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GENERALIZED FIXED POINT RESULT OF BANACH AND KANNAN TYPE IN S-MENGER SPACES

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Abstract: S-metric space is a relatively new concept in the literature and currently there is much attention being given to the generalization of S-metric spaces and fixed point theory in these spaces. Recently, the concept of S-Menger spaces was introduced in the literature as a generalization of both S-metric spaces and Menger spaces. Combinations of Banach and Kannan type contractions are very much important to find fixed point results and there are very few works on S-metric spaces that includes both of these type contractions. In this paper, we present a fixed point result in S-Menger spaces that includes both Banach type contractions and Kannan type contractions. We have also deduced some corollaries from our result and provided examples to validate our work.

Keywords and Phrases: S-metric space, Menger space, S-Menger space, Cauchy sequence, fixed point, t-norm.

2020 Mathematics Subject Classification: 47H10, 54H25, 54E70.

1. Introduction

The study of fixed point result originated in the literature due to Banach [3] for contractive type mappings. Since then, the theory has been extended in various directions. Some researchers have generalized contractive conditions, while the others have focused on the underlying space for the improvement of the theory. [6, 8, 12, 15, 27] and [28] are some examples of generalization of the space. In [11] Kannan introduced another type of contraction, commonly known as Kannan type contraction, which has also been an area of active research in fixed point theory.

Recently $D^\star$-metric spaces [28] had been used as a generalization of D-metric spaces [6]. Mustafa and Sims [15] proposed a generalization of metric spaces which is known as G-metric spaces. In [27] Sedghi et al. introduced the notion of S-metric spaces as a generalized version of both G-metric spaces and $D^\star$-metric spaces. Some of the recent works dealing with these spaces may be noted in [1, 4, 7, 9, 13, 16, 17, 19, 20, 21, 25, 26, 29] and [31].

Probabilistic generalization of metric spaces was proposed by K. Menger [12] in 1942 which was further extended by Schweizer and Sklar [23]. With the help of t-norm this spaces was further extended to Menger spaces. A comprehensive study of this spaces may be noted in [10] and [24]. The generalization of this space and the study of fixed points in this space continue to be an active area of research. Some recent references may be noted in [2, 14, 18].

Recently, in [22] S-Menger spaces were introduced as a probabilistic generalization of S-metric spaces by the present authors. Some basic properties of these spaces were discussed and a fixed point theorem was proved in that paper. In the present paper we have proved a fixed point result using combinations of Banach and Kannan type contraction in S-Menger spaces with the help of Ψ-function. Some corollaries has been deduced and examples are provided in support of our result.

2. Definitions and Mathematical Preliminaries

In this section, we have provided some preliminary definitions and lemmas along with some examples that are necessary for our main theorem.

Definition 2.1. S-metric Space [27]
Let X be a nonempty set. An S-metric on X is a function $S : X^3 \to [0, \infty)$ that satisfies the following conditions, for each $x, y, z, a \in X$,

(i) $S(x, y, z) \geq 0$,

(ii) $S(x, y, z) = 0$ *if and only if* $x = y = z$,

(iii) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.
The pair (X, S) is called an S-metric space.

Definition 2.2. [10, 24] *A mapping $F : \mathbb{R} \to \mathbb{R}^+$ is called a distribution function if it is non-decreasing and left continuous with $\inf_{t\in\mathbb{R}} F(t) = 0$ and $\sup_{t\in\mathbb{R}} F(t) = 1$, where $\mathbb{R}$ is the set of real numbers and $\mathbb{R}^+$ denotes the set of non-negative real numbers.*

Definition 2.3. t-norm [10, 24]
A t-norm is a function $T : [0, 1] \times [0, 1] \to [0, 1]$ which satisfies the following conditions

(i) $T(1, a) = a$,

(ii) $T(a, b) = T(b, a)$,

(iii) $T(c, d) \geq T(a, b)$ *whenever* $c \geq a$ *and* $d \geq b$,

(iv) $T(T(a, b), c) = T(a, T(b, c))$.

Definition 2.4. n-th order t-norm [30]
A mapping $T : [0, 1]^n \to [0, 1]$ is called n-th order t-norm if the following conditions are satisfied:

(i) $T(0, 0, 0, ..., 0,) = 0, T(a, 1, 1, ..., 1) = a$, *for all* $a \in [0, 1]$,

(ii) $T(a_1, a_2, a_3, ..., a_n) = T(a_2, a_1, a_3, ..., a_n) = T(a_2, a_3, a_1, ..., a_n)$
$..... = T(a_2, a_3, a_4, ..., a_n, a_1)$,

(iii) $a_i \geq b_i, i = 1, 2, 3...., n$ *implies* $T(a_1, a_2, a_3, ..., a_n) \geq T(b_1, b_2, b_3, ..., b_n)$

(iv) $T(T(a_1, a_2, a_3, ..., a_n), b_2, b_3, ..., b_n)$
$= T(a_1, T(a_2, a_3, ..., a_n, b_2), b_3, ..., b_n)$
$= T(a_1, a_2, T(a_3, a_4......, a_n, b_2, b_3), b_4....., b_n)$
....
$= T(a_1, a_2, a_3, ..., a_{n-1}, T(a_n, b_2, b_3, ..., b_n))$.

When $n = 2, 3$ then we have binary t-norm and 3-rd order t-norm respectively.

