \lceil \frac{n+2}{2} \rceil\}$, $a^k(P_m \vee P_n) = \lfloor \frac{m+2}{2} \rfloor + \lfloor \frac{n+2}{2} \rfloor + 2k - 4$.*

Proof. Let S be a k-strong defensive alliance. Let $U = \{u_i : 1 \leq i \leq m\}$ and $W = \{v_j : 1 \leq j \leq n\}$ be vertex sets of P_m and P_n such that u_i is adjacent to u_{i+1} for $1 \leq i \leq m-1$ and v_j is adjacent to v_{j+1} for $1 \leq j \leq n-1$. Note that no subset of U and W is a k-strong defensive alliance in $P_m \vee P_n$. Thus $S \cap U \neq \emptyset$ and $S \cap W \neq \emptyset$. Then by Theorem 2.10, it follows that $|S \cap U| \geq \lfloor \frac{m+2}{2} \rfloor + k - 2$ and $|S \cap W| \geq \lfloor \frac{n+2}{2} \rfloor + k - 2$. Thus $a^k(P_m + P_n) = \lfloor \frac{m+2}{2} \rfloor + \lfloor \frac{n+2}{2} \rfloor + 2k - 4$. Equality holds by noting that the set $\{u_1, \ldots, u_{\lfloor \frac{m+2}{2} \rfloor + k - 2}, v_1, \ldots, v_{\lfloor \frac{n+2}{2} \rfloor + k - 2}\}$ is a k-strong defensive alliance.

The following theorem can be proved similarly as above by using Theorem 2.10.

Theorem 4.3. *Let $m \geq 1$, $n > 1$ be any integers. Then*

1. $a^k(K_{m,n}) = \lfloor \frac{m}{2} \rfloor + \lfloor \frac{n}{2} \rfloor + 2k$ *for $m > 1$ and $k \leq min\{\lceil \frac{m}{2} \rceil, \lceil \frac{n}{2} \rceil\}$,*

2. $a^k(C_m \vee \overline{K_n}) = \lfloor \frac{n+2}{2} \rfloor + \lfloor \frac{m}{2} \rfloor + 2k - 2$ *for $m \geq 3$, $0 \leq k \leq \min\{\lceil \frac{n+2}{2} \rceil, \lceil \frac{m}{2} \rceil\}$.*

3. $a^k(P_m \vee \overline{K_n}) = \lfloor \frac{m}{2} \rfloor + \lfloor \frac{n+2}{2} \rfloor + 2k - 2$, *$m \geq 3, 0 \leq k \leq \min\{\lceil \frac{m}{2} \rceil, \lceil \frac{n+2}{2} \rceil\}$.*

4. $a^k(P_2 \vee \overline{K_n}) = \lfloor \frac{n+1}{2} \rfloor + 2k$ *for $n > 1$ and $0 \leq k \leq 1$.*

The Cartesian product $G_1 \times G_2$ of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph with vertex set $V_1 \times V_2$ and edge set $\{(u_i, v_i)(u_j, v_j) : u_i = u_j$ and $v_i v_j \in E_2$, or $v_i = v_j$ and $u_i u_j \in E_2\}$. The graphs of the form $P_m \times P_n$, $P_m \times C_n$ and $C_m \times C_n$ are called grid-like graphs.

Lemma 4.4. *A 1-strong defensive alliance in $P_m \times P_n$ contains a vertex of degree*

3.

Proof. Let $u_1, \ldots, u_m$ be vertices of P_m and $v_1, \ldots, v_n$ be vertices of P_n such that u_i is adjacent to u_{i+1} and v_j is adjacent to v_{j+1} for $1 \leq i \leq m-1$ and $1 \leq j \leq n-1$. Choose $v \in S$. Suppose $\deg(v) \neq 3$, then we have following cases.

Case-1. $\deg(v) = 2$. By Theorem 2.10, $N(v) \subseteq S$. Note that there is an adjacent vertex of v with degree 3.

Case-2. $\deg(v) = 4$. Then $v = (u_i, v_j)$ for some i, j with $1 < i < m$ and $1 < j < n$. By Theorem 2.10, $|N(v) \cap S| \geq 3$. Then either both (u_i, v_{j+1}) and (u_i, v_{j-1}) lie in S or both (u_{i+1}, v_j) and (u_{i-1}, v_j) lie in S. Since i and j vary over a finite set, it follows that at least one vertex of the form (u_t, v_n) or (u_t, v_1) or (u_1, v_t) or (u_n, v_t) lies in S. The vertices of this form have degree 3.

