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LAPLACE-SUMUDU INTEGRAL TRANSFORM ON TIME SCALES

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Abstract: In this paper we have extended double Laplace-Sumudu transform for time scales which can be applied to solve partial-integro dynamic equations and partial dynamic equations on time scales.

Keywords and Phrases: Laplace transform, Sumudu transform, time scales, dynamic equations.

2020 Mathematics Subject Classification: 26E70, 44A35, 26A33.

1. Introduction

Integral transforms have variety of applications as they convert differential and integral equations to more simpler algebraic expressions that can be solved easily [2, 8, 9]. Generalization of various integral transform have done for time scales $\mathbb{T}$ [4, 6, 7]. Initially for a function $f : \mathbb{T} \to \mathbb{C}$ Bohner and Peterson [5] have defined Laplace transform on time scale as

$$\mathscr{L}\{f\}(z) = \int_0^\infty e^\sigma_{\ominus z}(t, 0) f(t)\ \Delta t.$$

Further in 2012 Hassan Agwa, Fatma Abdelfatah Ali and Adem Kilicman [1] have generalized Sumudu transform on time scales as

$$\mathscr{S}\{f\}(u) = \frac{1}{u}\int_{t_0}^{\infty} e^{\sigma}_{\ominus\frac{1}{u}}(t,t_0)f(t)\ \Delta t.$$

Some classical integral transform are combined and are used to solve linear and non-linear fractional differential equations. In [2] authors have defined double Laplace-Sumudu Integral transform as

$$\mathscr{L}_x\mathscr{S}_t\{\phi(x,t)\} = \frac{1}{\sigma}\int_0^{\infty}\int_0^{\infty} e^{-\rho x - t/\sigma}\phi(x,t)dxdt.$$

In this paper we have extended double Laplace-Sumudu transform for time scale which can be applied to solve partial integro-dynamic equations on time scales.

Firstly we will recall concepts which we are going to use further.

Definition 1.1. [5] *The function $f:\mathbb{T}\to\mathbb{C}$ is called rd-continuous if it is continuous at right-dense points in $\mathbb{T}$ and left sided limit exists at left-dense points in $\mathbb{T}$.*

Definition 1.2. [5] *A function $f:\mathbb{T}\to\mathbb{C}$ is called regressive provided*

$$1+\mu(t)f(t)\neq 0 \ \text{ for all } \ t\in\mathbb{T}^k$$

Definition 1.3. [5] *Let $h>0$, $\mathbb{C}_h := \left\{z\in\mathbb{C} : z\neq -\frac{1}{h}\right\}$ is the set of Hilger complex numbers and for $z\in\mathbb{C}$ the Hilger real part of z is $\mathscr{R}e_h(z) := \frac{|zh+1|-1}{h}$.*

2. Main Results

In this section we extent classical definition of Laplace-Sumudu transform introduced in [2] as follows.

Definition 2.1. *Let $\mathbb{T}_1$ and $\mathbb{T}_2$ are time scales such that suprimum of both $\mathbb{T}_1$ and $\mathbb{T}_2$ is ∞ then for fixed $t_0\in\mathbb{T}_1$, $t_0'\in\mathbb{T}_2$ we define the Laplace-Sumudu transform of an rd-continuous function $f(t_1,t_2):\mathbb{T}_1\times\mathbb{T}_2\to\mathbb{C}$* as

$$\mathscr{L}_{t_1}\mathscr{S}_{t_2}\big[f(t_1,t_2)\big] = F(s,p) = \frac{1}{p}\int_{t_0}^{\infty}\int_{t_0'}^{\infty} e^{\sigma_1\sigma_2}_{\ominus s\ominus\frac{1}{p}}(t_1,t_2,t_0,t_0')f(t_1,t_2)\ \Delta t_1\ \Delta t_2$$

