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GENERAL RANDIĆ ENERGY OF SOME GRAPHS

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Abstract: For a graph G of order n, the general Randić matrix $GR(G) = [g_{ij}]$ is a symmetric matrix of order n in which $g_{ij} = (d_i d_j)^\alpha$, $\alpha \in \mathbb{R}$ if the vertices v_i and v_j are adjacent in G and 0, otherwise, where d_i is the degree of vertex v_i. The general Randić energy $E_{GR}(G)$ of G is the sum of the absolute values of the eigenvalues of $GR(G)$. In this paper, we compute the general Randić energy of the line graph of regular graph and the graph obtained by duplication of graph elements for regular graph. We also investigate general Randić equienergetic graphs.

Keywords and Phrases: General Randić matrix, General Randić eigenvalues, General Randić energy, General Randić equienergetic graphs.

2020 Mathematics Subject Classification: 05C50, 05C76.

1. Introduction

We begin with a simple connected graph G with the vertex set $V(G)$ and the edge set $E(G)$. Let d_i be the degree of a vertex v_i, for $i = 1, 2, ..., n$. The adjacency matrix $A(G) = [a_{ij}]$ of a graph G is a square matrix of order n, where

$$a_{ij} = \begin{cases} 1 & ;\ \text{if vertices } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & ;\ \text{otherwise} \end{cases}$$

Let $\lambda_1, \lambda_2, ..., \lambda_n$ are eigenvalues of $A(G)$, then they all real numbers with their sum is zero as $A(G)$ is a symmetric matrix. The set of eigenvalues with their multiplicities is known as spectrum of a graph and it is denoted by $Spec(G)$. The concept of graph energy was introduced by Gutman [8]. According to him energy of graph $\mathcal{E}(G)$ is defined as

$$\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i|$$

The number of positive, negative and zero eigenvalues of graph G be denoted by n^+, n^- and n^0 respectively. A brief account of spectra of graph and graph energy can be found in Balakrishnan [2], Cvetković *et al.* [6] and Li *et al.* [15].

In 1975, Randić [20] has defined Randić index as

$$R = \sum_{i \sim j} (d_i d_j)^{-\frac{1}{2}},$$

where the summation is taken over all pairs of adjacent vertices v_i and v_j. A brief account on Randić index can be found in [9, 13, 14, 21].

In 1998, Bollobás and Erdös [3] have generalized the concept of Randić index by replacing $-\frac{1}{2}$ power with any real number and named it as general Randić index which is denoted and defined as

$$R_\alpha = R_\alpha(G) = \sum_{i \sim j} (d_i d_j)^\alpha, \alpha \in \mathbb{R}$$

In 2010, Bozkurt *et al.* [4, 5] pointed out that the Randić index is purposeful to produce a graph matrix of order n named as Randić matrix $R(G) = [r_{ij}]$, where

$$r_{ij} = \begin{cases} (d_i d_j)^{-\frac{1}{2}} & ; \text{ if } v_i \text{ and } v_j \text{ are adjacent,} \\ 0 & ; \text{ otherwise} \end{cases}$$

The connection between the Randić matrix and the Randić index is obvious: The sum of all elements of $R(G)$ is equal to $2R$.

Let $R(G)$ be the Randić matrix with $\mu_1, \mu_2, ..., \mu_n$ are eigenvalues of matrix $R(G)$ then the Randić energy [4, 5] is defined as the sum of absolute values of Randić eigenvalues of graph G which is denoted as $RE(G)$. That is,

$$RE = RE(G) = \sum_{i=1}^{n} |\mu_i|$$

A detailed discussion on the Randić energy can be found in [1, 10, 22].