Definition 2.5. Menger space [10, 24]
A Menger space is a triplet (X, F, Δ) where X is a non empty set, F is a function defined on $X \times X$ to the set of distribution functions and Δ is a t-norm, such that the following are satisfied:

(i) $F_{x,y}(0) = 0$ *for all* $x, y \in X$,

(ii) $F_{x,y}(s) = 1$ *for all* $s > 0$ *and* $x, y \in X$ *if and only if* $x = y$,

(iii) $F_{x,y}(s) = F_{y,x}(s)$ *for all* $x, y \in X, s > 0$ *and*

(iv) $F_{x,y}(u + v) \geq \Delta(F_{x,z}(u), F_{z,y}(v))$ *for all* $u, v \geq 0$ *and* $x, y, z \in X$.

Now we go through the definition of S-Menger space, which was introduced by the present authors in [22].

Definition 2.6. S-Menger space [22]
The 3-tuple (X, F, T) *is said to be* S*-Menger space if* X *is a non-empty set,* F *is a functions defined on* X^3 *to the set of distribution function and* T *is a continuous third order t-norm such that the following conditions are satisfied:*

(i) $F_{x,y,z}(0) = 0$ *for all* $x, y, z \in X$,

(ii) $F_{x,x,y}(t) < 1$ *for* $t > 0$ *with* $x \neq y$,

(iii) $F_{x,y,z}(t) = 1$ *for all* $t > 0$, *if and only if* $x = y = z$,

(iv) $F_{x,y,z}(t) \geq T(F_{x,x,a}(t_1), F_{y,y,a}(t_2), F_{z,z,a}(t_3))$,
where $t = t_1 + t_2 + t_3$ *and* $t, t_1, t_2, t_3 > 0$, *for all* $x, y, z, a \in X$.

Example 2.7. Let $X = \{x_1, x_2, x_3, x_4\}$, $T(a, b, c)=$ min $\{a, b, c\}$, that is T is the 3rd order minimum t-norm and $F_{x,y,z}(t)$ be defined as,

(a) $$F_{x_1,x_2,x_3}(t) = F_{x_2,x_1,x_3}(t) = F_{x_2,x_4,x_1}(t) = F_{x_4,x_2,x_1}(t) = \begin{cases} 0, & \text{if } \ t \leq 0, \\ 0.65, & \text{if } 0 < t \leq 5, \\ 1, & \text{if } \ t > 5. \end{cases}$$

(b) $$F_{x_1,x_2,x_4}(t) = F_{x_2,x_1,x_4}(t) = F_{x_1,x_4,x_2}(t)$$
$$= F_{x_4,x_1,x_2}(t) = F_{x_3,x_4,x_2}(t) = F_{x_4,x_3,x_2}(t) = \begin{cases} 0, & \text{if } \ t \leq 0, \\ 0.60, & \text{if } 0 < t \leq 5, \\ 1, & \text{if } \ t > 5. \end{cases}$$

(c) $$F_{x_1,x_3,x_2}(t) = F_{x_3,x_1,x_2}(t) = F_{x_2,x_3,x_4}(t) = F_{x_3,x_2,x_4}(t) = \begin{cases} 0, & \text{if } \ t \leq 0, \\ 0.55, & \text{if } 0 < t \leq 5, \\ 1, & \text{if } \ t > 5. \end{cases}$$

(d) $$F_{x_1,x_3,x_4}(t) = F_{x_3,x_1,x_4}(t) = F_{x_1,x_4,x_3}(t) = F_{x_4,x_1,x_3}(t) = \begin{cases} 0, & \text{if } \ t \leq 0, \\ 0.75, & \text{if } 0 < t \leq 5, \\ 1, & \text{if } \ t > 5. \end{cases}$$

(e) $F_{x_2,x_3,x_1}(t) = F_{x_3,x_2,x_1}(t) = F_{x_2,x_4,x_3}(t) = F_{x_4,x_2,x_3}(t) = \begin{cases} 0, & \text{if} \quad t \leq 0, \\ 0.70, & \text{if } 0 < t \leq 5, \\ 1, & \text{if} \quad t > 5. \end{cases}$

(f) $F_{x_3,x_4,x_1}(t) = F_{x_4,x_3,x_1}(t) = \begin{cases} 0, & \text{if} \quad t \leq 0, \\ 0.80, & \text{if } 0 < t \leq 5, \\ 1, & \text{if} \quad t > 5. \end{cases}$

(g) $F_{x_1,x_1,x_2}(t) = F_{x_2,x_2,x_1}(t) = \begin{cases} 0, & \text{if} \quad t \leq 0, \\ 0.50, & \text{if } 0 < t \leq 5, \\ 1, & \text{if} \quad t > 5. \end{cases}$

(h) $F_{x_1,x_1,x_3}(t) = F_{x_3,x_3,x_1}(t) = \begin{cases} 0, & \text{if} \quad t \leq 0, \\ 0.55, & \text{if } 0 < t \leq 2, \\ 1, & \text{if} \quad t > 2. \end{cases}$

(i) $F_{x_1,x_1,x_4}(t) = F_{x_4,x_4,x_1}(t) = \begin{cases} 0, & \text{if} \quad t \leq 0, \\ 0.70, & \text{if } 0 < t \leq 4, \\ 1, & \text{if} \quad t > 4. \end{cases}$

(j) $F_{x_2,x_2,x_3}(t) = F_{x_3,x_3,x_2}(t) = \begin{cases} 0, & \text{if} \quad t \leq 0, \\ 0.50, & \text{if } 0 < t < 7, \\ 1, & \text{if} \quad t \geq 7. \end{cases}$

(k) $F_{x_2,x_2,x_4}(t) = F_{x_4,x_4,x_2}(t) = \begin{cases} 0, & \text{if} \quad t \leq 0, \\ 0.50, & \text{if } 0 < t < 6, \\ 1, & \text{if} \quad t \geq 6. \end{cases}$

(l) $F_{x_3,x_3,x_4}(t) = F_{x_4,x_4,x_3}(t) = \begin{cases} 0, & \text{if} \quad t \leq 0, \\ 0.55, & \text{if } 0 < t \leq 3, \\ 1, & \text{if} \quad t > 3. \end{cases}$

For any $x, y \in X$ we see that $F_{x,x,y}(t) = F_{y,y,x}(t) < 1$, where $x \neq y$. If $x = y$ then $F_{x,x,x}(t) = 1$ for all $x \in X, t > 0$. We can verified that (X, F, T) is a S-Menger space.

