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COMMON FIXED POINT RESULTS FOR FOUR SELF - MAPS SATISFYING CONTRACTIVE INEQUALITY OF INTEGRAL TYPE IN METRIC SPACES

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Abstract: This manuscript consists a common fixed point result for four weakly compatible self-maps $\hat{P}, \hat{Q}, \hat{S}, \hat{T}$ on a metric space (M, d^*) satisfying the following contractive inequality of integral type:

$$\int_0^{d^*(\hat{T}\mu,\hat{S}\nu)} \xi(t)dt \leq \beta(d^*(\mu,\nu)) \int_0^{\Delta_1(\mu,\nu)} \xi(t)dt,$$

where $(\xi, \beta) \in \xi_1 \times \xi_3$ and for all μ, ν in M.

$$\begin{aligned}\Delta_1(\mu,\nu) = max\{&d^*(\hat{T}\mu,\hat{S}\nu), d^*(\hat{T}\mu,\hat{P}\mu), d^*(\hat{S}\nu,\hat{Q}\nu),\\ &\frac{1}{2}[d^*(\hat{P}\mu,\hat{S}\nu)+d^*(\hat{Q}\nu,\hat{T}\mu)], \frac{d^*(\hat{P}\mu,\hat{T}\mu).d^*(\hat{Q}\nu,\hat{S}\nu)}{1+d^*(\hat{T}\mu,\hat{S}\nu)},\\ &\frac{d^*(\hat{P}\mu,\hat{S}\nu).d^*(\hat{Q}\nu,\hat{T}\mu)}{1+d^*(\hat{T}\mu,\hat{S}\nu)}, d^*(\hat{T}\mu,\hat{P}\mu)[\frac{1+d^*(\hat{T}\mu,\hat{Q}\nu)+d^*(\hat{S}\nu,\hat{P}\mu)}{1+d^*(\hat{T}\mu,\hat{P}\mu)+d^*(\hat{S}\nu,\hat{Q}\nu)}]\}.\end{aligned}$$

Also, some common fixed point results for the above mentioned weakly compatible self - maps along with E.A. property and (CLR) property are proved. A suitable illustrative example is also provided to support our result.

Keywords and Phrases: Fixed Point, Coincidence Point, Weakly Compatible Maps, E.A. Property, (CLR) Property.

2020 Mathematics Subject Classification: 47H10, 54H25.

1. Introduction

All around this paper we postulate that $R^+ = [0, +\infty), N_0 = \mathbb{N} \bigcup \{0\}$, where $\mathbb{N}$ stands for the set of positive integers and

- $\xi_1 = \{\xi | \xi : R^+ \to R^+$ saisfies that ξ is Lebesgue integrable, summable on each compact subset of R^+ and $\int_0^\delta \xi(t)dt > 0$ for each $\delta > 0\}$,
- $\xi_2 = \{\tau | \tau : R^+ \to [0, 1)$ satisfied that $\limsup_{s \to t} \tau(s) < 1$ for each $t \in R^+\}$,
- $\xi_3 = \{\tau | \tau \in \xi_2$ and $\limsup_{s \to +\infty} \tau(s) < 1\}$.

The concept of conractive mapping of integral type was introduced in 2002 by Branciari [2] and obtained the following fixed point result for the mapping:

Theorem 1.1. *Let (M, d^*) be a complete metric space and $\hat{T}$ be a self map on M satisfying $\int_0^{d^*(\hat{T}\mu, \hat{T}\nu)} \xi(t)dt \leq \beta \int_0^{d^*(\mu,\nu)} \xi(t)dt$ for all μ, ν in M, where $\beta \in (0, 1)$ is a constant and $\xi \in \xi_1$.*
Then $\hat{T}$ has a unique fixed point $b \in M$ such that $\lim_{n \to +\infty} \hat{T}^n \mu = b$ for each $\mu \in M$.

Definition 1.2. *A coincidence point of a pair of self - mappings $\hat{P}, \hat{Q} : M \to M$ is a point $\mu \in M$ for which $\hat{P}\mu = \hat{Q}\mu$. A common fixed point of a pair of self - mappings $\hat{P}, \hat{Q} : M \to M$ is a point $\mu \in M$ for which $\hat{P}\mu = \hat{Q}\mu = \mu$.*

The concept of weakly compatible maps was introduced in 1996 by Jungck [4], to study common fixed point theorems as follows:

Definition 1.3. [4] *Let (M, d^*) be a metric space. A pair of self - mappings $\hat{P}, \hat{Q} : M \to M$ is weakly compatible if they commute at their coincidence points, that is, if there exists $\mu \in M$ such that $\hat{P}\hat{Q}\mu = \hat{Q}\hat{P}\mu$, where μ is coincidence point of $\hat{P}$ and $\hat{Q}$.*

The conception of E.A. property was firstly explained by Aamri and EI Moutawakil [1] in 2002 as follows:

Definition 1.4. [1] *Let (M, d^*) be a metric space. Two self - mappings $\hat{P}, \hat{Q} : M \to M$ are said to satisfy the E.A. property, if there exists a sequence μ_n in M such that, $\lim_{n \to \infty} \hat{P}\mu_n = \lim_{n \to \infty} \hat{Q}\mu_n = t$, for some $t \in M$.*

The concept of (CLR) property was introduced by Sintunavarat *et al.* [6] in 2011 as follows:

Definition 1.5. [6] *let* (M, d^*) *be a metric space. Two self - mappings* $\hat{P}, \hat{Q}$: $M \to M$ *are said to satisfy the CLR property, if there exists a sequence* μ_n *in* M *such that,* $\lim_{n\to+\infty} \hat{P}\mu_n = \lim_{n\to+\infty} \hat{Q}\mu_n = \hat{P}t$*, for some* $t \in M$.

Lemma 1.6. [5] *Let* $\xi \in \xi_1$ *and* $\{\mu_n\}_{n\in\mathbb{N}}$ *be a non negative sequence with* $\lim_{n\to+\infty} \mu_n = b$*. Then*

$$\lim_{n\to+\infty} \int_0^{\mu_n} \xi(t)dt = \int_0^b \xi(t)dt$$

In 2011, Feng *et al.* [5] proved the following Theorem :

Theorem 1.7. *Let* A, B, S *and* T *be self maps on a metric space* (X, d) *such that*

$$(A, T) \textit{ and } (B, S) \textit{ are weakly compatible}; \tag{1.1}$$

$$TX \subseteq BX \textit{ and } SX \subseteq AX; \tag{1.2}$$

$$\textit{One of } AX, BX, CX \textit{ and } DX \textit{ is complete}; \tag{1.3}$$

$$\int_0^{d(Tx,Sy)} \xi(t)dt \le \beta(d(x,y)) \int_0^{M_1(x,y)} \xi(t)dt, \tag{1.4}$$

where $(\phi, \alpha) \in \phi_1 \times \phi_3$ *and for all* x, y *in* M.

$$\begin{aligned} M_1(x,y) = max\{&d(Ax,By), d(Ax,Tx), d(By,Sy), \frac{1}{2}[d(Ax,Sy) + d(Tx,By)], \\ &d(Ax,Tx)[\frac{1+d(Ax,By)}{1+d(By,Sy)}], d(By,Sy)[\frac{1+d(Ax,By)}{1+d(Ax,Tx)}], \\ &\frac{d^2(Ax,Tx))}{1+d(Tx,Sy)}, \frac{d^2(By,Sy)}{1+d(Tx,Sy)}, \\ &d(Ax,Tx)[\frac{1+d(Ax,Sy)+d(Tx,BY)}{1+d(Ax,By)+d(Tx,Sy)}], \\ &d(By,Sy)[\frac{1+d(Ax,Sy)+d(Tx,BY)}{1+d(Ax,By)+d(Tx,Sy)}]\}. \end{aligned}$$

Then :

(i) There exist $w, u \in X$ *such that* $Aw = Tw = Bu = Su$*;*

(ii) A, B, S *and* T *have a unique common fixed point in* X *if* T *and* A *as well as* S *and* B *are weakly compatible.*

2. Main Results

Theorem 2.1. *Let (M, d^*) be a metric space and let $\hat{P}, \hat{Q}, \hat{S}$ and $\hat{T}$ be self mappings on M satisfying the followings:*

$$\hat{T}M \subseteq \hat{Q}M, \hat{S}M \subseteq \hat{P}M; \tag{2.1}$$

$$(\hat{P}, \hat{T}) \, and \; (\hat{Q}, \hat{S}) \, are \; weakly \; compatible; \tag{2.2}$$

$$One \; of \; \hat{P}M, \hat{Q}M, \hat{S}M or \; \hat{T}M is \; complete; \tag{2.3}$$

$$\int_0^{d^*(\hat{T}\mu, \hat{S}\nu)} \xi(t)dt \leq \beta(d^*(\mu, \nu)) \int_0^{\Delta_1(\mu,\nu)} \xi(t)dt, \tag{2.4}$$

where $(\xi, \beta) \in \xi_1 \times \xi_3$ and for all μ, ν in M.

$$\Delta_1(\mu, \nu) = max\{d^*(\hat{T}\mu, \hat{S}\nu), d^*(\hat{T}\mu, \hat{P}\mu), d^*(\hat{S}\nu, \hat{Q}\nu), \frac{1}{2}[d^*(\hat{P}\mu, \hat{S}\nu) + d^*(\hat{Q}\nu, \hat{T}\mu)], \frac{d^*(\hat{P}\mu, \hat{T}\mu).d^*(\hat{Q}\nu, \hat{S}\nu)}{1 + d^*(\hat{T}\mu, \hat{S}\nu)}, \frac{d^*(\hat{P}\mu, \hat{S}\nu).d^*(\hat{Q}\nu, \hat{T}\mu)}{1 + d^*(\hat{T}\mu, \hat{S}\nu)}, d^*(\hat{T}\mu, \hat{P}\mu)[\frac{1 + d^*(\hat{T}\mu, \hat{Q}\nu) + d^*(\hat{S}\nu, \hat{P}\mu)}{1 + d^*(\hat{T}\mu, \hat{P}\mu) + d^*(\hat{S}\nu, \hat{Q}\nu)}]\}.$$

Then we prove the followings:

(i) There exist $a, b \in M$ such that $\hat{P}a = \hat{T}a = \hat{Q}b = \hat{S}b$;

(ii) $\hat{P}, \hat{Q}, \hat{S}$ and $\hat{T}$ have a unique common fixed point in M.

Proof. Let $\mu_0 \in M$ be an arbitrary point in M. From (2.1), we can construct two sequences μ_n and ν_n in M as follows:

$$\nu_{2n+1} = \hat{T}\mu_{2n} = \hat{Q}\mu_{2n+1}, \nu_{2n+2} = \hat{S}\mu_{2n+1} = \hat{P}\mu_{2n+2}, \text{ for all } n \in \mathbb{N}. \tag{2.5}$$

Since $\hat{T}M \subseteq \hat{Q}M, \hat{S}M \subseteq \hat{P}M$.
Now, we define $d^*_n = d^*(\nu_n, \nu_{n+1})$ for each $n \in \mathbb{N}$.