In [7], Gu *et al.* have introduced a concept of general Randić matrix and general Randić energy. The general Randić matrix $GR(G)$ of a graph G is a square matrix which is defined by $GR(G) = [g_{ij}]$, where

$$g_{ij} = \begin{cases} (d_i d_j)^\alpha & ; \text{ if vertices } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & ; \text{ otherwise} \end{cases} \text{, where } \alpha \in \mathbb{R}$$

For $\alpha = -\frac{1}{2}$, the above matrix reduces to Randić matrix, for $\alpha = 1$, it reduces to second Zagreb matrix [17] and for $\alpha = 0$ it reduces to adjacency matrix. The general Randić energy is defined as the sum of absolute values of eigenvalues of $GR(G)$.

$$E_{GR}(G) = \sum_{i=1}^{n} |\mu_i|$$

where μ_i's are eigen values of the general Randić matrix of graph G. The general Randić polynomial and general Randić energy of some standard graph families computed by Ramane and Gudodagi [19]. Some more results related to bounds on general Randić energy can be found in [7, 16].

Two non-isomorphic graphs G_1 and G_2 of the same order are said to be equienergetic graphs if $\mathcal{E}(G_1) = \mathcal{E}(G_2)$. Ramane *et al.* [18] have constructed infinitely many pairs of equienergetic graphs. In the context of equienergetic graphs, we define general Randić equienergetic graphs in which two non-isomorphic graphs are said to be general Randić equienergetic if they have same general Randić energy.

In this paper, we are going to prove some results on general Randić energy using some graph operations and also find some families of general Randić equienergetic graphs.

Proposition 1.1. [12] *Let M, N, P, $Q \in \mathbb{R}^{n\times n}$ be matrices, Q is invertible and*

$$S = \begin{bmatrix} M & N \\ P & Q \end{bmatrix}$$

then, $det(S) = det(Q) \cdot det[M - NQ^{-1}P]$

2. General Randić Energy of Line Graph

Definition 2.1. [11] *The line graph $L(G)$ of graph G has the edges of G as its vertices which are adjacent in $L(G)$ if and only if the corresponding edges are adjacent in G. Also, $L^2(G) = L(L(G))$.*

The following result relates the general Randić energy of graph G and its line graph.

Theorem 2.2. *Let G be an r-regular graph and $\lambda_1, \lambda_2, ..., \lambda_n$ are eigenvalues of G with no eigenvalues between $(2-r, r-2)$ except zero for each $i = 1, 2, ..., n$, then*

$$E_{GR}(L(G)) = (2r-2)^{2\alpha}\left[\mathcal{E}(G) + (r-2)(n^+ + n^0 - n^- + n)\right]$$

Proof. Let G be a r-regular simple and connected graph then its line graph $L(G)$ is a graph on $\frac{nr}{2}$ vertices and degree of each vertex is $2r-2$. Therefore, general Randić matrix of $L(G)$ is given by

$$GR(L(G)) = (2r-2)^{2\alpha} A(L(G))$$

where $A(L(G))$ is adjacency matrix of graph $L(G)$. Since, we know that [23], if $\lambda_1, \lambda_2, ..., \lambda_n$ are the eigenvalues of a r-regular graph G, then the eigenvalues of $L(G)$ are $\lambda_i + r - 2, i = 1, 2, 3, ..., n$ and -2, $\frac{n(r-2)}{2}$ times.

Therefore, $Spec(GR(L(G))) = \begin{pmatrix} (2r-2)^{2\alpha}(\lambda_i + r - 2) & -2(2r-2)^{2\alpha} \\ 1 & \frac{n(r-2)}{2} \end{pmatrix}$.

Also, $|\lambda_i| \geq (r-2)$, except zero for each $i = 1, 2, ..., n$, then

$$|\lambda_i + r - 2| = \begin{cases} |\lambda_i| + r - 2 & ; \text{if } \lambda_i > 0 \\ |\lambda_i| - r + 2 & ; \text{if } \lambda_i < 0 \\ r - 2 & ; \text{if } \lambda_i = 0 \end{cases}$$