Some more examples of S-Menger space :

Example 2.8. Let $X = \mathbb{R}$, $F_{x,y,z}(t)$ be defined as

$$F_{x,y,z}(t) = \frac{t}{t+|y-z|+|z-x|},$$

for all $x, y, z \in X$, $t \geq 0$ and $\Delta = 3^{rd}$ order minimum t-norm. Then (X, F, Δ) is an S-Menger space.

Example 2.9. Let $X = \mathbb{R}^n$ and $\|.\|$ a norm on X, $F_{x,y,z}(t)$ be defined as

$$F_{x,y,z}(t) = \frac{t}{t+\|y+z-2x\|+\|y-z\|},$$

for all $x, y, z \in X$, $t \geq 0$ and $\Delta = 3^{rd}$ order minimum t-norm. Then (X, F, Δ) is an S-Menger space.

Definition 2.10. *Let (X, F, T) be an S-Menger space. A sequence $\{x_n\} \subset X$ is said to be converge to some point $x \in X$ if given $\epsilon > 0, \lambda > 0$ we find a positive integer $N_{\epsilon,\lambda}$ such that for all $n > N_{\epsilon,\lambda}$*

$$F_{x_n,x_n,x}(\epsilon) \geq 1 - \lambda. \tag{2.1}$$

Definition 2.11. *Let (X, F, T) be an S-Menger space. A sequence $\{x_n\}$ is said to be Cauchy sequence if $F_{x_n,x_n,x_{n+p}}(t) \to 1$ as $n \to \infty$ for $p = 1, 2, 3....$ for each $t > 0$.*

Definition 2.12. *Let (X, F, T) be an S-Menger space. A sequence $\{x_n\}$ is said to be G-Cauchy sequence in X if given $\epsilon > 0, \lambda > 0$ there exists a positive integer $N_{\epsilon,\lambda}$ such that*

$$F_{x_n,x_n,x_m}(\epsilon) \geq 1 - \lambda, \quad \text{for all } m, n > N_{\epsilon,\lambda}. \tag{2.2}$$

Definition 2.13. *An S-Menger space (X, F, T) is said to be complete if every Cauchy sequence is convergent in X.*

We now propose some lemmas in S-Menger spaces which will be needed for our main theorem.

Lemma 2.14. *In every S-Menger space (X, F, T), where T is a continuous third order t-norm we have $F_{x,x,y}(t) = F_{y,y,x}(t)$ for $x, y \in X$ and for all $t > 0$.*

Proof. From definition of S-Menger space we can write, for all $t > 0$ and $x, y \in X$.

$$F_{x,x,y}(t) \geq T(F_{x,x,x}(\epsilon), F_{x,x,x}(\epsilon), F_{y,y,x}(t - 2\epsilon)).$$
$$= T(1, 1, F_{y,y,x}(t - 2\epsilon)).$$

Taking limit $\epsilon \to 0$, we have,

$$F_{x,x,y}(t) \geq T(1, 1, F_{y,y,x}(t)).$$
$$F_{x,x,y}(t) \geq F_{y,y,x}(t)). \tag{2.3}$$

Similarly we can prove,

$$F_{y,y,x}(t) \geq F_{x,x,y}(t)). \tag{2.4}$$

From the relations (2.3) and (2.4) we can write,
Therefore, $F_{x,x,y}(t) = F_{y,y,x}(t)$, for all $t > 0$ and $x, y \in X$.

Lemma 2.15. *Let (X, F, T) be an S-Menger space where T is a continuous third order t-norm. If the sequence $\{x_n\}$ in X converges to x, then $\{x_n\}$ is a Cauchy sequence.*

Proof. Since $\lim_{n\to\infty} x_n = x$, then for each $\lambda \in (0, 1)$ there exist $n_0 \in N$, for all $t > 0$ and $n, m = n + p \geq n_0 \in N$ where p is the positive integer, such that,

$$F_{x_n,x_n,x}(t) \geq 1 - \lambda$$

and

$$F_{x_m,x_m,x}(t) \geq 1 - \lambda.$$

Using triangular inequality we have,

$F_{x_n,x_n,x_m}(t) \geq T(F_{x_n,x_n,x}(\epsilon), F_{x_n,x_n,x}(\epsilon), F_{x_m,x_m,x}(t-2\epsilon)).$
Taking limit $\epsilon \to 0$ and $n \to \infty$ we get,
$F_{x_n,x_n,x_m}(t) \to 1.$
Therefore, $\{x_n\}$ is a Cauchy sequence.