On putting, $\mu = \mu_{2n}$ and $\nu = \mu_{2n+1}$ in (2.4) and using (2.5), we get

$$\Delta_1(\mu_{2n},\mu_{2n+1}) = max\{d^*(\hat{T}\mu_{2n},\hat{S}\mu_{2n+1}), d^*(\hat{T}\mu_{2n},\hat{P}\mu_{2n}), d^*(\hat{S}\mu_{2n+1},\hat{Q}\mu_{2n+1}),$$
$$\frac{1}{2}[d^*(\hat{P}\mu_{2n},\hat{S}\mu_{2n+1}) + d^*(\hat{Q}\mu_{2n+1},\hat{T}\mu_{2n})],$$
$$\frac{d^*(\hat{P}\mu_{2n},\hat{T}\mu_{2n}).d^*(\hat{Q}\mu_{2n+1},\hat{S}\mu_{2n+1})}{1+d^*(\hat{T}\mu_{2n},\hat{S}\mu_{2n+1})}, \frac{d^*(\hat{P}\mu_{2n},\hat{S}\mu_{2n+1}).d^*(\hat{Q}\mu_{2n+1},\hat{T}\mu_{2n})}{1+d^*(\hat{T}\mu_{2n},\hat{S}\mu_{2n+1})},$$
$$d^*(\hat{T}\mu_{2n},\hat{P}\mu_{2n})[\frac{1+d^*(\hat{T}\mu_{2n},\hat{Q}\mu_{2n+1}) + d^*(\hat{S}\mu_{2n+1},\hat{P}\mu_{2n})}{1+d^*(\hat{T}\mu_{2n},\hat{P}\mu_{2n}) + d^*(\hat{S}\mu_{2n+1},\hat{Q}\mu_{2n+1})}]\}.$$
$$= max\{d^*(\nu_{2n+1},\nu_{2n+2}), d^*(\nu_{2n+1},\nu_{2n}), d^*(\nu_{2n+2},\nu_{2n+1}),$$
$$\frac{1}{2}[d^*(\nu_{2n},\nu_{2n+2}) + d^*(\nu_{2n+1},\nu_{2n+1})],$$
$$\frac{d^*(\nu_{2n},\nu_{2n+1}).d^*(\nu_{2n+1},\nu_{2n+2})}{1+d^*(\nu_{2n+1},\nu_{2n+2})}, \frac{d^*(\nu_{2n},\nu_{2n+2}).d^*(\nu_{2n+1},\nu_{2n+1})}{1+d^*(\nu_{2n+1},\nu_{2n+2})},$$
$$d^*(\nu_{2n+1},\nu_{2n})[\frac{1+d^*(\nu_{2n+1},\nu_{2n+1}) + d^*(\nu_{2n+2},\nu_{2n})}{1+d^*(\nu_{2n+1},\nu_{2n}) + d^*(\nu_{2n+2},\nu_{2n+1})}]\}.$$
$$= max\{d^*_{2n}, d^*_{2n+1}\}.$$

If $d^*_{2n} < d^*_{2n+1}$

$$\Delta_1(\mu_{2n},\mu_{2n+1}) = d^*_{2n+1}.$$

And

$$0 < \int_0^{d^*_{2n+1}} \xi(t)dt$$
$$= \int_0^{d^*(\nu_{2n+1},\nu_{2n+2}} \xi(t)dt$$
$$= \int_0^{d^*(\hat{T}\mu_{2n},\hat{S}\mu_{2n+1})}$$
$$\leq \beta(d^*(\mu_{2n},\mu_{2n+1})) \int_0^{\Delta_1(\mu_{2n},\mu_{2n+1})} \xi(t)dt$$
$$= \beta(d^*(\mu_{2n},\mu_{2n+1})) \int_0^{d^*_{2n+1}} \xi(t)dt$$
$$< \int_0^{d^*_{2n+1}} \xi(t)dt.$$

a contradiction. Hence,
$d^*_{2n+1} < d^*_{2n} = \Delta_1(\mu_{2n},\mu_{2n+1})$ for all n in $\mathbb{N}$.

Similarly,
$d^*_{2n} < d^*_{2n-1} = \Delta_1(\mu_{2n-1}, \mu_{2n})$ for all n in $\mathbb{N}$,
which implies that

$$d^*_{n+1} < d^*_n, d^*_{2n} = \Delta_1(\mu_{2n}, \mu_{2n+1}), d^*_{2n-1} = \Delta_1(\mu_{2n}, \mu_{2n-1}) \tag{2.6}$$

which implies that $\{d^*_n\}$ is monotonic decreasing sequence bounded below and there exists a constant k such that,

$$\lim_{n\to+\infty} d^*_n = k \geq 0.$$

Suppose that $k > 0$. Then from (2.6) and Lemma 1.6, we get

$$\begin{aligned}
0 &< \int_0^k \xi(t)dt \\
&= \limsup_{n\to+\infty} \int_0^{d^*_{2n+1}} \xi(t)dt \\
&= \limsup_{n\to+\infty} \int_0^{d^*(\nu_{2n+1},\nu_{2n+2})} \xi(t)dt \\
&= \lim_{n\to+\infty} \int_0^{d^*(\hat{T}\mu_{2n},\hat{S}\mu_{2n+1})} \xi(t)dt \\
&\leq \limsup_{n\to+\infty} \beta d^*(\mu_{2n}, \mu_{2n+1}) \int_0^{\Delta_1(\mu_{2n},\mu_{2n+1})} \xi(t) \\
&= \limsup_{n\to+\infty} \beta d^*(\mu_{2n}, \mu_{2n+1}) \int_0^{d^*_{2n+1}} \xi(t)dt \\
&< \limsup_{n\to+\infty} \int_0^{d^*_{2n+1}} \xi(t)dt \\
&< \int_0^k \xi(t)dt,
\end{aligned}$$

which is a contradiction. Thus, $k = 0$, which implies that

$$\lim_{n\to+\infty} d^*_n = 0. \tag{2.7}$$

Now, we prove that $\{\nu_n\}$ is a cauchy sequence. For this it is sufficient to show that $\{\nu_{2n}\}$ is a cauchy sequence. Let, if possible $\{\nu_{2n}\}$ is not a cauchy sequence. Then, there exists $\epsilon > 0$ and $n, m > 0$ with $2m(\alpha) > 2n(\alpha) > 2\alpha$ satisfying

$$d^*(\nu_{2n(\alpha)}, \nu_{2m(\alpha)}) \geq \epsilon, \tag{2.8}$$

for all $\alpha \in N$.
where $2m(\alpha)$ is the least positive integer exceeding $2n(\alpha)$ satisfying (2.8). It follows that

$$d^*(\nu_{2n(\alpha)}, \nu_{2m(\alpha)}) \geq \epsilon, \text{ for all } \alpha \in \mathbb{N}.$$

Now, using (2.8) and triangular inequality, we obtain the following:

$$\begin{aligned}\epsilon &\leq d^*(\nu_{2n(\alpha)}, \nu_{2m(\alpha)}),\\ &\leq d^*(\nu_{2n(\alpha)}, (\nu_{2m(\alpha)-2}) + d^*(\nu_{2m(\alpha)-2}, \nu_{2m(\alpha)-1})\\ &+ d^*(\nu_{2m(\alpha)-1}, \nu_{2m(\alpha)})\\ &< \epsilon + d^*_{2m(\alpha)-2} + d^*_{2m(\alpha)-1}.\end{aligned} \tag{2.9}$$

And

$$\begin{aligned}|d^*(\nu_{2n(\alpha)}, \nu_{2m(\alpha)-1}) - d^*(\nu_{2n(\alpha)}, \nu_{2m(\alpha)})| &\leq d^*_{2m(\alpha)-1} \quad \text{for all} \quad \alpha \in \mathbb{N}.\\ |d^*(\nu_{2n(\alpha)+1}, \nu_{2m(\alpha)}) - d^*(\nu_{2n(\alpha)}, \nu_{2m(\alpha)})| &\leq d^*_{2n(\alpha)} \quad \text{for all} \quad \alpha \in \mathbb{N}.\\ |d^*(\nu_{2n(\alpha)+1}, \nu_{2m(\alpha)-1}) - d^*(\nu_{2n(\alpha)}, \nu_{2m(\alpha)-1})| &\leq d^*_{2n(\alpha)} \quad \text{for all} \quad \alpha \in \mathbb{N}.\end{aligned} \tag{2.10}$$

Letting $\alpha \to +\infty$ in (2.9) and (2.10) and using (2.7), we obtain

$$\begin{aligned}\lim_{\alpha\to+\infty} d^*(\nu_{2n(\alpha)}, \nu_{2m(\alpha)}) &= \lim_{\alpha\to+\infty} d^*(\nu_{2n(\alpha)}, \nu_{2m(\alpha)-1})\\ &= \lim_{\alpha\to+\infty} d^*(\nu_{2n(\alpha)+1}, \nu_{2m(\alpha)})\\ &= \lim_{\alpha\to+\infty} d^*(\nu_{2n(\alpha)+1}, \nu_{2m(\alpha)-1})\\ &= \epsilon.\end{aligned} \tag{2.11}$$