Hence, $E_{GR}(L(G))$

$$= (2r-2)^{2\alpha}\sum_{i=1}^{n}|\lambda_i + r - 2| + (2r-2)^{2\alpha}\sum_{i=1}^{\frac{n(r-2)}{2}}|-2|$$

$$= (2r-2)^{2\alpha}\left[\sum_{\lambda>0}(|\lambda_i| + r - 2) + \sum_{\lambda<0}(|\lambda_i| - r + 2) + \sum_{\lambda=0}(r-2) + n(r-2)\right]$$

$$= (2r-2)^{2\alpha}\left[\sum_{\lambda>0}|\lambda_i| + \sum_{\lambda<0}|\lambda_i| + (r-2)\left\{\sum_{\lambda_i>0}1 + \sum_{\lambda_i=0}1 - \sum_{\lambda_i<0}1\right\} + n(r-2)\right]$$

$$= (2r-2)^{2\alpha}\left[\mathcal{E}(G) + (r-2)(n^+ + n^0 - n^- + n)\right]$$

Corollary 2.3. *Let G_1 and G_2 be any 2-regular equienergetic graphs, then $L(G_1)$ and $L(G_2)$ are general Randić equienergetic graphs.*

Proof. Let G_1 and G_2 be any 2-regular equienergetic graphs, then from Theorem 2.2, $E_{GR}(L(G_i)) = 2^{2\alpha}\mathcal{E}(G_i)$, for $i = 1, 2$. Hence, $L(G_1)$ and $L(G_2)$ are general

Randić equienergetic graphs.

Corollary 2.4. *Let G_1 and G_2 be any r-regular equienergetic graphs of same order with no eigenvalues lies in $(2-r, r-2)$ except zero. Let n_1^+, n_1^0, n_1^- and n_2^+, n_2^0, n_2^- are number of positive, zero and negative eigenvalues of G_1 and G_2 respectively, then $L(G_1)$ and $L(G_2)$ are general Randić equienergetic graphs if and only if $n_1^+ + n_1^0 - n_1^- = n_2^+ + n_2^0 - n_2^-$.*
Proof. Follows from Theorem 2.2.

Theorem 2.5. *Let G be an r-regular graph and $\lambda_1, \lambda_2, ..., \lambda_n$ are eigenvalues of G with no eigenvalues lies in $(6-3r, 3r-6)$ except zero, then*

$$E_{GR}(L^2(G)) = (4r-6)^{2\alpha}\left[\mathcal{E}(G) + (3r-6)(n^+ + n^0 - n^-) + n(r-2)^2\right]$$

Proof. Let G be an r-regular, simple and connected graph, then its iterated line graph $L^2(G)$ is a graph on $\frac{nr(r-1)}{2}$ vertices and degree of each vertex is $4r-6$. Therefore, general Randić matrix of $L^2(G)$ can be written as

$$GR(L^2(G)) = (4r-6)^{2\alpha}A(L^2(G))$$

where $A(L^2(G))$ is adjacency matrix of graph $L^2(G)$. Since, we know that [18], if $\lambda_1, \lambda_2, ..., \lambda_n$ are the eigenvalues of a r-regular graph G, then the eigenvalues of $L^2(G)$ are $\lambda_i + 3r - 6, i = 1, 2, 3, ..., n$, $2r-6$, $\frac{n(r-2)}{2}$ times and -2, $\frac{n(r-2)}{2}$ times. Therefore, $Spec(GR(L^2(G)))$

$$= \begin{pmatrix} (4r-6)^{2\alpha}(\lambda_i + 3r - 6) & (4r-6)^{2\alpha}(2r-6) & (4r-6)^{2\alpha}(-2) \\ 1 & \frac{n(r-2)}{2} & \frac{n(r-2)}{2} \end{pmatrix}$$

So, $E_{GR}(L^2(G))$

$$= \sum_{i=1}^{n} |(4r-6)^{2\alpha}(\lambda_i + 3r - 6)| + \sum_{i=1}^{\frac{n(r-2)}{2}} \left|(4r-6)^{2\alpha}(2r-6)\right| + \sum_{i=1}^{\frac{n(r-2)}{2}} \left|(4r-6)^{2\alpha}(-2)\right|$$

Also, $|\lambda_i| \geq (3r-6)$, except zero for each $i = 1, 2, ..., n$. So,

$$|\lambda_i + 3r - 6| = \begin{cases} |\lambda_i| + 3r - 6 & ; \text{if } \lambda_i > 0 \\ |\lambda_i| - 3r + 6 & ; \text{if } \lambda_i < 0 \\ 3r - 6 & ; \text{if } \lambda_i = 0 \end{cases}$$