Lemma 2.16. *Let (X, F, T) be an S-Menger space and T is continuous third order t-norm. If t is a continuity point of $F_{x,x,y}(.)$ and sequence $\{x_n\}$ and $\{y_n\}$ are sequences in X, converging to x, y (respectively) then,*
$\lim_{n\to\infty} F_{x_n,x_n,y_n}(t) = F_{x,x,y}(t).$

Proof. Let $x, y \in X$, $t > 0$ continuity point of $F_{x,x,y}(.)$ and $\{x_n\}, \{y_n\}$ are sequences in X, converging to x, y (respectively). Then for every $0 < \epsilon < \frac{t}{4}$, we can write,

$$\begin{aligned} F_{x_n,x_n,y_n}(t) &\geq T(F_{x_n,x_n,x}(\epsilon), F_{x_n,x_n,x}(\epsilon), F_{y_n,y_n,x}(t-2\epsilon)), \\ &\geq T(F_{x_n,x_n,x}(\epsilon), F_{x_n,x_n,x}(\epsilon), T(F_{y_n,y_n,y}(\epsilon), F_{y_n,y_n,y}(\epsilon), F_{x,x,y}(t-4\epsilon))). \end{aligned}$$

For $n \to \infty$ we obtain,

$$\lim_{n\to\infty} F_{x_n,x_n,y_n}(t) \geq T(1, 1, T(1, 1, F_{x,x,y}(t-4\epsilon))) \geq F_{x,x,y}(t-4\epsilon). \tag{2.5}$$

Again we have,

$$\begin{aligned} F_{x,x,y}(t+4\epsilon) &\geq T(F_{x,x,x_n}(\epsilon), F_{x,x,x_n}(\epsilon), F_{y,y,x_n}(t+2\epsilon)), \\ &\geq T(F_{x,x,x_n}(\epsilon), F_{x,x,x_n}(\epsilon), T(F_{y,y,y_n}(\epsilon), F_{y,y,y_n}(\epsilon), F_{x_n,x_n,y_n}(t))). \end{aligned}$$

For $n \to \infty$ we can write,

$$F_{x,x,y}(t+4\epsilon) \geq T(1, 1, T(1, 1, \lim_{n\to\infty} F_{x_n,x_n,y_n}(t))) \geq \lim_{n\to\infty} F_{x_n,x_n,y_n}(t). \tag{2.6}$$

From the relations (2.5) and (2.6) we have,

$$F_{x,x,y}(t-4\epsilon) \leq \lim_{n\to\infty} F_{x_n,x_n,y_n}(t) \leq F_{x,x,y}(t+4\epsilon).$$

Since t is a continuity point of $F_{x,x,y}(.)$, the result follows.

Lemma 2.17. *The limit of convergent sequence in an S-Menger space (X, F, T), where T is a continuous third order t-norm, is unique.*

Proof. Let the sequence $\{x_n\}$ converges to x and y in the S-Menger space (X, F, T).
Therefore, for all $t > 0$,

$$\lim_{n\to\infty} F_{x,x,x_n}(t) = 1,$$

and

$$\lim_{n\to\infty} F_{y,y,x_n}(t) = 1.$$

Now for every $t_1, t_2, t_3, t > 0$ with $t = t_1 + t_2 + t_3$ we have, as $n \to \infty$,

$F_{x,x,x_n}(t_1) \to 1$
$F_{x,x,x_n}(t_2) \to 1$
$F_{y,y,x_n}(t_3) \to 1.$

The conclusion follows from the definition of S-Menger space that is,

$F_{x,x,y}(t) = F_{x,x,y}(t_1 + t_2 + t_3) \geq T(F_{x,x,x_n}(t_1), F_{x,x,x_n}(t_2), F_{y,y,x_n}(t_3))$.
Taking limit $n \to \infty$, we have from above inequality,
$F_{x,x,y}(t) \geq T(1, 1, 1) = 1$.
Therefore, $x = y$, hence the proof of the lemma is completed.

Lemma 2.18. *Let (X, F, T) be an S-Menger space with continuous third order t-norm T. Then $F_{x,x,y}(t)$ is non-decreasing with respect to t, for all $x, y \in X$.*
Proof. From the definition of S-Menger space we can write,
$F_{x,x,y}(t) \geq T(F_{x,x,a}(t_1), F_{x,x,a}(t_2), F_{y,y,a}(t_3))$,
where $t, t_1, t_2, t_3 > 0$ with $t = t_1 + t_2 + t_3$, and $x, y, a \in X$.
If we take $a = x$, then we get from the inequality,
$F_{x,x,y}(t) \geq T(F_{x,x,x}(t_1), F_{x,x,x}(t_2), F_{y,y,x}(t_3))$,
$F_{x,x,y}(t) \geq T(1, 1, F_{y,y,x}(t_3))$.
So, $F_{x,x,y}(t) \geq F_{y,y,x}(t_3)$, [where $t > t_3$]
Therefore, F is non-decreasing with respect to $t > 0$.
We now give the definition of Ψ-functions. This type of functions will be used in our result.

Definition 2.19. Ψ-function [5]
A function $\psi : [0, 1]^4 \to [0, 1]$ is said to be a Ψ-function if

(i) $\psi(0, 0, 0, 0) = 0$, and $\psi(1, 1, 1, 1) = 1$

(ii) ψ-is monotone increasing and continuous,

(iii) $\psi(x, x, x, x) > x$ for all $0 < x < 1$.

An example of this type of Ψ-function is
$\psi(x_1, x_2, x_3, x_4) = \frac{a\sqrt{x_1}+b\sqrt{x_2}+c\sqrt{x_3}+d\sqrt{x_4}}{a+b+c+d}$ where a, b, c and d are positive real numbers.

3. Main Results

We now present our main theorem, which utilizes the Ψ-function.