Now, on putting $\mu = \mu_{2n(\alpha)}$ and $\nu = \mu_{2m(\alpha)-1}$ in (2.4), using (2.11), Lemma 1.6 and $(\xi, \beta) \in \xi_1 \times \xi_3$, we obtain that

$$\begin{aligned}\Delta_1(\mu_{2n(\alpha)},\mu_{2m(\alpha)-1}) = max\{&d^*(\hat{T}\mu_{2n(\alpha)}, \hat{S}\mu_{2m(\alpha)-1}),\\ &d^*(\hat{T}\mu_{2n(\alpha)}, \hat{P}\mu_{2n(\alpha)}), d^*(\hat{S}\mu_{2m(\alpha)-1}, \hat{Q}\mu_{2m(\alpha)-1}),\\ &\frac{1}{2}[d^*(\hat{P}\mu_{2n(\alpha)}, \hat{S}\mu_{2m(\alpha)-1}) + d^*(\hat{Q}\mu_{2m(\alpha)-1}, \hat{T}\mu_{2n(\alpha)})],\\ &\frac{d^*(\hat{P}\mu_{2n(\alpha)}, \hat{T}\mu_{2n(\alpha)}).d^*(\hat{Q}\mu_{2m(\alpha)-1}, \hat{S}\mu_{2m(\alpha)-1})}{1 + d^*(\hat{T}\mu_{2n(\alpha)}, \hat{S}\mu_{2m(\alpha)-1})},\\ &\frac{d^*(\hat{P}\mu_{2n(\alpha)}, \hat{S}\mu_{2m(\alpha)-1}).d^*(\hat{Q}\mu_{2m(\alpha)-1}, \hat{T}\mu_{2n(\alpha)})}{1 + d^*(\hat{T}\mu_{2n(\alpha)}, \hat{S}\mu_{2m(\alpha)-1})},\end{aligned}$$

$$d^*(\hat{T}\mu_{2n(\alpha)},\hat{P}\mu_{2n(\alpha)})[\frac{1+d^*(\hat{T}\mu_{2n(\alpha)},\hat{Q}\mu_{2m(\alpha)-1})+d^*(\hat{S}\mu_{2m(\alpha)-1},\hat{P}\mu_{2n(\alpha)})}{1+d^*(\hat{T}\mu_{2n(\alpha)},\hat{P}\mu_{2n(\alpha)})+d^*(\hat{S}\mu_{2m(\alpha)-1},\hat{Q}\mu_{2m(\alpha)-1})}]\}.$$

$$= max\{d^*(\nu_{2n(\alpha)+1},\nu_{2m(\alpha)}),d^*(\nu_{2n(\alpha)+1},\nu_{2n(\alpha)}),d^*(\nu_{2m(\alpha)},\nu_{2m(\alpha)-1}),$$

$$\frac{1}{2}[d^*(\nu_{2n(\alpha)},\nu_{2m(\alpha)})+d^*(\nu_{2m(\alpha-1)},\nu_{2n(\alpha)+1})],$$

$$\frac{d^*(\nu_{2n(\alpha)},\nu_{2n(\alpha)+1}).d^*(\nu_{2m(\alpha)-1},\nu_{2m(\alpha)})}{1+d^*(\nu_{2n(\alpha)+1},\nu_{2m(\alpha)})},\frac{d^*(\nu_{2n(\alpha)},\nu_{2m(\alpha)}).d^*(\nu_{2m(\alpha)-1},\nu_{2n(\alpha)+1})}{1+d^*(\nu_{2n(\alpha)+1},\nu_{2m(\alpha)})},$$

$$d^*(\nu_{2n(\alpha)+1},\nu_{2n(\alpha)})[\frac{1+d^*(\nu_{2n(\alpha)+1},\nu_{2m(\alpha)-1})+d^*(\nu_{2m(\alpha)},\nu_{2n(\alpha)})}{1+d^*(\nu_{2n(\alpha)+1},\nu_{2n(\alpha)})+d^*(\nu_{2m(\alpha)},\nu_{2m(\alpha)-11})}]\}.$$

$$= max\{\epsilon,0,0,\frac{1}{2}[\epsilon+\epsilon],0,\frac{\epsilon.\epsilon}{1+0+\epsilon},0\}.$$

$$= \epsilon \quad as \quad \alpha\to+\infty.$$

And

$$\begin{aligned}
0 &< \int_0^{\epsilon}\xi(t)dt \\
&= \limsup_{\alpha\to+\infty}\int_0^{d^*(\nu_{2n(\alpha)+1},\nu_{2m(\alpha)}}\xi(t)dt \\
&= \lim_{\alpha\to+\infty}\int_0^{d^*(\hat{T}\mu_{2n(\alpha)},\hat{S}\mu_{2m(\alpha)-1})}\xi(t)dt \\
&\leq \limsup_{\alpha\to+\infty}\beta[d^*(\mu_{2n(\alpha)},\mu_{2m(\alpha)-1})\int_0^{\Delta_1(\mu_{2n(\alpha)},\mu_{2m(\alpha)-1})}\xi(t)dt] \\
&= \limsup_{\alpha\to+\infty}\beta d^*(\mu_{2n(\alpha)},\mu_{2m(\alpha)-1})\limsup_{\alpha\to+\infty}\int_0^{\Delta_1(\mu_{2n(\alpha)},\mu_{2m(\alpha)-1})}\xi(t)dt \\
&< \int_0^{\epsilon}\xi(t)dt
\end{aligned}$$

which is impossible. Hence $\{\nu_n\}$is a cauchy sequence.
Without loss of generality, let us assume that $\hat{P}M$ is complete subspace of M. Therefore, there exists $c\in\hat{P}M$ such that $\lim_{n\to+\infty}\nu_{2n}=c$.
Now, there exists $d\in M$ such that $c=\hat{P}d$. Also, we can obtain that

$$\begin{aligned}
c &= \lim_{n\to+\infty}\nu_{2n} \\
&= \lim_{n\to+\infty}\hat{T}\mu_{2n-1} \\
&= \lim_{n\to+\infty}\hat{Q}\mu_{2n}
\end{aligned}$$

$$= \lim_{n\to+\infty} \hat{S}\mu_{2n-1}$$
$$= \lim_{n\to+\infty} \hat{P}\mu_{2n} \tag{2.12}$$

Now, we prove that $\hat{T}d = c$. On putting $\mu = d$, $\nu = \mu_{2n+1}$ in (2.4), using (2.12), Lemma 1.6 and $(\xi, \beta) \in \xi_1 \times \xi_3$, we obtain that

$$\begin{aligned}\Delta_1(d, \mu_{2n+1}) = max\{&d^*(\hat{T}d, \hat{S}\mu_{2n+1}), d^*(\hat{T}d, \hat{P}d), d^*(\hat{S}\mu_{2n+1}, \hat{Q}\mu_{2n+1}),\\ &\frac{1}{2}[d^*(\hat{P}d, \hat{S}\mu_{2n+1}) + d^*(\hat{Q}\mu_{2n+1}, \hat{T}d)],\\ &\frac{d^*(\hat{P}d, \hat{T}d).d^*(\hat{Q}\mu_{2n+1}, \hat{S}\mu_{2n+1})}{1 + d^*(\hat{T}d, \hat{S}\mu_{2n+1})}, \frac{d^*(\hat{P}d, \hat{S}\mu_{2n+1}).d^*(\hat{Q}\mu_{2n+1}, \hat{T}d)}{1 + d^*(\hat{T}d, \hat{S}\mu_{2n+1})},\\ &d^*(\hat{T}d, \hat{P}d)[\frac{1 + d^*(\hat{T}d, \hat{Q}\mu_{2n+1}) + d^*(\hat{S}\mu_{2n+1}, \hat{P}d)}{1 + d^*(\hat{T}, \hat{P}d) + d^*(\hat{S}\mu_{2n+1}, \hat{Q}\mu_{2n+1})}]\}.\end{aligned}$$

Taking limit as $n \to +\infty$

$$\begin{aligned}\lim_{n\to+\infty} \Delta_1(d, \mu_{2n+1}) &= max\{d^*(\hat{T}d, c), d^*(\hat{T}d, c), d^*(c, c), \frac{1}{2}[d^*(c, c) + d^*(c, \hat{T}d)],\\ &\frac{d^*(c, \hat{T}d).d^*(c, c)}{1 + d^*(\hat{T}d, c)}, \frac{d^*(c, c).d^*(c, \hat{T}d)}{1 + d^*(\hat{T}d, c)}, d^*(\hat{T}d, c)[\frac{1 + d^*(\hat{T}d, c) + d^*(c, c)}{1 + d^*(\hat{T}d, c) + d^*(c, c)}]\}.\\ &= max\{d^*(\hat{T}d, c), d^*(\hat{T}d, c), 0, \frac{1}{2}[0 + d^*(c, \hat{T}d)], 0, 0, d^*(\hat{T}d, c)[\frac{1 + d^*(\hat{T}d, c) + 0}{1 + d^*(\hat{T}d, c) + 0}]\}.\\ &= d^*(\hat{T}d, c).\end{aligned}$$

And

$$\begin{aligned}0 &< \int_0^{d^*(\hat{T}d, c)} \xi(t)dt\\ &= \lim_{n\to+\infty} \int_0^{d^*(\hat{T}d, \hat{S}\mu_{2n+1})} \xi(t)dt\\ &\leq \limsup_{n\to+\infty} \beta[d^*(d, \mu_{2n+1}) \int_0^{\Delta_1(d, \mu_{2n+1})} \xi(t)dt]\\ &= \limsup_{n\to+\infty} [\beta d^*(d, \mu_{2n+1})] \limsup_{n\to+\infty} \int_0^{\Delta_1(d, \mu_{2n+1})} \xi(t)dt\\ &< \int_0^{d^*(\hat{T}d, c)} \xi(t)dt,\end{aligned}$$

a contradiction. Hence $\hat{T}d = c$. Since $\hat{T}M \subseteq \hat{Q}M$, therefore, there exists a point $b \in M$ with $c = \hat{Q}b = \hat{T}d$.
Now, we prove that $\hat{S}b = c$. Let, if possible $\hat{S}b \neq c$.

Now, in the view of (2.4), (2.12), Lemma 1.6 and $(\xi, \beta) \in \xi_1 \times \xi_3$, we obtain that

$$\begin{aligned}\Delta_1(\mu_{2n}, b) = max\{& d^*(\hat{T}\mu_{2n}, \hat{S}b), d^*(\hat{T}\mu_{2n}, \hat{P}\mu_{2n}), d^*(\hat{S}b, \hat{Q}b),\\ & \frac{1}{2}[d^*(\hat{P}\mu_{2n}, \hat{S}b) + d^*(\hat{Q}b, \hat{T}\mu_{2n})],\\ & \frac{d^*(\hat{P}\mu_{2n}, \hat{T}\mu_{2n}).d^*(\hat{Q}b, \hat{S}b)}{1 + d^*(\hat{T}\mu_{2n}, \hat{S}b)}, \frac{d^*(\hat{P}\mu_{2n}, \hat{S}b).d^*(\hat{Q}b, \hat{T}\mu_{2n})}{1 + d^*(\hat{T}\mu_{2n}, \hat{S}b)},\\ & d^*(\hat{T}\mu_{2n}, \hat{P}\mu_{2n})[\frac{1 + d^*(\hat{T}\mu_{2n}, \hat{Q}b) + d^*(\hat{S}b, \hat{P}\mu_{2n})}{1 + d^*(\hat{T}\mu_{2n}, \hat{P}\mu_{2n}) + d^*(\hat{S}b, \hat{Q}b)}]\}.\end{aligned}$$

Letting limit as $n \to +\infty$, we have

$$\begin{aligned}\lim_{n\to+\infty} \Delta_1(\mu_{2n}, b) &= max\{d^*(c, \hat{S}b), d^*(c, c), d^*(\hat{S}b, c), \frac{1}{2}[d^*(c, \hat{S}b) + d^*(c, c)],\\ & \frac{d^*(c, c).d^*(c, \hat{S}b)}{1 + d^*(c, \hat{S}b)}, \frac{d^*(c, \hat{S}b).d^*(c, c)}{1 + d^*(c, \hat{S}b)},\\ & d^*(c, c)[\frac{1 + d^*(c, c) + d^*(\hat{S}b, c)}{1 + d^*(c, c) + d^*(\hat{S}b, c)}]\}.\\ &= max\{d^*(c, \hat{S}b), 0, d^*(\hat{S}b, c), \frac{1}{2}[d^*(c, \hat{S}b) + 0], 0, 0, 0\}.\\ &= d^*(c, \hat{S}b)\end{aligned}$$