We have extended definition 1.2 in [7] as follows

Definition 2.2. *The function $f(t_1,t_2):\mathbb{T}_1\times\mathbb{T}_2\to\mathbb{C}$ is said to be of exponential type I if there exists constants $\mathcal{M}, c_1, c_2>0$ such that $|f(t_1,t_2)| \leq \mathcal{M}\ e^{c_1t_1+c_2t_2}$. Further f is said to be of exponential type II if there exists constants $\mathcal{M}, c_1, c_2, >0$*

such that $|f(t_1,t_2)| \leq \mathcal{M}\ e_{c_1\oplus c_2}(t_1,t_2,t_0,t_0')$

Theorem 2.1. [**Existence Theorem**] *If* $f(t_1,t_2):\mathbb{T}_1\times\mathbb{T}_2\to\mathbb{C}$ *is an rd-continuous function on every finite intervals* $\mathbb{T}_1\cap(t_0,\lambda_1)$ *and* $\mathbb{T}_2\cap(t_0',\lambda_2)$ *and is of exponential type II, then Laplace-Sumudu transform of* $f(t_1,t_2)$ *exists for all regressive* s *and* $\frac{1}{p}$ *provided,* $\mathscr{R}e_{\mu_1}(s)>c_1,\ \mathscr{R}e_{\mu_2}(\frac{1}{p})>c_2$

Proof.

$$\begin{aligned}
\big|\mathscr{L}_{t_1}\mathscr{S}_{t_2}\big[f(t_1,t_2)\big]\big| &= \left|\frac{1}{p}\int_{t_0}^{\infty}\int_{t_0'}^{\infty} e^{\sigma_1\sigma_2}_{\ominus s\ominus\frac{1}{p}}(t_1,t_2,t_0,t_0')\ f(t_1,t_2)\ \Delta t_1\ \Delta t_2\right| \\
&\leq \frac{1}{p}\int_{t_0}^{\infty}\int_{t_0'}^{\infty} e^{\sigma_1\sigma_2}_{\ominus s\ominus\frac{1}{p}}(t_1,t_2,t_0,t_0')\ \big|f(t_1,t_2)\big|\ \Delta t_1\ \Delta t_2 \\
&\leq \frac{1}{p}\int_{t_0}^{\infty}\int_{t_0'}^{\infty} e^{\sigma_1\sigma_2}_{\ominus s\ominus\frac{1}{p}}(t_1,t_2,t_0,t_0')\ \mathcal{M}e_{c_1\oplus c_2}(t_1,t_2,t_0,t_0')\ \Delta t_1\ \Delta t_2 \\
&\leq \frac{\mathcal{M}}{p}\int_{t_0}^{\infty}\int_{t_0'}^{\infty} \frac{e_{\ominus s\ominus\frac{1}{p}}(t_1,t_2,t_0,t_0')}{\big(1+\mu_1 s\big)\big(1+\mu_2\frac{1}{p}\big)}\ e_{c_1\oplus c_2}(t_1,t_2,t_0,t_0')\ \Delta t_1\ \Delta t_2 \\
&= \mathcal{M}\int_{t_0}^{\infty}\frac{e_{c_1\ominus s}(t_1,t_0)}{\big(1+\mu_1 s\big)}\ \Delta t_1\ \cdot\frac{1}{p}\int_{t_0'}^{\infty}\frac{e_{c_2\ominus\frac{1}{p}}(t_2,t_0')}{\big(1+\mu_2\frac{1}{p}\big)}\ \Delta t_2 \\
&= \frac{\mathcal{M}}{\big(s-c_1\big)\big(1-c_2 p\big)}
\end{aligned}$$

provided $\lim\limits_{t_1\to\infty} e_{c_1\ominus s}(t_1,t_0)\to 0$ and $\lim\limits_{t_2\to\infty} e_{c_2\ominus\frac{1}{p}}(t_2,t_0')\to 0$ with $\mathscr{R}e_{\mu_1}(s)>c_1,\ \ \mathscr{R}e_{\mu_2}(\frac{1}{p})>c_2$

We give Laplace-Sumudu transform of some elementary functions on time scales in tabular form.