Theorem 3.1. *Let (X, F, T) be a complete S-Menger space where T is minimum third ordered t-norm and $f : X \to X$ be a mapping satisfying the following conditions:*
there are $k \in (0, 1)$ and $\psi \in \Psi$, such that,

$$F_{fx,fy,fz}(t) \geq \psi(F_{x,y,z}(\tfrac{t}{k}), F_{x,x,fx}(\tfrac{t}{k}), F_{y,y,fy}(\tfrac{t}{k}), F_{z,z,fz}(\tfrac{t}{k})) \quad (3.1)$$

for all $x, y, z \in X, t > 0$ and

$$F_{x,x,fz}(t) > 0, \forall x, z \in X \text{ and for all } t > 0.$$

Then f has a unique fixed point.
Proof. Let x_0 be any arbitrary point in X. Now we define the sequence $\{x_n\}$ in X

by $x_n = fx_{n-1},\ n \in \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers. We are supposing $x_n \neq x_{n-1}, \forall n \in \mathbb{N}$, otherwise existence of a fixed point is obvious.
Taking, $x = x_{n-1}, y = x_{n-1}$ and $z = x_n$ we can write from (3.1),
$F_{fx_{n-1},fx_{n-1},fx_n}(t) \geq \psi(F_{x_{n-1},x_{n-1},x_n}(\frac{t}{k}), F_{x_{n-1},x_{n-1},fx_{n-1}}(\frac{t}{k}), F_{x_{n-1},x_{n-1},fx_{n-1}}(\frac{t}{k}),$
$F_{x_n,x_n,fx_n}(\frac{t}{k})),$
So,
$F_{x_n,x_n,x_{n+1}}(t) \geq \psi(F_{x_{n-1},x_{n-1},x_n}(\frac{t}{k}), F_{x_{n-1},x_{n-1},x_n}(\frac{t}{k}), F_{x_{n-1},x_{n-1},x_n}(\frac{t}{k}), F_{x_n,x_n,x_{n+1}}(\frac{t}{k})).$(3.2)
We claim that, $F_{x_n,x_n,x_{n+1}}(\frac{t}{k}) \geq F_{x_{n-1},x_{n-1},x_n}(\frac{t}{k})$, for all $t > 0$ and $n \geq 1$.
If not, let for some $t_1 > 0$, $n \geq 1$, $F_{x_n,x_n,x_{n+1}}(\frac{t_1}{k}) \neq 1$ and

$$F_{x_n,x_n,x_{n+1}}(\tfrac{t_1}{k}) < F_{x_{n-1},x_{n-1},x_n}(\tfrac{t_1}{k}).$$

Then we can write from (3.2),

$$\begin{aligned} F_{x_n,x_n,x_{n+1}}(t_1) &\geq \psi(F_{x_n,x_n,x_{n+1}}(\tfrac{t_1}{k}), F_{x_n,x_n,x_{n+1}}(\tfrac{t_1}{k}), F_{x_n,x_n,x_{n+1}}(\tfrac{t_1}{k}), F_{x_n,x_n,x_{n+1}}(\tfrac{t_1}{k}). \\ &> F_{x_n,x_n,x_{n+1}}(\tfrac{t_1}{k}) \text{ [by the properties of } \psi \text{ function].} \\ &\geq F_{x_n,x_n,x_{n+1}}(t_1), \text{ which is a contradiction, as } 0 < k < 1. \end{aligned}$$

Therefore, for all $t > 0$ and $n \geq 1$,

$$F_{x_n,x_n,x_{n+1}}(\tfrac{t}{k}) \geq F_{x_{n-1},x_{n-1},x_n}(\tfrac{t}{k}). \tag{3.3}$$

Now for $t > 0$, using (3.3) in (3.2) we can get,

$$\begin{aligned} F_{x_n,x_n,x_{n+1}}(t) &\geq \psi(F_{x_{n-1},x_{n-1},x_n}(\tfrac{t}{k}), F_{x_{n-1},x_{n-1},x_n}(\tfrac{t}{k}), F_{x_{n-1},x_{n-1},x_n}(\tfrac{t}{k}), F_{x_{n-1},x_{n-1},x_n}(\tfrac{t}{k})). \\ &> F_{x_{n-1},x_{n-1},x_n}(\tfrac{t}{k})). \end{aligned}$$

By repeated use of above inequality we can get, for all $t > 0$,

$$\begin{aligned} F_{x_n,x_n,x_{n+1}}(t) &\geq (F_{x_{n-1},x_{n-1},x_n}(\tfrac{t}{k}) \\ &\geq (F_{x_{n-2},x_{n-2},x_{n-1}}(\tfrac{t}{k^2}) \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &\geq (F_{x_0,x_0,x_1}(\tfrac{t}{k^n}). \end{aligned} \tag{3.4}$$

Now for $t > 0$ and $n \to \infty$ we have from (3.4),

$$\lim_{n\to\infty} F_{x_n,x_n,x_{n+1}}(t) = 1. \tag{3.5}$$

We now claim that $\{x_n\}$ is a Cauchy sequence. If not then there exists $\epsilon > 0, \lambda > 0$ and subsequences $\{x_{m(r)}\}$ and $\{x_{n(r)}\}$ such that $n(r) > m(r) > r$ with $r > N_1$ [where N_1 is a positive integer], we have

$$F_{x_{m(r)},x_{m(r)},x_{n(r)}}(\epsilon) < 1 - \lambda. \tag{3.6}$$

$$F_{x_{m(r)},x_{m(r)},x_{n(r)-1}}(\epsilon) \geq 1 - \lambda. \tag{3.7}$$

From (3.6) we can write,

$$\begin{aligned} 1 - \lambda > F_{x_{m(r)},x_{m(r)},x_{n(r)}}(\epsilon) &= F_{fx_{m(r)-1},fx_{m(r)-1},fx_{n(r)-1}}(\epsilon) \\ &\geq \psi(F_{x_{m(r)-1},x_{m(r)-1},x_{n(r)-1}}(\tfrac{\epsilon}{k}), F_{x_{m(r)-1},x_{m(r)-1},x_{m(r)}}(\tfrac{\epsilon}{k}), \\ &\quad F_{x_{m(r)-1},x_{m(r)-1},x_{m(r)}}(\tfrac{\epsilon}{k}), F_{x_{n(r)-1},x_{n(r)-1},x_{n(r)}}(\tfrac{\epsilon}{k}). \end{aligned} \tag{3.8}$$