And

$$\begin{aligned} 0 &< \int_0^{d^*(c,\hat{S}b)} \xi(t)dt\\ &= \limsup_{n\to+\infty} \int_0^{d^*(\hat{T}\mu_{2n},\hat{S}b)} \xi(t)dt\\ &\leq \limsup_{n\to+\infty} \beta[d^*(\mu_{2n}, b) \int_0^{\Delta_1(\mu_{2n},b)} \xi(t)dt]\\ &= \limsup_{n\to+\infty}[\beta d^*(\mu_{2n}, b)] \limsup_{n\to+\infty} \int_0^{\Delta_1(\mu_{2n},b)} \xi(t)dt\\ &< \int_0^{d^*(c,\hat{S}b)} \xi(t)dt,\end{aligned}$$

which is a contradiction. Hence $\hat{S}b = \hat{Q}b = c$ and $\hat{T}d = \hat{P}d = c$. Next we prove that $\hat{P}, \hat{Q}, \hat{S}$ and $\hat{T}$ have a unique common fixed point. Since $(\hat{P}, \hat{T})$ and $(\hat{Q}, \hat{S})$ are weakly compatible. Therefore,

$$\hat{P}c = \hat{P}\hat{T}d = \hat{T}\hat{P}d = \hat{T}c.$$

$$\hat{Q}c = \hat{Q}\hat{S}b = \hat{S}\hat{Q}b = \hat{S}c.$$

Now, we show that $\hat{T}c = \hat{S}c$. Let, if possible $\hat{T}c \neq \hat{S}c$. Now, from (2.4), and $(\xi, \beta) \in \xi_1 \times \xi_3$, we obtain that

$$\begin{aligned}
\Delta_1(c,c) = max\{ & d^*(\hat{T}c, \hat{S}c), d^*(\hat{T}c, \hat{P}c), d^*(\hat{S}c, \hat{Q}c), \frac{1}{2}[d^*(\hat{P}c, \hat{S}c) + d^*(\hat{Q}c, \hat{T}c)], \\
& \frac{d^*(\hat{P}c, \hat{T}c).d^*(\hat{Q}c, \hat{S}c)}{1 + d^*(\hat{T}c, \hat{S}c)}, \frac{d^*(\hat{P}c, \hat{S}c).d^*(\hat{Q}c, \hat{T}c)}{1 + d^*(\hat{T}c, \hat{S}c)}, \\
& d^*(\hat{T}c, \hat{P}c)[\frac{1 + d^*(\hat{T}c, \hat{Q}c) + d^*(\hat{S}c, \hat{P}c)}{1 + d^*(\hat{T}c, \hat{P}c) + d^*(\hat{S}c, \hat{Q}c)}]\}. \\
& = max\{d^*(\hat{T}c, \hat{S}c), 0, 0, \frac{1}{2}[d^*(\hat{T}c, \hat{S}c) + d^*(\hat{S}c, \hat{T}c)], 0, \frac{d^*(\hat{T}c, \hat{S}c).d^*(\hat{S}c, \hat{T}c)}{1 + d^*(\hat{T}c, \hat{S}c)}, 0\}. \\
& = d^*(\hat{T}c, \hat{S}c).
\end{aligned}$$

And

$$\begin{aligned}
0 &< \int_0^{d^*(\hat{T}c,\hat{S}c)} \xi(t)dt \\
&\leq \beta d^*(c, c) \int_0^{\Delta_1(c,c)} \xi(t)dt \\
&= \beta(0) \int_0^{d^*(\hat{T}c,\hat{S}c)} \xi(t)dt \\
&< \int_0^{d^*(\hat{T}c,\hat{S}c)} \xi(t)dt,
\end{aligned}$$

which is a contradiction. Hence $\hat{S}c = \hat{T}c = c$. That is $\hat{P}c = \hat{Q}c = \hat{S}c = \hat{T}c$.
Now, let, if possible $\hat{T}c \neq c$.
Now, in the view of (2.4) and $(\xi, \beta) \in \xi_1 \times \xi_3$, we obtain that

$$\begin{aligned}
\Delta_1(c, b) = max\{ & d^*(\hat{T}c, \hat{S}b), d^*(\hat{T}c, \hat{P}c), d^*(\hat{S}b, \hat{Q}b), \frac{1}{2}[d^*(\hat{P}c, \hat{S}b) + d^*(\hat{Q}b, \hat{T}c)], \\
& \frac{d^*(\hat{P}c, \hat{T}c).d^*(\hat{Q}b, \hat{S}b)}{1 + d^*(\hat{T}c, \hat{S}b)}, \frac{d^*(\hat{P}c, \hat{S}b).d^*(\hat{Q}b, \hat{T}c)}{1 + d^*(\hat{T}c, \hat{S}b)},
\end{aligned}$$

$$d^*(\hat{T}c,\hat{P}c)[\frac{1+d^*(\hat{T}c,\hat{Q}b)+d^*(\hat{S}c,\hat{P}c)}{1+d^*(\hat{T}c,\hat{P}c)+d^*(\hat{S}b,\hat{Q}b)}]\}.$$
$$= max\{d^*(\hat{T}c,c),0,0,d^*(\hat{T}c,c),0,\frac{d^*(\hat{T}c,c).d^*(c,\hat{T}c)}{1+d^*(\hat{T}c,c)},0\}.$$
$$= d^*(\hat{T}c,c).$$

And

$$\begin{aligned} 0 &< \int_0^{d^*(\hat{T}c,c)} \xi(t)dt \\ &= \int_0^{d^*(\hat{T}c,\hat{S}b)} \xi(t)dt \\ &\le \beta d^*(c,b)\int_0^{\Delta_1(c,b)} \xi(t)dt \\ &= \beta(d^*(c,b))\int_0^{d^*(\hat{T}c,c)} \xi(t)dt \\ &< \int_0^{d^*(\hat{T}c,c)} \xi(t)dt, \end{aligned}$$

which is not possible. Therefore, $\hat{T}c = c$. Hence $\hat{P}c = \hat{Q}c = \hat{S}c = \hat{T}c = c$ which implies that $\hat{P},\hat{Q},\hat{S}$ and $\hat{T}$ have a fixed point, that is, c. Now, for the uniqueness, suppose that $\hat{P},\hat{Q},\hat{S}$ and $\hat{T}$ have two fixed points r and s such that $r \neq s$.
Now, in the view of (2.4) and $(\xi,\beta) \in \xi_1 \times \xi_3$, we obtain that

$$\Delta_1(r,s) = max\{d^*(\hat{T}r,\hat{S}s), d^*(\hat{T}r,\hat{P}r), d^*(\hat{S}s,\hat{Q}s), \frac{1}{2}[d^*(\hat{P}r,\hat{S}s)+d^*(\hat{Q}s,\hat{T}r)],$$
$$\frac{d^*(\hat{P}r,\hat{T}r).d^*(\hat{Q}s,\hat{S}s)}{1+d^*(\hat{T}r,\hat{S}s)}, \frac{d^*(\hat{P}r,\hat{S}s).d^*(\hat{Q}s,\hat{T}r)}{1+d^*(\hat{T}r,\hat{S}s)},$$
$$d^*(\hat{T}r,\hat{P}r)[\frac{1+d^*(\hat{T}r,\hat{Q}s)+d^*(\hat{S}r,\hat{P}r)}{1+d^*(\hat{T}r,\hat{P}r)+d^*(\hat{S}s,\hat{Q}s)}]\}.$$
$$= max\{d^*(r,s), d^*(r,r), d^*(s,s), \frac{1}{2}[d^*(r,s)+d^*(s,r)],$$
$$\frac{d^*(r,r).d^*(s,s)}{1+d^*(r,s)}, \frac{d^*(r,s).d^*(s,r)}{1+d^*(r,s)}, d^*(r,r)[\frac{1+d^*(r,s)+d^*(r,r)}{1+d^*(r,r)+d^*(s,s)}]\}.$$
$$= max\{d^*(r,s),0,0,d^*(r,s),0,\frac{d^*(r,s).d^*(s,r)}{1+d^*(r,s)},0\}.$$
$$= d^*(r,s).$$

And

$$\begin{aligned} 0 &< \int_0^{d^*(r,s)} \xi(t)dt \\ &= \int_0^{d^*(\hat{T}r,\hat{S}s)} \xi(t)dt \\ &\leq \beta d^*(r,s) \int_0^{\Delta_1(r,s)} \xi(t)dt \\ &= \beta(d^*(r,s)) \int_0^{d^*(r,s)} \xi(t)dt \\ &< \int_0^{d^*(r,s)} \xi(t)dt, \end{aligned}$$

which is a contradiction. Hence $\hat{P}, \hat{Q}, \hat{S}$ and $\hat{T}$ have a unique common fixed point. This completes the proof of the theorem.

Theorem 2.2. *Let (M, d^*) be a metric space and let $\hat{P}, \hat{Q}, \hat{S}$ and $\hat{T}$ be self mappings on M satisfying $(2.1), (2.4)$ and the followings:*

$$\textit{Pairs} \quad (\hat{P},\hat{T})\,\textit{and}\,(\hat{Q},\hat{S})\,\textit{are weakly compatible}; \tag{2.13}$$

$$\textit{Pair} \quad (\hat{P},\hat{T})\,\textit{or}\,(\hat{Q},\hat{S})\,\textit{satisfy the E.A. property}; \tag{2.14}$$

if one of $\hat{P}M, \hat{Q}M, \hat{S}M$ or $\hat{T}M$ is complete. Then, $\hat{P}, \hat{Q}, \hat{S}$ and $\hat{T}$ have a unique common fixed point in M.

Proof. Suppose the pair $(\hat{P}, \hat{T})$ satisfy the E.A. property. Then there exists a sequence $\{\mu_n\}$ in M such that,

$$\lim_{n\to+\infty} \hat{P}\mu_n = \lim_{n\to+\infty} \hat{T}\mu_n = c, \text{for some} c \in M$$

Since $\hat{T}M \subseteq \hat{Q}M$, therefore there exists a sequence $\{\nu_n\}$ in M such that

$$\hat{T}\mu_n = \hat{Q}\nu_n.$$

Hence, $\lim_{n\to+\infty} \hat{Q}\nu_n = c$.
Now, we shall prove that $\lim_{n\to+\infty} \hat{S}\nu_n = c$.
Let, if possible $\lim_{n\to\infty} \hat{S}\nu_n = d \neq c$.