Hence, $E_{GR}(L^2(G))$

$$= (4r-6)^{2\alpha}\left[\sum_{\lambda_i>0}(|\lambda_i|+3r-6)+\sum_{\lambda_i<0}(|\lambda_i|-3r+6)+\sum_{\lambda_i=0}(3r-6)\right.$$
$$\left.+\frac{n(r-2)}{2}(2r-4)\right]$$
$$= (4r-6)^{2\alpha}\left[\sum_{\lambda_i>0}|\lambda_i|+\sum_{\lambda_i<0}|\lambda_i|+(3r-6)\left(\sum_{\lambda_i>0}1+\sum_{\lambda_i=0}1-\sum_{\lambda_i<0}1\right)+n(r-2)^2\right]$$
$$= (4r-6)^{2\alpha}\left[\mathcal{E}(G)+(3r-6)(n^+ + n^0 - n^-)+n(r-2)^2\right]$$

Corollary 2.6. *Let G_1 and G_2 be any 2-regular equienergetic graphs, then $L^2(G_1)$ and $L^2(G_2)$ are general Randić equienergetic graphs.*
Proof. Let G_1 and G_2 be any 2-regular equienergetic graphs, then from Theorem 2.5, $E_{GR}(L^2(G_i)) = 2^{2\alpha}\mathcal{E}(G_i)$, for $i = 1, 2$. Hence, $L^2(G_1)$ and $L^2(G_2)$ are general Randić equienergetic graphs.

Corollary 2.7. *Let G_1 and G_2 be any r-regular equienergetic graphs of same order with no eigenvalues lies in $(6-3r, 3r-6)$ except zero. Let n_1^+, n_1^0, n_1^- and n_2^+, n_2^0, n_2^- are number of positive, zero and negative eigenvalues of G_1 and G_2 respectively, then $L^2(G_1)$ and $L^2(G_2)$ are general Randić equienergetic graphs if and only if $n_1^+ + n_1^0 - n_1^- = n_2^+ + n_2^0 - n_2^-$.*
Proof. Follows from Theorem 2.5.

3. General Randić Energy of Graphs Obtained by Duplication of Graph Elements

Definition 3.1. [24] *Let G be a graph with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$, then duplication of a vertex v_k by a new edge $e = v'v''$ in a graph G produces a new graph G_1 such that $N(v') = \{v_k, v''\}$ and $N(v'') = \{v_k, v'\}$.*

Theorem 3.2. *Let G be an r-regular graph with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ and G_1 be the graph obtained from G by duplicating each vertex of G by a new edge, then $E_{GR}(G_1)$*

$$= 2^{2\alpha}n+\sum_{\lambda_i\le 2}\sqrt{[(r+2)^{2\alpha}\lambda_i+2^{2\alpha}]^2+2^{2\alpha+2}(r+2)^{2\alpha}(2-\lambda_i)}+\sum_{\lambda_i>2}[(r+2)^{2\alpha}\lambda_i+2^{2\alpha}]$$

Proof. Let G be an r-regular graph with $v_1, v_2, \cdots, v_n$ are vertices of a graph G. Now, we duplicate all the vertices of the given graph together by the edges

$e_1, e_2, \cdots, e_n$ respectively, such that $e_1 = v_1'v_1''$, $e_2 = v_2'v_2''$, $\cdots$, $e_n = v_n'v_n''$ to obtain graph G_1, then the general Randić matrix of G_1 is given by

$$GR(G_1) = \begin{bmatrix} (r+2)^{2\alpha}A(G) & B_{n\times 2n} \\ B^T_{2n\times n} & I_n \otimes 2^{2\alpha}A(K_2) \end{bmatrix}$$

where $A(G)$ is the adjacency matrix of given graph G and B

$$= \begin{bmatrix} [2(r+2)]^\alpha & [2(r+2)]^\alpha & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & [2(r+2)]^\alpha & [2(r+2)]^\alpha & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & [2(r+2)]^\alpha & [2(r+2)]^\alpha \end{bmatrix}$$