By (3.5) for $0 < \lambda_1 < \lambda < 1$, it is possible to find a positive integer N_2 such that for all $r > N_2$,

$$F_{x_{m(r)-1},x_{m(r)-1},x_{m(r)}}(\tfrac{\epsilon}{k}) \geq 1 - \lambda_1. \tag{3.9}$$

$$F_{x_{n(r)-1},x_{n(r)-1},x_{n(r)}}(\tfrac{\epsilon}{k}) \geq 1-\lambda_1. \tag{3.10}$$

As $0 < k < 1$, we have $\frac{\epsilon}{k} > \epsilon$ and make a choice of positive numbers β_1 and β_2 such that $\frac{\epsilon}{k} \geq \epsilon + \beta_1 + \beta_2$.

Now,

$$\begin{aligned} F_{x_{m(r)-1},x_{m(r)-1},x_{n(r)-1}}(\tfrac{\epsilon}{k}) &\geq F_{x_{m(r)-1},x_{m(r)-1},x_{n(r)-1}}(\beta_1+\beta_2+\epsilon) \\ &\geq T(F_{x_{m(r)-1},x_{m(r)-1},x_{m(r)}}(\beta_1), F_{x_{m(r)-1},x_{m(r)-1},x_{m(r)}}(\beta_2), \\ &\qquad F_{x_{n(r)-1},x_{n(r)-1},x_{m(r)}}(\epsilon)) \\ &= T(F_{x_{m(r)-1},x_{m(r)-1},x_{m(r)}}(\beta_1), F_{x_{m(r)-1},x_{m(r)-1},x_{m(r)}}(\beta_2), \\ &\qquad F_{x_{m(r)},x_{m(r)},x_{n(r)-1}}(\epsilon)). \end{aligned} \tag{3.11}$$

Let $0 < \lambda_2 < \lambda < 1$ be chosen. Then by (3.5) there exists a positive integer N_3 such that for all $r > N_3$,

$$F_{x_{m(r)-1},x_{m(r)-1},x_{m(r)}}(\beta_1) \geq 1-\lambda_2. \tag{3.12}$$

$$F_{x_{m(r)-1},x_{m(r)-1},x_{m(r)}}(\beta_2) \geq 1-\lambda_2. \tag{3.13}$$

Using (3.7), (3.12) and (3.13) in (3.11) for all $r > \max\{N_1, N_2, N_3\}$,

$$F_{x_{m(r)-1},x_{m(r)-1},x_{n(r)-1}}(\tfrac{\epsilon}{k}) \geq T(1-\lambda_2, 1-\lambda_2, 1-\lambda).$$

As $0 < \lambda_2 < \lambda < 1$ and T is min. t-norm, we have,

$$T(1-\lambda_2, 1-\lambda_2, 1-\lambda) = 1-\lambda.$$

$$\therefore F_{x_{m(r)-1},x_{m(r)-1},x_{n(r)-1}}(\tfrac{\epsilon}{k}) \geq 1-\lambda. \tag{3.14}$$

Using (3.9), (3.10) and (3.14) in (3.8) we get,

$$\begin{aligned} 1-\lambda > F_{x_{m(r)},x_{m(r)},x_{n(r)}}(\epsilon) &\geq \psi(1-\lambda, 1-\lambda_1, 1-\lambda_1, 1-\lambda_1), \\ &\geq \psi(1-\lambda, 1-\lambda, 1-\lambda, 1-\lambda) \text{ [since } 0 < \lambda_1 < \lambda < 1], \\ &> 1-\lambda \text{ [by the properties of } \psi] \end{aligned}$$

which is a contradiction.

Hence $\{x_n\}$ is a Cauchy sequence.

Since (X, F, T) be a complete S-Menger space, so $\lim_{n\to\infty} x_n = u$ for some $u \in X$. We now show that $fu = u$.

First we show that $F_{u,u,fu}(t) = 1$ for every continuity point $t > 0$ of $F_{u,u,fu}(.)$. Indeed, let us suppose that for some continuity point $t > 0$ of $F_{u,u,fu}(.)$ we have, $F_{u,u,fu}(t) < 1$. Then $0 < F_{u,u,fu}(t) < 1$. Putting $x = x_n, y = x_n$ and $z = u$ in inequality (3.1) we can write,

$$F_{fx_n,fx_n,fu}(t) \geq \psi(F_{x_n,x_n,u}(\tfrac{t}{k}), F_{x_n,x_n,fx_n}(\tfrac{t}{k}), F_{x_n,x_n,fx_n}(\tfrac{t}{k}), F_{u,u,fu}(\tfrac{t}{k})),$$

$$F_{x_{n+1},x_{n+1},fu}(t) \geq \psi(F_{x_n,x_n,u}(\tfrac{t}{k}), F_{x_n,x_n,x_{n+1}}(\tfrac{t}{k}), F_{x_n,x_n,x_{n+1}}(\tfrac{t}{k}), F_{u,u,fu}(\tfrac{t}{k})).$$