Now, from (2.4), Lemma 1.6 and $(\xi, \beta) \in \xi_1 \times \xi_3$, we obtain that

$$\begin{aligned}\Delta_1(\mu_n, \nu_n) = & \, max\{d^*(\hat{T}\mu_n, \hat{S}\nu_n), d^*(\hat{T}\mu_n, \hat{P}\nu_n), d^*(\hat{S}\nu_n, \hat{Q}\nu_n), \\ & \frac{1}{2}[d^*(\hat{P}\mu_n, \hat{S}\nu_n) + d^*(\hat{Q}\nu_n, \hat{T}\mu_n)], \\ & \frac{d^*(\hat{P}\mu_n, \hat{T}\mu_n).d^*(\hat{Q}\nu_n, \hat{S}nu_n)}{1 + d^*(\hat{T}\mu_n, \hat{S}\nu_n)}, \frac{d^*(\hat{P}\mu_n, \hat{S}\nu_n).d^*(\hat{Q}\nu_n, \hat{T}\mu_n)}{1 + d^*(\hat{T}\mu_n, \hat{S}\nu_n)}, \\ & d^*(\hat{T}\mu_n, \hat{P}\mu_n)[\frac{1 + d^*(\hat{T}\mu_n, \hat{Q}\nu_n) + d^*(\hat{S}\nu_n, \hat{P}\mu_{2n})}{1 + d^*(\hat{T}\mu_{2n}, \hat{P}\mu_n) + d^*(\hat{S}\nu_n, \hat{Q}\nu_n)}]\}.\end{aligned}$$

Taking limit as$n \to \infty$, we get

$$\begin{aligned}\lim_{n\to\infty} \Delta_1(\mu_n \nu_n) = & \, max\{d^*(c, d), d^*(c, c), d^*(d, c), \frac{1}{2}[d^*(c, d) + d^*(c, c)], \\ & \frac{d^*(c, c).d^*(c, d)}{1 + d^*(c, d)}, \frac{d^*(c, d).d^*(c, c)}{1 + d^*(c, d)}, d^*(c, c)[\frac{1 + d^*(c, d) + d^*(d, c)}{1 + d^*(c, c) + d^*(d, c)}]\}. \\ = & \, max\{d^*(c, d), 0, d^*(d, c), \frac{1}{2}[d^*(c, d) + 0], 0, 0, 0\}. \\ = & \, d^*(c, d).\end{aligned}$$

And

$$\begin{aligned} 0 &< \int_0^{d^*(c,d)} \xi(t)dt \\ &= \limsup_{n\to+\infty} \int_0^{d^*(\hat{T}\mu_n, \hat{S}\nu_n)} \xi(t)dt \\ &\leq \limsup_{n\to+\infty} [\beta d^*(\mu_n, \nu_n)] \limsup_{n\to+\infty} \int_0^{\Delta_1(\mu_n,\nu_n)} \xi(t)dt \\ &< \int_0^{d^*(c,d)} \xi(t)dt,\end{aligned}$$

a contradiction. Thus c = d, that is, $\lim_{n\to+\infty} \hat{S}\nu_n = c$.
Now, suppose that $\hat{Q}M$ is closed subspace of M . Then $c = \hat{Q}b$, for some b in M. Subsequently, we have

$$\lim_{n\to+\infty} \hat{S}\nu_n = \lim_{n\to+\infty} \hat{P}\mu_n = \lim_{n\to+\infty} \hat{T}\mu_n = \lim_{n\to+\infty} \hat{Q}\nu_n = c = \hat{Q}b.$$

Now, we show that $\hat{S}b = \hat{Q}b$.
Let, if possible $\hat{S}b \neq \hat{Q}b$.

Now, from (2.4), Lemma 1.6 and $(\xi,\beta)\in\xi_1\times\xi_3$, we obtain that

$$\begin{aligned}\Delta_1(\mu_n,b) = & \, max\{d^*(\hat{T}\mu_n,\hat{S}b), d^*(\hat{T}\mu_n,\hat{P}\mu_n), d^*(\hat{S}b,\hat{Q}b),\\ & \frac{1}{2}[d^*(\hat{P}\mu_n,\hat{S}b)+d^*(\hat{Q}b,\hat{T}\mu_n)],\\ & \frac{d^*(\hat{P}\mu_n,\hat{T}\mu_n).d^*(\hat{Q}b,\hat{S}b)}{1+d^*(\hat{T}\mu_n,\hat{S}b)}, \frac{d^*(\hat{P}\mu_n,\hat{S}b).d^*(\hat{Q}b,\hat{T}\mu_n)}{1+d^*(\hat{T}\mu_n,\hat{S}b)},\\ & d^*(\hat{T}\mu_n,\hat{P}\mu_n)[\frac{1+d^*(\hat{T}\mu_n,\hat{Q}b)+d^*(\hat{S}b,\hat{P}\mu_n)}{1+d^*(\hat{T}\mu_n,\hat{P}\mu_n)+d^*(\hat{S}b,\hat{Q}b)}]\}.\end{aligned}$$

Letting limit as $n\to+\infty$, we have

$$\begin{aligned}\lim_{n\to+\infty}\Delta_1(\mu_n,b) = & \, max\{d^*(c,\hat{S}b), d^*(c,c), d^*(\hat{S}b,c), \frac{1}{2}[d^*(c,\hat{S}b)+d^*(c,c)],\\ & \frac{d^*(c,c).d^*(c,\hat{S}b)}{1+d^*(c,\hat{S}b)}, \frac{d^*(c,\hat{S}b).d^*(c,c)}{1+d^*(c,\hat{S}b)},\\ & d^*(c,c)[\frac{1+d^*(c,c)+d^*(\hat{S}b,c)}{1+d^*(c,c)+d^*(\hat{S}b,c)}]\}.\\ = & \, max\{d^*(c,\hat{S}b), 0, d^*(\hat{S}b,c), \frac{1}{2}[d^*(c,\hat{S}b)+0], 0, 0, 0\}.\\ = & \, d^*(c,\hat{S}b)\end{aligned}$$

And

$$\begin{aligned} 0 &< \int_0^{d^*(c,\hat{S}b)}\xi(t)dt\\ &= \limsup_{n\to+\infty}\int_0^{d^*(\hat{T}\mu_n,\hat{S}b)}\xi(t)dt\\ &\le \limsup_{n\to+\infty}\beta[d^*(\mu_n,b)\int_0^{\Delta_1(\mu_n,b)}\xi(t)dt]\\ &= \limsup_{n\to+\infty}[\beta d^*(\mu_n,b)]\limsup_{n\to+\infty}\int_0^{\Delta_1(\mu_n,b)}\xi(t)dt\\ &< \int_0^{d^*(c,\hat{S}b)}\xi(t)dt,\end{aligned}$$

which is impossible. Hence $\hat{S}b=\hat{Q}b=c$.
Since the pair $(\hat{Q},\hat{S})$ is weakly compatible. Therefore, $\hat{Q}\hat{S}b=\hat{S}\hat{Q}b$, implies that

$$\hat{Q}\hat{S}b=\hat{Q}\hat{Q}b=\hat{S}\hat{S}b=\hat{S}\hat{Q}b.$$

Since $\hat{S}M \subseteq \hat{P}M$, there exists $a \in M$, such that

$$\hat{S}b = \hat{P}a = c.$$

Now, we claim that $\hat{P}a = \hat{T}a$
Now, from (2.4) and $(\xi, \beta) \in \xi_1 \times \xi_3$, we obtain that

$$\begin{aligned}
\Delta_1(a,b) = & max\{d^*(\hat{T}a,\hat{S}b), d^*(\hat{T}a,\hat{P}a), d^*(\hat{S}b,\hat{Q}b), \frac{1}{2}[d^*(\hat{P}a,\hat{S}b)+d^*(\hat{Q}b,\hat{T}a)],\\
& \frac{d^*(\hat{P}a,\hat{T}a).d^*(\hat{Q}b,\hat{S}b)}{1+d^*(\hat{T}a,\hat{S}b)}, \frac{d^*(\hat{P}a,\hat{S}b).d^*(\hat{Q}b,\hat{T}a)}{1+d^*(\hat{T}a,\hat{S}b)},\\
& d^*(\hat{T}a,\hat{P}a)[\frac{1+d^*(\hat{T}a,\hat{Q}b)+d^*(\hat{S}a,\hat{P}a)}{1+d^*(\hat{T}a,\hat{P}a)+d^*(\hat{S}b,\hat{Q}b)}]\}.\\
& = max\{d^*(\hat{T}a,c), d^*(\hat{T}a,c), d^*(c,c), \frac{1}{2}[d^*(c,c)+d^*(c,\hat{T}a)],\\
& \frac{d^*(c,\hat{T}a).d^*(c,c)}{1+d^*(c,c)}, \frac{d^*(c,\hat{T}a).d^*(c,c)}{1+d^*(c,c)}, d^*(\hat{T}a,c)[\frac{1+d^*(\hat{T}a,c)+d^*(c,c)}{1+d^*(\hat{T}a,c)+d^*(c,c)}]\}.\\
& = max\{d^*(\hat{T}a,c), d^*(\hat{T}a,c), \frac{1}{2}[0+d^*(\hat{T}a,c)]0,0,d^*(\hat{T}a,c)\}.\\
& = d^*(\hat{T}a,c).
\end{aligned}$$

And

$$\begin{aligned}
0 &< \int_0^{d^*(\hat{T}a,c)} \xi(t)dt\\
&= \int_0^{d^*(\hat{T}a,\hat{S}b)} \xi(t)dt\\
&\leq \beta d^*(a,b) \int_0^{\Delta_1(a,b)} \xi(t)dt\\
&= \beta(d^*(a,b)) \int_0^{d^*(\hat{T}a,c)} \xi(t)dt < \int_0^{d^*(\hat{T}a,c)} \xi(t)dt,
\end{aligned}$$

which is a contradiction. Hence $\hat{P}a = \hat{S}b = \hat{T}a$. Thus, we have $\hat{S}b = \hat{Q}b = \hat{P}a = \hat{T}a$. Since the pair $(\hat{P}, \hat{T})$ is weakly compatible. Therefore, $\hat{T}\hat{P}a = \hat{T}\hat{T}a = \hat{P}\hat{P}a = \hat{P}\hat{T}a$.
Now, we claim that $\hat{S}b$ is common fixed point of $\hat{P}, \hat{Q}, \hat{S}$ and $\hat{T}$.