The characteristic polynomial of $GR(G_1)$ is given by $\phi(G_1 : x)$

$$\begin{aligned}
&= |xI_{3n} - GR(G_1)| \\
&= \begin{vmatrix} xI_n - (r+2)^{2\alpha}A(G) & B \\ B^T & I_n \otimes (xI_2 - 2^{2\alpha}A(K_2)) \end{vmatrix} \\
&= |I_n \otimes (xI_2 - 2^{2\alpha}A(K_2))|\ |xI_n - (r+2)^{2\alpha}A(G) - B(I_n \otimes (xI_2 - 2^{2\alpha}A(K_2)))^{-1}B^T| \\
&= (x^2 - 2^{4\alpha})^n |xI_n - (r+2)^{2\alpha}A(G) - B((xI_2 - 2^{2\alpha}A(K_2))^{-1} \otimes I_n^{-1})B^T| \\
&= (x^2 - 2^{4\alpha})^n \left| xI_n - (r+2)^{2\alpha}A(G) - B\left(\frac{1}{x^2 - 2^{4\alpha}}(xI_2 + 2^{2\alpha}A(K_2)) \otimes I_n\right) B^T \right| \\
&= |(x^2 - 2^{4\alpha})(xI_n - (r+2)^{2\alpha}A(G)) - B((xI_2 + 2^{2\alpha}A(K_2)) \otimes I_n)B^T|
\end{aligned}$$

Take $\gamma = [2(r+2)]^\alpha$ then, $B((xI_2 + 2^{2\alpha}A(K_2)) \otimes I_n)B^T$

$$= \begin{bmatrix} \gamma & \gamma & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \gamma & \gamma & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \gamma & \gamma \end{bmatrix} \begin{bmatrix} x & 2^{2\alpha} & 0 & 0 & \cdots & 0 & 0 \\ 2^{2\alpha} & x & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & x & 2^{2\alpha} & \cdots & 0 & 0 \\ 0 & 0 & 2^{2\alpha} & x & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & x & 2^{2\alpha} \\ 0 & 0 & 0 & 0 & \cdots & 2^{2\alpha} & x \end{bmatrix} \begin{bmatrix} \gamma & 0 & \cdots & 0 \\ \gamma & 0 & \cdots & 0 \\ 0 & \gamma & \cdots & 0 \\ 0 & \gamma & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \gamma \\ 0 & 0 & \cdots & \gamma \end{bmatrix}$$

$$= \begin{bmatrix} (x+2^{2\alpha})\gamma & (x+2^{2\alpha})\gamma & & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \\ \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & \cdots & (x+2^{2\alpha})\gamma & (x+2^{2\alpha})\gamma & \end{bmatrix} \begin{bmatrix} \gamma & 0 & \cdots & 0 \\ \gamma & 0 & \cdots & 0 \\ 0 & \gamma & \cdots & 0 \\ 0 & \gamma & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \gamma \\ 0 & 0 & \cdots & \gamma \end{bmatrix}$$

$$= \begin{bmatrix} 2(x+2^{2\alpha})\gamma^2 & 0 & 0 & \cdots & 0 \\ 0 & 2(x+2^{2\alpha})\gamma^2 & 0 & \cdots & 0 \\ 0 & 0 & 2(x+2^{2\alpha})\gamma^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2(x+2^{2\alpha})\gamma^2 \end{bmatrix}$$

$$= 2(x+2^{2\alpha})[2(r+2)]^{2\alpha} I_n$$

by continuing proof of theorem

$$\begin{aligned} \phi(G_1 : x) &= |(x^2 - 2^{4\alpha})(xI_n - (r+2)^{2\alpha}A(G)) - 2(x+2^{2\alpha})[2(r+2)]^{2\alpha} I_n| \\ &= (x+2^{2\alpha})^n |(x-2^{2\alpha})(xI_n - (r+2)^{2\alpha}A(G)) - 2^{2\alpha+1}(r+2)^{2\alpha} I_n| \end{aligned}$$

If $\lambda_1, \lambda_2, ..., \lambda_n$ are eigenvalues of $A(G)$, then

$$\phi(G_1 : x) = (x+2^{2\alpha})^n \prod_{i=1}^{n} \left[x^2 - x((r+2)^{2\alpha}\lambda_i + 2^{2\alpha}) - 2^{2\alpha}(r+2)^{2\alpha}(2-\lambda_i)\right]$$