$f(t_1,t_2)$	1	$h_n(t_1,t_0)h_m(t_2,t_0')$	$e_{a\oplus b}(t_1,t_2,t_0,t_0')$	$e_{i(a\oplus b)}(t_1,t_2,t_0,t_0')$	$e_{-i(a\oplus b)}(t_1,t_2,t_0,t_0')$
$\mathscr{L}_{t_1}\mathscr{S}_{t_2}\big[f(t_1,t_2)\big]$	$\frac{1}{s}$	$\frac{p^m}{s^{n+1}}$	$\frac{1}{(s-a)(1-bp)}$	$\frac{1}{(s-ia)(1-ibp)}$	$\frac{1}{(s+ia)(1+ibp)}$

$f(t_1,t_2)$	$sin_{a\oplus b}(t_1,t_2,t_0,t_0')$	$cos_{a\oplus b}(t_1,t_2,t_0,t_0')$	$sinh_{a\oplus b}(t_1,t_2,t_0,t_0')$	$cosh_{a\oplus b}(t_1,t_2,t_0,t_0')$
$\mathscr{L}_{t_1}\mathscr{S}_{t_2}\big[f(t_1,t_2)\big]$	$\frac{a+bsp}{(s^2+a^2)(1+b^2p^2)}$	$\frac{s-abp}{(s^2+a^2)(1+b^2p^2)}$	$\frac{(a+bsp)}{(s^2-a^2)(1-b^2p^2)}$	$\frac{(s+abp)}{(s^2-a^2)(1-b^2p^2)}$

3. Basic Derivative Properties

Following derivative properties are useful for the applications of Laplace-Sumudu transform.

Theorem 3.1. *Let* $f(t_1,t_2):\mathbb{T}_1\times\mathbb{T}_2\to\mathbb{C}$ *is an rd-continuous function such that*

$$f^{\Delta_1}(t_1,t_2)=\frac{\partial f(t_1,t_2)}{\Delta_1 t_1},\quad f^{\Delta_1^2}(t_1,t_2)=\frac{\partial^2 f(t_1,t_2)}{\Delta_1 t_1^2},$$

$$f^{\Delta_2}(t_1,t_2)=\frac{\partial f(t_1,t_2)}{\Delta_2 t_2},\quad f^{\Delta_2^2}(t_1,t_2)=\frac{\partial^2 f(t_1,t_2)}{\Delta_2 t_2^2},$$

are also rd-continuous then

(1) $\mathscr{L}_{t_1}\mathscr{S}_{t_2}\big[f^{\Delta_1}(t_1,t_2)\big]=s\ \mathscr{L}_{t_1}\mathscr{S}_{t_2}\big[f(t_1,t_2)\big]-\mathscr{S}_{t_2}\big[f(t_0,t_2)\big]$

(2) $\mathscr{L}_{t_1}\mathscr{S}_{t_2}\big[f^{\Delta_2}(t_1,t_2)\big]=\frac{1}{p}\mathscr{L}_{t_1}\mathscr{S}_{t_2}\big[f(t_1,t_2)\big]-\frac{1}{p}\ \mathscr{L}_{t_1}\big[f(t_1,t_0')\big]$

(3) $\mathscr{L}_{t_1}\mathscr{S}_{t_2}\big[f^{\Delta_1^2}(t_1,t_2)\big]=s^2\ \mathscr{L}_{t_1}\mathscr{S}_{t_2}\big[f(t_1,t_2)\big]-s\ \mathscr{S}_{t_2}\big[f(t_0,t_2)\big]-\mathscr{S}_{t_2}\big[f^{\Delta_1}(t_0,t_2)\big]$

(4) $\mathscr{L}_{t_1}\mathscr{S}_{t_2}\big[f^{\Delta_2^2}(t_1,t_2)\big]=\frac{1}{p^2}\mathscr{L}_{t_1}\mathscr{S}_{t_2}\big[f(t_1,t_2)\big]-\frac{1}{p^2}\mathscr{L}_{t_1}\big[f(t_1,t_0')\big]-\frac{1}{p}\mathscr{L}_{t_1}\big[f^{\Delta_2}(t_1,t_0')\big]$

Proof.