Suppose that, $\hat{S}\hat{S}b = \hat{S}b$.
Now, from (2.4) and $(\xi, \beta) \in \xi_1 \times \xi_3$, we obtain that

$$\begin{aligned}
\Delta_1(a, \hat{S}b) &= max\{d^*(\hat{T}a, \hat{S}\hat{S}b), d^*(\hat{T}a, \hat{P}a), d^*(\hat{S}\hat{S}b, \hat{Q}\hat{S}b), \\
&\frac{1}{2}[d^*(\hat{P}a, \hat{S}\hat{S}b) + d^*(\hat{Q}\hat{S}b, \hat{T}a)], \quad \frac{d^*(\hat{P}a, \hat{T}a).d^*(\hat{Q}\hat{S}b, \hat{S}\hat{S}b)}{1 + d^*(\hat{T}a, \hat{S}\hat{S}b)}, \\
&\frac{d^*(\hat{P}a, \hat{S}\hat{S}b).d^*(\hat{Q}\hat{S}b, \hat{T}a)}{1 + d^*(\hat{T}a, \hat{S}\hat{S}b)}, d^*(\hat{T}a, \hat{P}a)[\frac{1 + d^*(\hat{T}a, \hat{Q}\hat{S}b) + d^*(\hat{S}a, \hat{P}a)}{1 + d^*(\hat{T}a, \hat{P}a) + d^*(\hat{S}\hat{S}b, \hat{Q}\hat{S}b)}]\}. \\
&= max\{d^*(\hat{S}b, \hat{S}\hat{S}b), d^*(\hat{S}b, \hat{S}b), d^*(\hat{S}\hat{S}b, \hat{S}\hat{S}b), \frac{1}{2}[d^*(\hat{S}b, \hat{S}\hat{S}b) + d^*(\hat{S}\hat{S}b, \hat{S}b)], \\
&\frac{d^*(\hat{S}b, \hat{S}b).d^*(\hat{S}\hat{S}b, \hat{S}\hat{S}b)}{1 + d^*(\hat{S}b, \hat{S}\hat{S}b)}, \frac{d^*(\hat{S}b, \hat{S}\hat{S}b).d^*(\hat{S}\hat{S}b, \hat{S}b)}{1 + d^*(\hat{S}b, \hat{S}\hat{S}b)}, \\
&d^*(\hat{S}b, \hat{S}b)[\frac{1 + d^*(\hat{S}b, \hat{S}\hat{S}b) + d^*(\hat{S}\hat{S}b, \hat{S}b)}{1 + d^*(\hat{S}b, \hat{S}b) + d^*(\hat{S}\hat{S}b, \hat{S}\hat{S}b)}]\}. \\
&= max\{d^*(\hat{S}b, \hat{S}\hat{S}b), 0, 0, d^*(\hat{S}b, \hat{S}\hat{S}b), 0, d^*(\hat{S}b, \hat{S}\hat{S}b), 0\}. \\
&= d^*(\hat{S}b, \hat{S}\hat{S}b).
\end{aligned}$$

And

$$\begin{aligned}
0 &< \int_0^{d^*(\hat{S}b, \hat{S}\hat{S}b)} \xi(t)dt \\
&= \int_0^{d^*(\hat{T}a, \hat{S}\hat{S}b)} \xi(t)dt \\
&\leq \beta d^*(a, \hat{S}b) \int_0^{\Delta_1(a, \hat{S}b)} \xi(t)dt \\
&= \beta(d^*(a, \hat{S}b)) \int_0^{d^*(\hat{S}b, \hat{S}\hat{S}b)} \xi(t)dt < \int_0^{d^*(\hat{S}b, \hat{S}\hat{S}b)} \xi(t)dt,
\end{aligned}$$

which is a contradiction. Hence $\hat{S}\hat{S}b = \hat{Q}\hat{S}b = \hat{S}b$. This implies $\hat{S}b$ is the common fixed point of $\hat{Q}$ and $\hat{S}$.
Similarly, we can prove that $\hat{T}a$ is the common fixed point of $\hat{P}$ and $\hat{T}$. Since $\hat{S}b = \hat{T}a$. This shows that $\hat{S}b$ is the common fixed point of $\hat{P}$, $\hat{Q}, \hat{S}$ and $\hat{T}$. If we assume that $\hat{P}M$ is closed subset of M, the proof is similar. Similarly, we can prove the theorem for cases when $\hat{T}M$ or $\hat{S}M$ is closed subset of M. Since $\hat{T}M \subseteq \hat{Q}M$ and $\hat{S}M \subseteq \hat{P}M$. Now, we shall prove the uniqueness of common fixed

point. Suppose that $\hat{P}$, $\hat{Q}, \hat{S}$ and $\hat{T}$ have two fixed points r and s such that $r \neq s$. Now, in the view of (2.4) and $(\xi, \beta) \in \xi_1 \times \xi_3$, we obtain that

$$\begin{aligned}
\Delta_1(r,s) = & max\{d^*(\hat{T}r,\hat{S}s), d^*(\hat{T}r,\hat{P}r), d^*(\hat{S}s,\hat{Q}s), \frac{1}{2}[d^*(\hat{P}r,\hat{S}s) + d^*(\hat{Q}s,\hat{T}r)], \\
& \frac{d^*(\hat{P}r,\hat{T}r).d^*(\hat{Q}s,\hat{S}s)}{1+d^*(\hat{T}r,\hat{S}s)}, \frac{d^*(\hat{P}r,\hat{S}s).d^*(\hat{Q}s,\hat{T}r)}{1+d^*(\hat{T}r,\hat{S}s)}, \\
& d^*(\hat{T}r,\hat{P}r)[\frac{1+d^*(\hat{T}r,\hat{Q}s)+d^*(\hat{S}r,\hat{P}r)}{1+d^*(\hat{T}r,\hat{P}r)+d^*(\hat{S}s,\hat{Q}s)}]\}. \\
= & max\{d^*(r,s), d^*(r,r), d^*(s,s), \frac{1}{2}[d^*(r,s)+d^*(s,r)], \\
& \frac{d^*(r,r).d^*(s,s)}{1+d^*(r,s)}, \frac{d^*(r,s).d^*(s,r)}{1+d^*(r,s)}, d^*(r,r)[\frac{1+d^*(r,s)+d^*(r,r)}{1+d^*(r,r)+d^*(s,s)}]\}. \\
= & max\{d^*(r,s), 0, 0, d^*(r,s), 0, \frac{d^*(r,s).d^*(s,r)}{1+d^*(r,s)}, 0\}. \\
= & d^*(r,s).
\end{aligned}$$

And

$$\begin{aligned}
0 &< \int_0^{d^*(r,s)} \xi(t)dt \\
&= \int_0^{d^*(\hat{T}r,\hat{S}s)} \xi(t)dt \\
&\leq \beta d^*(r,s) \int_0^{\Delta_1(r,s)} \xi(t)dt \\
&= \beta(d^*(r,s)) \int_0^{d^*(\hat{T}r,s)} \xi(t)dt \\
&< \int_0^{d^*(r,s)} \xi(t)dt,
\end{aligned}$$

which is a contradiction. Hence $\hat{P}, \hat{Q}, \hat{S}$ and $\hat{T}$ have a unique common fixed point.

Theorem 2.3. *Let (M, d^*) be a metric space and let $\hat{P}, \hat{Q}, \hat{S}$ and $\hat{T}$ be self mappings on M satisfying* (2.2), (2.4) *and the followings:*

$$\hat{T}M \subseteq \hat{Q}M \, and \; pair \, (\hat{P},\hat{T}) satisfies \, (CLR_{\hat{P}}) \; property \; or; \tag{2.15}$$

$$\hat{S}M \subseteq \hat{P}M \, and \; pair \, (\hat{Q},\hat{S}) satisfies (CLR_{\hat{Q}}) \; property \; . \tag{2.16}$$

Then, $\hat{P}, \hat{Q}, \hat{S}$ and $\hat{T}$ have a unique common fixed point in M.

Proof. Without loss of generality, we assume that $\hat{T}M \subseteq \hat{Q}M$ and the pair $(\hat{P}, \hat{T})$ satisfy the $(CLR_{\hat{P}})$ property. Then there exists a sequence $\{\mu_n\} \in M$ such that,

$$\lim_{n\to+\infty} \hat{P}\mu_n = \lim_{n\to+\infty} \hat{T}\mu_n = \hat{P}c,$$

for some $c \in M$.

Since $\hat{T}M \subseteq \hat{Q}M$, therefore there exists a sequence ν_n in M such that

$$\hat{T}\mu_n = \hat{Q}\nu_n.$$

Hence $\lim_{n\to+\infty} \hat{Q}\nu_n = \hat{P}c$.

Now, we shall prove that $\lim_{n\to+\infty} \hat{S}\nu_n = \hat{P}c$.

Let, if possible $\lim_{n\to+\infty} \hat{S}\nu_n = d \neq \hat{P}c$.

Now, in the view of (2.4), Lemma 1.6 and $(\xi, \beta) \in \xi_1 \times \xi_3$, we obtain that

$$\begin{aligned}\Delta_1(\mu_n, \nu_n) = max\{&d^*(\hat{T}\mu_n, \hat{S}\nu_n), d^*(\hat{T}\mu_n, \hat{P}\nu_n), d^*(\hat{S}\nu_n, \hat{Q}\nu_n),\\ &\frac{1}{2}[d^*(\hat{P}\mu_n, \hat{S}\nu_n) + d^*(\hat{Q}\nu_n, \hat{T}\mu_n)],\\ &\frac{d^*(\hat{P}\mu_n, \hat{T}\mu_n).d^*(\hat{Q}\nu_n, \hat{S}nu_n)}{1 + d^*(\hat{T}\mu_n, \hat{S}\nu_n)}, \frac{d^*(\hat{P}\mu_n, \hat{S}\nu_n).d^*(\hat{Q}\nu_n, \hat{T}\mu_n)}{1 + d^*(\hat{T}\mu_n, \hat{S}\nu_n)},\\ &d^*(\hat{T}\mu_n, \hat{P}\mu_n)[\frac{1 + d^*(\hat{T}\mu_n, \hat{Q}\nu_n) + d^*(\hat{S}\nu_n, \hat{P}\mu_{2n})}{1 + d^*(\hat{T}\mu_{2n}, \hat{P}\mu_n) + d^*(\hat{S}\nu_n, \hat{Q}\nu_n)}]\}.\end{aligned}$$

Taking limit as $n \to +\infty$, we get

$$\begin{aligned}\lim_{n\to+\infty} \Delta_1(\mu_n\nu_n) &= max\{d^*(\hat{P}c, d), d^*(\hat{P}c, \hat{P}c), d^*(d, \hat{P}c), \frac{1}{2}[d^*(\hat{P}c, d) + d^*(\hat{P}c, \hat{P}c)],\\ &\frac{d^*(\hat{P}c, \hat{P}c).d^*(\hat{P}c, d)}{1 + d^*(\hat{P}c, d)}, \frac{d^*(\hat{P}c, d).d^*(\hat{P}c, \hat{P}c)}{1 + d^*(\hat{P}c, d)},\\ &d^*(\hat{P}c, \hat{P}c)[\frac{1 + d^*(\hat{P}c, d) + d^*(d, \hat{P}c)}{1 + d^*(\hat{P}c, \hat{P}c) + d^*(d, \hat{P}c)}]\}.\\ &= max\{d^*(\hat{P}c, d), 0, d^*(d, \hat{P}c), \frac{1}{2}[d^*(\hat{P}c, d) + 0], 0, 0, 0\}.\\ &= d^*(\hat{P}c, d).\end{aligned}$$