So, the roots of above characteristic polynomial are given by

$$x = -2^{2\alpha}(n\,\text{times}),$$

$$x = \frac{[(r+2)^{2\alpha}\lambda_i + 2^{2\alpha}] \pm \sqrt{[(r+2)^{2\alpha}\lambda_i + 2^{2\alpha}]^2 + 2^{2\alpha+2}(r+2)^{2\alpha}(2-\lambda_i)}}{2}$$

for each $i = 1, 2, \cdots, n$. Now, we have two possibilities for the calculation of positive eigenvalues,

$$[(r+2)^{2\alpha}\lambda_i + 2^{2\alpha}] \le \sqrt{[(r+2)^{2\alpha}\lambda_i + 2^{2\alpha}]^2 + 2^{2\alpha+2}(r+2)^{2\alpha}(2-\lambda_i)}\ ;\ \text{if } \lambda_i \le 2$$
$$[(r+2)^{2\alpha}\lambda_i + 2^{2\alpha}] > \sqrt{[(r+2)^{2\alpha}\lambda_i + 2^{2\alpha}]^2 + 2^{2\alpha+2}(r+2)^{2\alpha}(2-\lambda_i)}\ ;\ \text{if } \lambda_i > 2$$

Hence, $E_{GR}(G_1)$

$$= \sum_{i=1}^{3n} |\mu_i| = \sum_{i=1}^{n} |-2^{2\alpha}|$$

$$+ \sum_{i=1}^{n} \left| \frac{[(r+2)^{2\alpha}\lambda_i + 2^{2\alpha}] \pm \sqrt{[(r+2)^{2\alpha}\lambda_i + 2^{2\alpha}]^2 + 2^{2\alpha+2}(r+2)^{2\alpha}(2-\lambda_i)}}{2} \right|$$

$$= 2^{2\alpha}n + \sum_{\lambda_i \leq 2} \left[\frac{[(r+2)^{2\alpha}\lambda_i + 2^{2\alpha}] + \sqrt{[(r+2)^{2\alpha}\lambda_i + 2^{2\alpha}]^2 + 2^{2\alpha+2}(r+2)^{2\alpha}(2-\lambda_i)}}{2} \right.$$

$$\left. + \frac{\sqrt{[(r+2)^{2\alpha}\lambda_i + 2^{2\alpha}]^2 + 2^{2\alpha+2}(r+2)^{2\alpha}(2-\lambda_i)} - [(r+2)^{2\alpha}\lambda_i + 2^{2\alpha}]}{2} \right]$$

$$+ \sum_{\lambda_i > 2} \left[\frac{[(r+2)^{2\alpha}\lambda_i + 2^{2\alpha}] + \sqrt{[(r+2)^{2\alpha}\lambda_i + 2^{2\alpha}]^2 + 2^{2\alpha+2}(r+2)^{2\alpha}(2-\lambda_i)}}{2} \right.$$

$$\left. + \frac{[(r+2)^{2\alpha}\lambda_i + 2^{2\alpha}] - \sqrt{[(r+2)^{2\alpha}\lambda_i + 2^{2\alpha}]^2 + 2^{2\alpha+2}(r+2)^{2\alpha}(2-\lambda_i)}}{2} \right]$$

$$= 2^{2\alpha}n + \sum_{\lambda_i \leq 2} \sqrt{[(r+2)^{2\alpha}\lambda_i + 2^{2\alpha}]^2 + 2^{2\alpha+2}(r+2)^{2\alpha}(2-\lambda_i)} + \sum_{\lambda_i > 2} [(r+2)^{2\alpha}\lambda_i + 2^{2\alpha}]$$

Definition 3.3. [24] *Let G be a graph with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$, then duplication of an edge $e = v_iv_j$ by a vertex v' in a graph G produces a new graph G_1 such that $N(v') = \{v_i, v_j\}$.*