(1) $\mathscr{L}_{t_1}\mathscr{S}_{t_2}\big[f^{\Delta_1}(t_1,t_2)\big]=s\ \mathscr{L}_{t_1}\mathscr{S}_{t_2}\big[f(t_1,t_2)\big]-\mathscr{S}_{t_2}\big[f(t_0,t_2)\big]$

$$\begin{aligned}
&\mathscr{L}_{t_1}\mathscr{S}_{t_2}\big[f^{\Delta_1}(t_1,t_2)\big]\\
&=\frac{1}{p}\int_{t_0}^{\infty}\int_{t_0'}^{\infty}e^{\sigma_1\sigma_2}_{\ominus s\ominus\frac{1}{p}}(t_1,t_2,t_0,t_0')\ f^{\Delta_1}(t_1,t_2)\ \Delta t_1\ \Delta t_2\\
&=\frac{1}{p}\int_{t_0'}^{\infty}e^{\sigma_2}_{\ominus\frac{1}{p}}(t_2,t_0')\left[\ \int_{t_0}^{\infty}e^{\sigma_1}_{\ominus s}(t_1,t_0)\ f^{\Delta_1}(t_1,t_2)\ \Delta t_1\right]\Delta t_2\\
&=\frac{1}{p}\int_{t_0'}^{\infty}e^{\sigma_2}_{\ominus\frac{1}{p}}(t_2,t_0')\left[\ \int_{t_0}^{\infty}\Big[\Big(e_{\ominus s}(t_1,t_2)\ f(t_1,t_2)\Big)^{\Delta_1}-\Big(f(t_1,t_2)e^{\Delta_1}_{\ominus s}(t_1,t_2)\Big)\Big]\ \Delta t_1\right]\Delta t_2\\
&=\frac{1}{p}\int_{t_0'}^{\infty}e^{\sigma_2}_{\ominus\frac{1}{p}}(t_2,t_0')\left[-f(t_0,t_2)\ +s\int_{t_0}^{\infty}e^{\sigma_1}_{\ominus s}(t_1,t_0)f(t_1,t_2)\ \Delta t_1\right]\Delta t_2\\
&=-\frac{1}{p}\int_{t_0'}^{\infty}e^{\sigma_2}_{\ominus\frac{1}{p}}(t_2,t_0')\ f(t_0,t_2)\ \Delta t_2\\
&+\ s\cdot\frac{1}{p}\int_{t_0}^{\infty}\int_{t_0'}^{\infty}e^{\sigma_1}_{\ominus s}(t_1,t_0)e^{\sigma_1,\sigma_2}_{\ominus s\ominus\frac{1}{p}}(t_1,t_2,t_0,t_0')f(t_1,t_2)\ \Delta t_1\ \Delta t_2\\
&=\ s\ \mathscr{L}_{t_1}\mathscr{S}_{t_2}\big[f(t_1,t_2)\big]\ -\ \mathscr{S}_{t_2}\big[f(t_0,t_2)\big]
\end{aligned}$$