And

$$0 \;<\; \int_0^{d^*(\hat{P}c,d)} \xi(t)dt$$

$$
\begin{aligned}
&= \limsup_{n\to+\infty} \int_0^{d^*(\hat{T}\mu_n,\hat{S}\nu_n)} \xi(t)dt \\
&\leq \limsup_{n\to+\infty}[\beta d^*(\mu_n,\nu_n)] \limsup_{n\to+\infty} \int_0^{\Delta_1(\mu_n,\nu_n)} \xi(t)dt \\
&< \int_0^{d^*(\hat{P}c,d)} \xi(t)dt,
\end{aligned}
$$

which is ridiculous. Thus $\hat{P}c = d$, that is,

$$\lim_{n\to+\infty} \hat{S}\nu_n = \hat{P}c.$$

Subsequently, we have

$$\lim_{n\to+\infty} \hat{S}\nu_n = \lim_{n\to+\infty} \hat{P}\mu_n = \lim_{n\to+\infty} \hat{T}\mu_n = \lim_{n\to+\infty} \hat{Q}\nu_n = \hat{P}c = d.$$

Now, we show that $\hat{T}c = d$.
Let, if possible $\hat{T}c \neq d$.
Now, in the light of (2.4), Lemma 1.6 and $(\xi,\beta) \in \xi_1 \times \xi_3$, we obtain that

$$
\begin{aligned}
\Delta_1(c,\nu_n) = max\{&d^*(\hat{T}c,\hat{S}\nu_n), d^*(\hat{T}c,\hat{P}\nu_n), d^*(\hat{S}\nu_n,\hat{Q}\nu_n), \\
&\frac{1}{2}[d^*(\hat{P}c,\hat{S}\nu_n) + d^*(\hat{Q}\nu_n,\hat{T}c)], \\
&\frac{d^*(\hat{P}c,\hat{T}c).d^*(\hat{Q}\nu_n,\hat{S}nu_n)}{1+d^*(\hat{T}c,\hat{S}\nu_n)}, \frac{d^*(\hat{P}c,\hat{S}\nu_n).d^*(\hat{Q}\nu_n,\hat{T}c)}{1+d^*(\hat{T}c,\hat{S}\nu_n)}, \\
&d^*(\hat{T}c,\hat{P}c)[\frac{1+d^*(\hat{T}c,\hat{Q}\nu_n)+d^*(\hat{S}\nu_n,\hat{P}c)}{1+d^*(\hat{T}c,\hat{P}c)+d^*(\hat{S}\nu_n,\hat{Q}\nu_n)}]\}.
\end{aligned}
$$

Taking limit as $n \to +\infty$, *we get*

$$
\begin{aligned}
\lim_{n\to+\infty} \Delta_1(c,\nu_n) &= max\{d^*(\hat{T}c,d), d^*(\hat{T}c,d), d^*(d,d), \frac{1}{2}[d^*(d,d)+d^*(d,\hat{T}c)], \\
&\frac{d^*(d,\hat{T}c).d^*(d,d)}{1+d^*(\hat{T}c,d)}, \frac{d^*(d,d).d^*(d,\hat{T}c)}{1+d^*(\hat{T}c,d)}, d^*(\hat{T}c,d)[\frac{1+d^*(\hat{T}c,d)+d^*(d,d)}{1+d^*(\hat{T}c,d)+d^*(d,d)}]\}. \\
&= max\{d^*(\hat{T}c,d), d^*(\hat{T}c,d), 0, \frac{1}{2}[0+d^*(d,\hat{T}c)], 0, 0, d^*(\hat{T}c,d)\}. \\
&= d^*(\hat{T}c,d).
\end{aligned}
$$

And

$$0 \;<\; \int_0^{d^*(\hat{T}c,d)} \xi(t)dt$$

$$
\begin{aligned}
&= \limsup_{n\to+\infty}\int_0^{d^*(\hat{T}c,\hat{S}\nu_n)}\xi(t)dt \\
&\leq \limsup_{n\to+\infty}[\beta d^*(c,\nu_n)]\limsup_{n\to+\infty}\int_0^{\Delta_1(c,\nu_n)}\xi(t)dt \\
&< \int_0^{d^*(\hat{T}c,d)}\xi(t)dt,
\end{aligned}
$$

a contradiction. Thus $\hat{T}c = d = \hat{P}c$. Since the pair $(\hat{P},\hat{T})$ is weakly compatible, it follows that $\hat{T}d = \hat{P}d$.
Also, since $\hat{T}M \subseteq \hat{Q}M$, then, there exists a in M , such that $\hat{T}c = \hat{Q}a$, that is $\hat{Q}a = d$.
Now, we show that $\hat{S}a = d$. Let, if possible $\hat{S}a \neq d$.
Now, in the light of (2.4), Lemma 1.6 and $(\xi,\beta) \in \xi_1 \times \xi_3$, we obtain that

$$
\begin{aligned}
\Delta_1(\mu_n,a) = max\{ & d^*(\hat{T}\mu_n,\hat{S}a), d^*(\hat{T}\mu_n,\hat{P}\mu_n), d^*(\hat{S}a,\hat{Q}a), \\
& \frac{1}{2}[d^*(\hat{P}\mu_n,\hat{S}a) + d^*(\hat{Q}a,\hat{T}\mu_n)], \\
& \frac{d^*(\hat{P}\mu_n,\hat{T}\mu_n).d^*(\hat{Q}a,\hat{S}a)}{1+d^*(\hat{T}\mu_n,\hat{S}a)}, \frac{d^*(\hat{P}\mu_n,\hat{S}a).d^*(\hat{Q}a,\hat{T}\mu_n)}{1+d^*(\hat{T}\mu_n,\hat{S}a)}, \\
& d^*(\hat{T}\mu_n,\hat{P}\mu_n)[\frac{1+d^*(\hat{T}\mu_n,\hat{Q}a)+d^*(\hat{S}a,\hat{P}\mu_n)}{1+d^*(\hat{T}\mu_n,\hat{P}\mu_n)+d^*(\hat{S}a,\hat{Q}a)}]\}.
\end{aligned}
$$

Taking limit as $n \to +\infty$, we get

$$
\begin{aligned}
\lim_{n\to+\infty}\Delta_1(\mu_n,a) &= max\{d^*(d,\hat{S}a), d^*(d,d), d^*(\hat{S}a,d), \frac{1}{2}[d^*(d,\hat{S}a)+d^*(d,d)], \\
&\frac{d^*(d,d).d^*(d,\hat{S}a)}{1+d^*(d,\hat{S}a)}, \frac{d^*(d,\hat{S}a).d^*(d,d)}{1+d^*(d,\hat{S}a)}, d^*(d,d)[\frac{1+d^*(d,d)+d^*(\hat{S}a,d)}{1+d^*(d,d)+d^*(\hat{S}a,d)}]\}. \\
&= max\{d^*(d,\hat{S}a), 0, d^*(\hat{S}a,d), \frac{1}{2}[d^*(d,\hat{S}a)+0], 0, 0, 0\}. \\
&= d^*(d,\hat{S}a).
\end{aligned}
$$

And

$$
\begin{aligned}
0 &< \int_0^{d^*(d,\hat{S}a)}\xi(t)dt \\
&= \limsup_{n\to\infty}\int_0^{d^*(\hat{T}\mu_n,\hat{S}a)}\xi(t)dt
\end{aligned}
$$

$$\begin{aligned}
&\leq \limsup_{n\to+\infty}[\beta d^*(\mu_n, a)] \limsup_{n\to+\infty} \int_0^{\Delta_1(\mu_n,a)} \xi(t)dt \\
&< \int_0^{d^*(d,\hat{S}a)} \xi(t)dt,
\end{aligned}$$

a contradiction. Thus $\hat{S}a = d = \hat{Q}a$.
Since the pair $(\hat{Q}, \hat{S})$ is weakly compatible, it follows that $\hat{S}d = \hat{Q}d$ Now, we show that $\hat{S}d = \hat{T}d$. Let, if possible $\hat{S}d \neq \hat{T}d$.
Now, in the view of (2.4), and $(\xi, \beta) \in (\xi_1 \times \xi_3)$, we obtain that

$$\begin{aligned}
\Delta_1(d,d) = max\{&d^*(\hat{T}d, \hat{S}d), d^*(\hat{T}d, \hat{P}d), d^*(\hat{S}d, \hat{Q}d), \frac{1}{2}[d^*(\hat{P}d, \hat{S}d) + d^*(\hat{Q}d, \hat{T}d)], \\
&\frac{d^*(\hat{P}d, \hat{T}d).d^*(\hat{Q}d, \hat{S}d)}{1 + d^*(\hat{T}d, \hat{S}d)}, \frac{d^*(\hat{P}d, \hat{S}d).d^*(\hat{Q}d, \hat{T}d)}{1 + d^*(\hat{T}d, \hat{S}d)}, \\
&d^*(\hat{T}d, \hat{P}d)[\frac{1 + d^*(\hat{T}d, \hat{Q}d) + d^*(\hat{S}d, \hat{P}d)}{1 + d^*(\hat{T}d, \hat{P}d) + d^*(\hat{S}d, \hat{Q}d)}]\}. \\
= max\{&d^*(\hat{T}d, \hat{S}d), d^*(\hat{T}d, \hat{T}d), d^*(\hat{S}d, \hat{S}d), \frac{1}{2}[d^*(\hat{T}d, \hat{S}d) + d^*(\hat{S}d, \hat{T}d)], \\
&\frac{d^*(\hat{T}d, \hat{T}d).d^*(\hat{S}d, \hat{S}d)}{1 + d^*(\hat{T}d, \hat{S}d,)}, \frac{d^*(\hat{T}d, \hat{S}d).d^*(\hat{S}d, \hat{T}d)}{1 + d^*(\hat{T}d, \hat{S}d)}, \\
&d^*(\hat{T}d, \hat{T}d)[\frac{1 + d^*(\hat{T}d, \hat{S}d) + d^*(\hat{S}d, \hat{T}d)}{1 + d^*(\hat{T}d, \hat{T}d) + d^*(\hat{S}d, \hat{S}d)}]\}. \\
= max\{&d^*(\hat{T}d, \hat{S}d), 0, 0, d^*(\hat{T}d, \hat{S}d), 0, \frac{d^*(\hat{T}d, \hat{S}d).d^*(\hat{S}d, \hat{T}d)}{1 + d^*(\hat{T}d, \hat{S}d)}, 0\}. \\
= d^*&(\hat{T}d, \hat{S}d).
\end{aligned}$$