Theorem 3.4. *Let G be an r-regular graph order n and size m with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ and G_1 be the graph obtained from G by duplicating each edge of G by a new vertex, then*

$$E_{GR}(G_1) = \sum_{i=1}^{n} \sqrt{(2r)^{4\alpha}\lambda_i^2 + 4(4r)^{2\alpha}(\lambda_i + r)}$$

Proof. Let G be an r-regular graph with $v_1, v_2, \cdots, v_n$ be the vertices and $e_1, e_2, \cdots, e_m$ be the edges of G. Now, duplicate all the edges $e_1, e_2, \cdots, e_m$ by considering the new vertices $e'_1, e'_2, \cdots, e'_m$ respectively to obtained a graph G_1. The general Randić matrix of G_1 is given by

$$GR(G_1) = \begin{bmatrix} (2r)^{2\alpha}A(G) & (4r)^{\alpha}X(G) \\ (4r)^{\alpha}X(G)^T & O_n \end{bmatrix}$$

where $A(G)$ and $X(G)$ are adjacency matrix and incidence matrix of given graph G respectively. The characteristic polynomial of above matrix is given by

$$\begin{aligned}
\phi(G_1 : x) &= |xI_{n+m} - GR(G_1)| \\
&= \begin{vmatrix} xI_n - (2r)^{2\alpha}A(G) & (4r)^{\alpha}X(G) \\ (4r)^{\alpha}X(G)^T & xI_m \end{vmatrix} \\
&= |xI_m|\,|xI_n - (2r)^{2\alpha}A(G) - (4r)^{\alpha}X(G)(xI_m)^{-1}(4r)^{\alpha}X(G)^T| \\
&= x^m \left| xI_n - (2r)^{2\alpha}A(G) - \frac{1}{x}(4r)^{2\alpha}X(G)X(G)^T \right| \\
&= x^{m-n}\left| x^2I_n - x(2r)^{2\alpha}A(G) - (4r)^{2\alpha}(A + rI_n)\right|
\end{aligned}$$

Now, if $\lambda_1, \lambda_2, \cdots, \lambda_n$ are eigenvalues of $A(G)$, then characteristic polynomial is given by

$$\phi(G_1 : x) = x^{m-n}\prod_{i=1}^{n}\left(x^2 - x(2r)^{2\alpha}\lambda_i - (4r)^{2\alpha}(\lambda_i + r)\right)$$

The roots of above characteristic polynomial are given by

$$x = 0(m - n\,\text{times}),\ x = \frac{(2r)^{2\alpha}\lambda_i \pm \sqrt{(2r)^{4\alpha}\lambda_i^2 + 4\cdot(4r)^{2\alpha}(\lambda_i + r)}}{2}$$

For each $i = 1, 2, \cdots, n$. Also, $(2r)^{2\alpha}\lambda_i \leq \sqrt{(2r)^{4\alpha}\lambda_i^2 + 4\cdot(4r)^{2\alpha}(\lambda_i + r)}$. Hence,

$$\begin{aligned}
E_{GR}(G_1) &= \sum_{i=1}^{n}\left|\frac{(2r)^{2\alpha}\lambda_i \pm \sqrt{(2r)^{4\alpha}\lambda_i^2 + 4\cdot(4r)^{2\alpha}(\lambda_i + r)}}{2}\right| \\
&= \sum_{i=1}^{n}\left(\frac{(2r)^{2\alpha}\lambda_i + \sqrt{(2r)^{4\alpha}\lambda_i^2 + 4\cdot(4r)^{2\alpha}(\lambda_i + r)}}{2}\right. \\
&\qquad\qquad \left. + \frac{\sqrt{(2r)^{4\alpha}\lambda_i^2 + 4\cdot(4r)^{2\alpha}(\lambda_i + r)} - (2r)^{2\alpha}\lambda_i}{2}\right) \\
&= \sum_{i=1}^{n}\sqrt{(2r)^{4\alpha}\lambda_i^2 + 4\cdot(4r)^{2\alpha}(\lambda_i + r)}
\end{aligned}$$

6. Conclusion

We have obtained the general Randić energy of various regular graphs. Some new classes of general Randić equienergetic graphs are also investigated.

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