(2) $\mathscr{L}_{t_1}\mathscr{S}_{t_2}\big[f^{\Delta_2}(t_1,t_2)\big] = \frac{1}{p}\mathscr{L}_{t_1}\mathscr{S}_{t_2}\big[f(t_1,t_2)\big] \; - \; \frac{1}{p}\,\mathscr{L}_{t_1}\big[f(t_1,t_0')\big]$

$$
\begin{aligned}
&\mathscr{L}_{t_1}\mathscr{S}_{t_2}\big[f^{\Delta_2}(t_1,t_2)\big]\\
&= \frac{1}{p}\int_{t_0}^{\infty}\int_{t_0'}^{\infty} e^{\sigma_1\sigma_2}_{\ominus s\ominus\frac{1}{p}}(t_1,t_2,t_0,t_0')\; f^{\Delta_2}(t_1,t_2)\;\Delta t_1\;\Delta t_2\\
&= \int_{t_0}^{\infty} e^{\sigma_1}_{\ominus s}(t_1,t_0)\left[\frac{1}{p}\int_{t_0'}^{\infty}\Big[\Big(e_{\ominus\frac{1}{p}}(t_2,t_0')f(t_1,t_2)\Big)^{\Delta_2} - \Big(e^{\Delta_2}_{\ominus\frac{1}{p}}(t_2,t_0')f(t_1,t_2)\Big)\Big]\Delta t_2\right]\Delta t_1\\
&= \int_{t_0}^{\infty} e^{\sigma_1}_{\ominus s}(t_1,t_0)\left[\frac{-f(t_1,t_0')}{p} \;+\; \frac{1}{p^2}\int_{t_0'}^{\infty}\frac{e_{\ominus\frac{1}{p}}(t_2,t_0')}{\Big(1+\mu_2\frac{1}{p}\Big)}\; f(t_1,t_2)\;\Delta t_2\right]\Delta t_1\\
&= \int_{t_0}^{\infty} e^{\sigma_1}_{\ominus s}(t_1,t_0)\left[\frac{-f(t_1,t_0')}{p} + \frac{1}{p^2}\int_{t_0'}^{\infty} e^{\sigma_2}_{\ominus\frac{1}{p}}(t_2,t_0')\; f(t_1,t_2)\;\Delta t_2\right]\Delta t_1\\
&= -\frac{1}{p}\int_{t_0}^{\infty} e^{\sigma_1}_{\ominus s}(t_1,t_0)f(t_1,t_0')\;\Delta t_1\\
&+\; \frac{1}{p}\cdot\frac{1}{p}\int_{t_0}^{\infty}\int_{t_0'}^{\infty} e^{\sigma_1\sigma_2}_{\ominus s\ominus\frac{1}{p}}(t_1,t_2,t_0,t_0')\; f(t_1,t_2)\;\Delta t_1\;\Delta t_2\\
&= \frac{1}{p}\mathscr{L}_{t_1}\mathscr{S}_{t_2}\big[f(t_1,t_2)\big] - \frac{1}{p}\mathscr{L}_{t_1}\big[f(t_1,t_0')\big]
\end{aligned}
$$

(3) $\mathscr{L}_{t_1}\mathscr{S}_{t_2}\big[f^{\Delta_1^2}(t_1,t_2)\big] = s^2\,\mathscr{L}_{t_1}\mathscr{S}_{t_2}\big[f(t_1,t_2)\big] - s\,\mathscr{S}_{t_2}\big[f(t_0,t_2)\big] - \mathscr{S}_{t_2}\big[f^{\Delta_1}(t_0,t_2)\big]$