And

$$\begin{aligned}
0 &< \int_0^{d^*(\hat{T}d,\hat{S}d)} \xi(t)dt \\
&\leq \beta d^*(d,d) \int_0^{\Delta_1(d,d)} \xi(t)dt \\
&= \beta(d^*(d,d)) \int_0^{d^*(\hat{T}d,\hat{S}d)} \xi(t)dt \\
&< \int_0^{d^*(\hat{T}d,\hat{S}d)} \xi(t)dt,
\end{aligned}$$

a contradiction. Hence $\hat{T}d = \hat{S}d$. Thus $\hat{P}d = \hat{Q}d = \hat{S}d = \hat{T}d$.
Now, we show that $d = \hat{S}d$. Let, if possible $d \neq \hat{S}d$.
Now, in the view of (2.4), and $(\xi, \beta) \in (\xi_1 \times \xi_3)$, we obtain that

$$\begin{aligned}
\Delta_1(d,d) = max\{&d^*(\hat{T}c,\hat{S}d), d^*(\hat{T}c,\hat{P}c), d^*(\hat{S}d,\hat{Q}d), \frac{1}{2}[d^*(\hat{P}c,\hat{S}d) + d^*(\hat{Q}d,\hat{T}c)], \\
&\frac{d^*(\hat{P}c,\hat{T}c).d^*(\hat{Q}d,\hat{S}d)}{1+d^*(\hat{T}c,\hat{S}d)}, \frac{d^*(\hat{P}c,\hat{S}d).d^*(\hat{Q}d,\hat{T}c)}{1+d^*(\hat{T}c,\hat{S}d)}, \\
&d^*(\hat{T}c,\hat{P}c)[\frac{1+d^*(\hat{T}c,\hat{Q}d)+d^*(\hat{S}d,\hat{P}c)}{1+d^*(\hat{T}c,\hat{P}c)+d^*(\hat{S}d,\hat{Q}d)}]\}. \\
= max\{&d^*(d,\hat{S}d), d^*(d,d), d^*(\hat{S}d,\hat{S}d), \frac{1}{2}[d^*(d,\hat{S}d) + d^*(\hat{S}d,d)], \\
&\frac{d^*(d,d).d^*(\hat{S}d,\hat{S}d)}{1+d^*(d,\hat{S}d,)}, \frac{d^*(d,\hat{S}d).d^*(\hat{S}d,d)}{1+d^*(d,\hat{S}d)}, \\
&d^*(d,d)[\frac{1+d^*(d,\hat{S}d)+d^*(\hat{S}d,d)}{1+d^*(d,d)+d^*(\hat{S}d,\hat{S}d)}]\}. \\
= max\{&d^*(d,\hat{S}d), 0, 0, d^*(d,\hat{S}d), 0, \frac{d^*(d,\hat{S}d).d^*(\hat{S}d,d)}{1+d^*(d,\hat{S}d)}, 0\}. \\
= d^*&(d,\hat{S}d).
\end{aligned}$$

And

$$\begin{aligned}
0 &< \int_0^{d^*(d,\hat{S}d)} \xi(t)dt \\
&= \int_0^{d^*(\hat{T}c,\hat{S}d)} \xi(t)dt \\
&\leq \beta d^*(c,d) \int_0^{\Delta_1(c,d)} \xi(t)dt \\
&= \beta(d^*(c,d)) \int_0^{d^*(d,\hat{S}d)} \xi(t)dt \\
&< \int_0^{d^*(d,\hat{S}d)} \xi(t)dt,
\end{aligned}$$

a contradiction. Hence $d = \hat{S}d$, which shows that $d = \hat{P} = \hat{Q}d = \hat{S}d = \hat{T}d$. This shows that d is the common fixed point of $\hat{P}, \hat{Q}, \hat{S}$ and $\hat{T}$. Now, we shall prove the

uniqueness of common fixed point. Suppose that $\hat{P}$, $\hat{Q}, \hat{S}$ and $\hat{T}$ have two fixed points r and s such that $r \neq s$.
Now, in the view of (2.4) and $(\xi, \beta) \in \xi_1 \times \xi_3$, we obtain that

$$\begin{aligned}
\Delta_1(r,s) &= max\{d^*(\hat{T}r,\hat{S}s), d^*(\hat{T}r,\hat{P}r), d^*(\hat{S}s,\hat{Q}s), \frac{1}{2}[d^*(\hat{P}r,\hat{S}s) + d^*(\hat{Q}s,\hat{T}r)],\\
&\quad \frac{d^*(\hat{P}r,\hat{T}r).d^*(\hat{Q}s,\hat{S}s)}{1+d^*(\hat{T}r,\hat{S}s)}, \frac{d^*(\hat{P}r,\hat{S}s).d^*(\hat{Q}s,\hat{T}r)}{1+d^*(\hat{T}r,\hat{S}s)},\\
&\quad d^*(\hat{T}r,\hat{P}r)[\frac{1+d^*(\hat{T}r,\hat{Q}s)+d^*(\hat{S}r,\hat{P}r)}{1+d^*(\hat{T}r,\hat{P}r)+d^*(\hat{S}s,\hat{Q}s)}]\}.\\
&= max\{d^*(r,s), d^*(r,r), d^*(s,s), \frac{1}{2}[d^*(r,s)+d^*(s,r)],\\
&\quad \frac{d^*(r,r).d^*(s,s)}{1+d^*(r,s)}, \frac{d^*(r,s).d^*(s,r)}{1+d^*(r,s)}, d^*(r,r)[\frac{1+d^*(r,s)+d^*(r,r)}{1+d^*(r,r)+d^*(s,s)}]\}.\\
&= max\{d^*(r,s), 0, 0, d^*(r,s), 0, \frac{d^*(r,s).d^*(s,r)}{1+d^*(r,s)}, 0\}.\\
&= d^*(r,s).
\end{aligned}$$

And

$$\begin{aligned}
0 &< \int_0^{d^*(r,s)} \xi(t)dt\\
&= \int_0^{d^*(\hat{T}r,\hat{S}s)} \xi(t)dt\\
&\leq \beta d^*(r,s) \int_0^{\Delta_1(r,s)} \xi(t)dt\\
&= \beta(d^*(r,s)) \int_0^{d^*(\hat{T}r,s)} \xi(t)dt\\
&< \int_0^{d^*(r,s)} \xi(t)dt,
\end{aligned}$$

which is a contradiction. Hence $\hat{P}, \hat{Q}, \hat{S}$ and $\hat{T}$ have a unique common fixed point.

Example 2.4. Let $\hat{P}, \hat{Q}, \hat{S}$ and $\hat{T}$ be self mappings on M. $M = \mathbb{R}^+$ be endowed with the Euclidean metric $\Delta_1(r,s) = \|r-s\|$ for all $r, s \in M$. Let $\beta : \mathbb{R}^+ \to [0,1), \xi : \mathbb{R}^+ \to \mathbb{R}^+$ and $\hat{P}, \hat{Q}, \hat{S}$ and $\hat{T}$ are defined by

$\hat{P}r = \frac{r}{3} + \frac{2}{3}, \quad \hat{Q}r = r^2, \quad \hat{S}r = 1,$

$$\hat{T}r = \begin{cases} 1 & \text{if } r \in M - \{\frac{1}{6}\} \\ \frac{35}{36} & \text{if } r = \frac{1}{6} \end{cases}$$

$\beta(t) = \frac{1+2t}{3+4t}, \quad \xi(t) = 3t$, for all $t \in \mathbb{R}$.
Clearly, $\hat{T}M \subseteq \hat{Q}M, \hat{S}M \subseteq \hat{P}M$, also $\beta(t) \in [\frac{1}{3}, \frac{1}{2})$.
Since $\hat{P}\hat{Q}(1) = \hat{Q}\hat{P}(1) = 1$, implies that the pair $(\hat{P}, \hat{Q})$ is weakly compatible and $\hat{S}\hat{T}(1) = \hat{T}\hat{S}(1) = 1$, implies that the pair $(\hat{S}, \hat{T})$ is weakly compatible. Hence (2.1)-(2.3) satisfied.
Now, we check condition (2.4). For this we have two cases:
Case 1. $r \in M - \{\frac{1}{6}\}, s \in \mathbb{R}^+$. It is easy to see that

$$\int_0^{d^*(\hat{T}r,\hat{S}s)} \xi(t)dt \leq [\beta d^*(r,s) \int_0^{\Delta_1(r,s)} \xi(t)dt].$$

Case 2.$r = \frac{1}{6}$, clearly

$$\Delta_1(r,s) \geq d^*(\hat{P}r, \hat{T}r) = |\frac{13}{18} - \frac{35}{36}| = \frac{1}{4}.$$

$$\begin{aligned} \int_0^{d^*(\hat{T}r,\hat{S}s)} \xi(t)dt &= \int_0^{\frac{1}{36}} \xi(t)dt \\ &= \frac{1}{864} \\ &< \frac{1}{3}.\frac{3}{32} \\ &\leq \beta d^*(\frac{1}{6}, s) \int_0^{\frac{1}{4}} \xi(t)dt \\ &\leq \beta d^*(r,s) \int_0^{\Delta_1(r,s)} \xi(t)dt. \end{aligned}$$

Hence, (2.4) holds. Also, $\lim_{n\to+\infty} \hat{P}\mu_n = \lim\limits_{n\to+\infty} \frac{1}{3}(\frac{n+1}{1}) + \frac{2}{3} = 1$
And $\lim_{n\to+\infty} \hat{T}\mu_n = 1$.
Hence $\lim_{n\to+\infty} \hat{P}\mu_n = \lim_{n\to+\infty} \hat{T}\mu_n$ implies that $(\hat{P}, \hat{T})$ satisfies the E.A. property.

Also, we can easily see that $\lim_{n\to+\infty} \hat{Q}\mu_n = \lim_{n\to+\infty} \hat{S}\mu_n = 1$ which shows that the pair $(\hat{Q}, \hat{S})$ satisfies the E.A. property.
$\lim_{n\to+\infty} \hat{P}\mu_n = \lim_{n\to+\infty} \hat{T}\mu_n = 1 = \hat{P}(1)$ implies that $(\hat{P}, \hat{T})$ satisfies the $(CLR_{\hat{P}})$ property.
Also, we can easily see that $\lim_{n\to+\infty} \hat{Q}\mu_n = \lim_{n\to+\infty} \hat{S}\mu_n = 1 = \hat{Q}(1)$ which shows that the pair $(\hat{Q}, \hat{S})$ satisfies the $(CLR_{\hat{Q}})$ property.

Hence all the conditions of Theorems 2.1, 2.2 and 2.3 are satisfied. Hence $\hat{P}, \hat{Q}, \hat{S}$ and $\hat{T}$ have a unique common fixed point. Here, 1 is the common fixed point.

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