Remark 4.5. *In Case-2 of Lemma 4.4, note that S intersects every row or every column of $P_m \times P_n$ such that at least two vertices of every column or every row belong to S. Thus $|S| \geq min\{2m, 2n\}$.*

Theorem 4.6. *For integers $m, n > 1$, $a^1(P_m \times P_n) = \min\{2m, 2n\}$. Further $P_m \times P_n$ does not contain k-strong defensive alliance for $k \geq 2$.*

Proof. Let $u_1, u_2, \ldots, u_m$ be the vertices of P_m with u_i adjacent to u_{i+1} for $1 \leq i \leq m-1$ and $v_1, v_2, \ldots, v_n$ be the vertices of P_n with v_j adjacent to v_{j+1} for $1 \leq j \leq n-1$. Let S be a 1-strong defensive alliance in $P_m \times P_n$. Suppose $2 \leq \deg(v) \leq 3\ \forall\ v \in S$. By Theorem 2.10, $|N(v) \cap S| \geq 2\ \forall\ v \in S$. Then it follows that $S = \{(u_1, v_j), (u_m, v_j), (u_i, v_1), (u_i, v_n) : 1 \leq i \leq m, 2 \leq j \leq n-1\}$. Then $|S| = 2m + 2n - 4$. Suppose there is a vertex $u \in S$ with $deg(u) = 4$. Then by Theorem 2.10, $|N(u) \cap S| \geq 3$. Then by Remark 4.5, $|S| \geq \min\{2m, 2n\}$. Since $S_1 = \{(u_i, v_1), (u_i, v_2) : 1 \leq i \leq m\}$ and $S_2 = \{(u_1, v_j), (u_2, v_j) : 1 \leq j \leq n\}$ are 1-strong defensive alliances, we get $a^1(P_m \times P_n) = \min\{2m, 2n, 2m + 2n - 4\} = min\{2m, 2n\}$. For $k \geq 2$, suppose S is a k-strong defensive alliance. Since S is also a 1-strong defensive alliance, by Lemma 4.4, it contains a vertex of degree 3. Let $v \in S$ with $deg(v) = 3$. Then by Theorem 2.10, $N(v) \subseteq S$. By repeating this argument finitely many times, it follows that S contains a vertex of degree 2, which contradicts Theorem 2.11.

Theorem 4.7. *For any integers $m \geq 3$ and $n \geq 2$, $a^1(C_m \times P_n) = \min\{m, 2n\}$.*

Proof. Let $u_1, u_2, \ldots, u_m$ be the vertices of C_m with u_i adjacent to u_{i+1} for $1 \leq i \leq m-1$, u_m adjacent to u_1 and $v_1, v_2, \ldots, v_n$ be the vertices of P_n with v_j adjacent to v_{j+1} for $1 \leq j \leq n$. Let S be a 1-strong defensive alliance. If S does not contain any vertex of degree 4, then by Theorem 2.10, S must be one of the following.

1. $S = \{(u_1, v_j), (u_m, v_j), (u_i, v_1), (u_i, v_n) : 1 \leq i \leq m, 2 \leq j \leq n-1\}$.

2. $S = \{(u_i, v_1) : 1 \leq i \leq m\}$.
3. $S = \{(u_i, v_n) : 1 \leq i \leq m\}$.

Then $|S| \geq m$. Suppose S has a vertex of degree 4. Then similar to Remark 4.5, we get $|S| \geq \min\{2m, 2n\}$. Thus $a^1(C_m \times P_n) \geq \min\{m, 2m, 2n\} = min\{m, 2n\}$. To achieve the equality, note that the sets $S_1 = \{(u_i, v_1) : 1 \leq i \leq m\}$ and $S_2 = \{(u_1, v_j), (u_2, v_j) : 1 \leq j \leq n\}$ are 1-strong defensive alliances.

The following theorem can be proved similarly.

Theorem 4.8. *For integers $m, n \geq 3$, $a^1(C_m \times C_n) = \min\{2m, 2n\}$.*

Note that every vertex in $C_m \times C_n$ and $C_m \times P_n$ is of degree at most 4. Then the proofs of following results are similar to that of Theorem 4.1.

Theorem 4.9. *For $m, n \geq 3$, $a^2(C_m \times P_n) = mn$. Further there is no k-strong defensive alliance in $C_m \times P_n$ for $k \geq 3$.*

Theorem 4.10. *For $m \geq 3$ and $n > 1$, $a^2(C_m \times C_n) = mn$. Further there is no k-strong defensive alliance in $C_m \times C_n$ for $k \geq 3$.*

Figure 3, Figure 4 and Figure 5 illustrate minimal 1-strong defensive alliances in grid like graphs. The vertices of both S_1 and S_2 are minimal 1-strong defensive alliances in each of the graphs of Figure 3, Figure 4 and Figure 5.

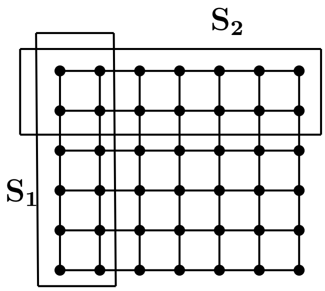

Figure 3: $P_6 \times P_7$

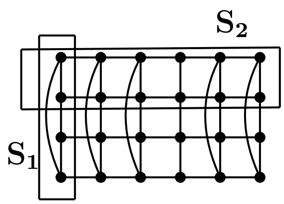

Figure 4: $C_4 \times P_6$

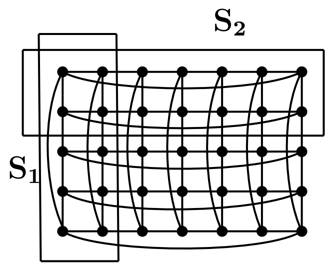

Figure 5: $C_5 \times C_7$

5. Conclusion

In this paper, the existence and some properties of k-strong defensive alliances are discussed. The k-strong defensive alliance numbers are determined for some classes of graphs. One may think to determine $A^k(G), \gamma_{a^k}(G), \gamma_{A^k}(G)$ for some classes of graphs.

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