Limiting $n \to \infty$ both the side of the above inequality we get,

$$F_{u,u,fu}(t) \geq \psi(F_{u,u,u}(\tfrac{t}{k}), F_{u,u,u}(\tfrac{t}{k}), F_{u,u,u}(\tfrac{t}{k}), F_{u,u,fu}(\tfrac{t}{k})).$$

$$\begin{aligned} F_{u,u,fu}(t) &\geq \psi(1, 1, 1, F_{u,u,fu}(\tfrac{t}{k})) \\ &\geq \psi(F_{u,u,fu}(\tfrac{t}{k}), F_{u,u,fu}(\tfrac{t}{k}), F_{u,u,fu}(\tfrac{t}{k}), F_{u,u,fu}(\tfrac{t}{k})) \\ &> F_{u,u,fu}(\tfrac{t}{k}) \text{ [by the properties of } \psi] \\ &\geq F_{u,u,fu}(t), \text{ which is a contradiction, as } 0 < k < 1. \end{aligned}$$

Hence, $F_{u,u,fu}(t) = 1$ for every continuity point $t > 0$ of $F_{u,u,fu}(.)$. As the set of discontinuity points of $F_{u,u,fu}$ is at most countable, the equality $F_{u,u,fu}(t) = 1$ actually holds for all $t > 0$. This implies $fu = u$.

For uniqueness, if possible let u and v be two distinct fixed points of f. Then $F_{u,u,v}(t) < 1$ for some $t > 0$, and

$$\begin{aligned} F_{u,u,v}(t) &= F_{fu,fu,fv}(t) \\ &\geq \psi(F_{u,u,v}(\tfrac{t}{k}), F_{u,u,fu}(\tfrac{t}{k}), F_{u,u,fu}(\tfrac{t}{k}), F_{v,v,fv}(\tfrac{t}{k})) \\ &\geq \psi(F_{u,u,v}(\tfrac{t}{k}), F_{u,u,u}(\tfrac{t}{k}), F_{u,u,u}(\tfrac{t}{k}), F_{v,v,v}(\tfrac{t}{k})) \\ &\geq \psi(F_{u,u,v}(\tfrac{t}{k}), 1, 1, 1) \\ &\geq \psi(F_{u,u,v}(\tfrac{t}{k}), F_{u,u,v}(\tfrac{t}{k}), F_{u,u,v}(\tfrac{t}{k}), F_{u,u,v}(\tfrac{t}{k})) \\ &> F_{u,u,v}(\tfrac{t}{k}) \text{ [by the properties of } \psi] \\ &\geq F_{u,u,v}(t), \text{ again we arrived at a contradiction, as } 0 < k < 1. \end{aligned}$$

Therefore, $v = u$. By this proof of the theorem is completed.

From theorem 3.1. we get the following corollaries.

Corollary 3.2. *Let (X, S, T) be a complete S-Menger space where T is minimum third ordered t-norm and $f : X \to X$ be a self mapping satisfying the following conditions :*
there is $k \in (0, 1)$ such that,

$$F_{fx,fx,fy}(t) \geq \psi(F_{x,x,y}(\tfrac{t}{k}), F_{x,x,fx}(\tfrac{t}{k}), F_{y,y,fy}(\tfrac{t}{k})) \tag{3.15}$$

for all $x, y \in X, t > 0$, where ψ is a Ψ-function and

$$F_{x,x,fy}(t) > 0, \forall x, y \in X \text{ and for all } t > 0.$$

Then f has a unique fixed point.

Corollary 3.3. *Let (X, S, T) be a complete S-Menger space where T is minimum third ordered t-norm and $f : X \to X$ be a self mapping satisfying the following conditions :*
there is $k \in (0, 1)$ such that,

$$F_{fx,fx,fy}(t) \geq \psi(F_{x,x,y}(\tfrac{t}{k}), F_{y,y,fy}(\tfrac{t}{k})) \tag{3.16}$$

for all $x, y \in X, t > 0$, where ψ is a Ψ-function and

$$F_{x,x,fy}(t) > 0, \forall x, y \in X \text{ and for all } t > 0.$$

Then f has a unique fixed point.

Now we give some examples which validate Theorem 3.1.

Example 3.4. Let $X = [0, 1]$, F be defined as $F_{x,y,z}(t) = e^{-\frac{|x-y|+|y-z|+|z-x|}{t}}$ where all $x, y, z \in X$, $t > 0$, $T(a, b, c) = \min\{a, b, c\}$ then (X, F, T) is a complete S-Menger space. Let us take the self mapping $f : X \to X$, defined by $fx = \frac{x}{7}$. Then taking $\frac{2}{3} < k < 1$ and $\psi(x_1, x_2, x_3, x_4) = \frac{min\{x_1,x_2\}+(x_3)^{\frac{1}{2}}\times(x_4)^{\frac{1}{4}}}{2}$, f satisfies all the conditions of Theorem: 3.1 and we have $x = 0$ is the unique fixed point of f.

Example 3.5. Let $X = [0, 1]$, F be defined as $F_{x,y,z}(t)=e^{-\frac{|x-z|+|y-z|}{t}}$ where all $x, y, z \in X$, $t > 0$, $T(a, b, c)$ =min $\{a, b, c\}$ then (X, F, T) is a complete S-Menger space. Let us take the self mapping $f : X \to X$, defined by $fx = \frac{x}{25}$. Then taking $\frac{1}{6} < k < 1$ and $\psi(x_1, x_2, x_3, x_4) = \frac{min\{x_1,x_2\}+(x_3)^{\frac{1}{2}}\times(x_4)^{\frac{1}{4}}}{2}$, f satisfies all the conditions of Theorem: 3.1 and we have $x = 0$ is the unique fixed point of f.

4. Conclusion

S-Metric spaces and their extension, S-Menger spaces are relatively new concepts, which were introduced by the present authors. In this study, we have proved a fixed point theorem in S-Menger spaces, which extends and generalizes some existing fixed point results in the literature. Our result can also be applied to other spaces such as b-metric spaces, intuitionistic fuzzy metric spaces, Menger spaces, as well as non-linear programming and game theory. We have used the minimum t-norm to establish our result and it leads us to further investigate which types of t-norms can replace the minimum t-norm in our main theorem. This theorem is also an example of the use of Ψ-function in finding fixed point results.

References

[1] Afra J. M., Double contraction in S-metric spaces, Int. J. Math. Anal., 9 (3) (2015), 117-125.

[2] Agnihotri Swati, Dubey K. K. and Gupta V. K., Common fixed point of compatible type (K) mappings in fuzzy metric space, South East Asian J. of Math. & Math. Sc., 18 (2) (2022), 245-258.

[3] Banach S., Sur les operations dans les ensembles abstraits el leur application aux equations integrals, Fund. Math., 3 (1922), 133-181.

[4] Chaipornjareansri S., Fixed point theorems for F_w-contractions in complete S-metric spaces, Thai Journal of Mathematics, 2016 (2016), 98-109.

[5] Choudhury B. S. and Das K., Fixed points of generalized Kannan type mappings in generalized Menger spaces, Commoun. Korean Maths. Soc., 24 (2009), 529-537.

[6] Dhage B. C., A study of some fixed point theorem, Ph. D. thesis, Marathwada University, Aurangabad, India, 1984.

[7] Ege M. E. and Alaca C., Some properties of modular S-metric spaces and its fixed point results, J. Comput. Anal. Appl., 20 (1) (2016), 24-34.

[8] Gahler S., 2-metriche raume und ihre topologische strukture, Math. Nachr., 26 (1963), 115-148.

[9] Gholidahneh A., Sedghi S., Dosenovic T. and Radenovic S., Ordered S-metric spaces and coupled common fixed point theorems of integral type contraction, Mathematics Interdisciplinary Research, 2 (2017), 71-84.

[10] Hazic O. and Pap E., Fixed Point Theory in Probabilistic Metric Spaces, Kluwer Academic Publisher, 2001.

[11] Kannan R., Some results on fixed points, Bulletin of the Calcutta Mathematical Society, 60 (1968), 71-76.

[12] Menger K., Statistical metrices, Proc. Natl. Acad. Sci., USA, 28 (1942), 535-537.

[13] Mlaiki N., Özgür N. Y. and Taş N., New fixed-point theorems on an S-metric space via simulation functions, Σ Mathematics, 7 (2019), 583.

[14] Mohanraj R. and Malliga Devi V., Fixed point of Kannan results in fuzzy metric spaces, South East Asian J. of Math. & Math. Sc., 17 (2) (2021), 97-106.

[15] Mustafa Z. and Sims B., A new approach to generalized metric space, Journal of Nonlinear Convex Analysis, 7 (2006), 289-297.

[16] Özgür N. Y. and Taş N., Some new contractive mappings on S-metric spaces and their relationships with the mapping, Mathematical Science, 11 (1) (2017), 7-16.

[17] Özgür N. Y. and Taş N., Some fixed point theorems on S-metric spaces, Matematicki Vesnik, 69 (1) (2017), 39-52.

[18] Poovaragavan D. and Jeyaraman M., Common fixed point theorems in right complete dislocated quasi G-fuzzy metric spaces, South East Asian J. of Math. & Math. Sc., 18(1) (2022), 235-246.

[19] Rajesh C. and Baskaran B., Some results in fixed point theorems for asymptotically regular mappings in G-matric spaces, South East Asian J. of Math. & Math. Sc., 14 (2) (2018), 45-50.

[20] Reena and Singh Balbir, Common fixed point theorems for non compatible mappings in G-metric space, South East Asian J. of Math. & Math. Sc., 18 (1) (2022), 247-256.

[21] Saluja G. S., Fixed point theorems for cyclic contractions in S-metric spaces involving C-class function, Mathematica Moravica, 26 (1) (2022), 57-76.

[22] Sarkar K. K., Das K. and Pramanik A., Fixed point result in S-Menger spaces, Zeichen Journal, 7 (9) (2021), 186-196.

[23] Schweizer B. and Sklar A., Statistical metric spaces, Pacific J. Math., 10 (1960), 313-334.

[24] Schweizer B. and Sklar A., Probabilistic Metric Spaces, Elsevier, North-Holland, (1983).

[25] Sedghi S., Altun I., Shobe N. and Salahshour M. A., Some properties of S-metric spaces and fixed point results, Kyungpook Math. J., 54 (2014), 113-122.

[26] Sedghi S. and Dung N. V., Fixed point theorems on S-metric spaces, Matematicki Vesnik, 66 (1) (2014), 113-124.

[27] Sedghi S., Shobe N. and Aliouche A., A generalization of fixed point theorems in S-metric spaces, Matematicki Vesnik, 64 (3) (2012), 258-266.

[28] Sedghi S., Shobe N. and Zhou H., A common fixed point theorem in $D^\star$-metric spaces, Fixed Point Theory and Applications, Vol. 2007, Article ID 27906, 13 pages.

[29] Sedghi S., Shobkolaei N., Shahraki M. and Dosenovic T., Common fixed point of four maps in S-metric spaces, Mathematical Sciences, 12 (2018), 137-143.

[30] Shi Y., Ren L. and Wang X., The extension of fixed point theorems for set valued mapping, J. Appl. Math. & Computing, 13 (2003), 277-286.

[31] Taş N. and Özgür N. Y., Common fixed points of continuous mappings on S-metric spaces, Mathematical Notes, 104 (2018), 587-600.