$$
\begin{aligned}
&\mathscr{L}_{t_1}\mathscr{S}_{t_2}\big[f^{\Delta_1^2}(t_1,t_2)\big]\\
&= \frac{1}{p}\int_{t_0}^{\infty}\int_{t_0'}^{\infty} e^{\sigma_1\sigma_2}_{\ominus s\ominus\frac{1}{p}}(t_1,t_2,t_0,t_0')\; f^{\Delta_1^2}(t_1,t_2)\;\Delta t_1\;\Delta t_2\\
&= \frac{1}{p}\int_{t_0'}^{\infty} e^{\sigma_2}_{\ominus\frac{1}{p}}(t_2,t_0')\left[\int_{t_0}^{\infty}\Big[\Big(e_{\ominus s}(t_1,t_0)f^{\Delta_1}(t_1,t_2)\Big)^{\Delta_1} - \Big(e^{\Delta_1}_{\ominus s}(t_1,t_0)f^{\Delta_1}(t_1,t_2)\Big)\Delta t_1\Big]\right]\Delta t_2\\
&= \frac{1}{p}\int_{t_0'}^{\infty} e^{\sigma_2}_{\ominus\frac{1}{p}}(t_2,t_0')\left[-f^{\Delta_1}(t_0,t_2) \;+\; s\int_{t_0}^{\infty} e^{\sigma_1}_{\ominus s}(t_1,t_0)f^{\Delta_1}(t_1,t_2)\Delta t_1\right]\Delta t_2\\
&= -\frac{1}{p}\int_{t_0'}^{\infty} e^{\sigma_2}_{\ominus\frac{1}{p}}(t_2,t_0')f^{\Delta_1}(t_0,t_2)\Delta t_2\\
&\qquad +\frac{s}{p}\int_{t_0'}^{\infty} e^{\sigma_2}_{\ominus\frac{1}{p}}(t_2,t_0')\left[-f(t_0,t_2) \;+s\int_{t_0}^{\infty} e^{\sigma_1}_{\ominus s}(t_1,t_0)\;f(t_1,t_2)\;\Delta t_1\right]\Delta t_2\\
&= -\frac{1}{p}\int_{t_0'}^{\infty} e^{\sigma_2}_{\ominus\frac{1}{p}}(t_2,t_0')f^{\Delta_1}(t_0,t_2)\Delta t_2\\
&-\frac{s}{p}\int_{t_0'}^{\infty} e^{\sigma_2}_{\ominus\frac{1}{p}}(t_2,t_0')f(t_0,t_2)\;\Delta t_2 + \frac{s^2}{p}\int_{t_0'}^{\infty}\int_{t_0}^{\infty} e^{\sigma_1\sigma_2}_{\ominus s\ominus\frac{1}{p}}(t_1t_2,t_0,t_0')f(t_1,t_2)\Delta t_1\Delta t_2\\
&= s^2\mathscr{L}_{t_1}\mathscr{S}_{t_2}\big[f(t_1,t_2)\big] \;-s\mathscr{S}_{t_2}\big[f(t_0,t_2)\big] \;-\mathscr{S}_{t_2}\big[f^{\Delta_1}(t_0,t_2)\big]
\end{aligned}
$$

(4) $\mathscr{L}_{t_1}\mathscr{S}_{t_2}\big[f^{\Delta_2^2}(t_1,t_2)\big] = \frac{1}{p^2}\mathscr{L}_{t_1}\mathscr{S}_{t_2}\big[f(t_1,t_2)\big] - \frac{1}{p^2}\mathscr{L}_{t_1}\big[f(t_1,t_0')\big] - \frac{1}{p}\mathscr{L}_{t_1}\big[f^{\Delta_2}(t_1,t_0')\big]$

$$\begin{aligned}
&\mathscr{L}_{t_1}\mathscr{S}_{t_2}\big[f^{\Delta_2^2}(t_1,t_2)\big]\\
&= \frac{1}{p}\int_{t_0}^{\infty}\int_{t_0'}^{\infty} e^{\sigma_1\sigma_2}_{\ominus s\ominus\frac{1}{p}}(t_1,t_2,t_0,t_0')\ f^{\Delta_2^2}(t_1,t_2)\ \Delta t_1\ \Delta t_2\\
&= \frac{1}{p}\int_{t_0}^{\infty} e^{\sigma_1}_{\ominus s}(t_1,t_0)\left[\int_{t_0'}^{\infty}\Big[\Big(e_{\ominus\frac{1}{p}}(t_2,t_0')f^{\Delta_2}(t_1,t_2)\Big)^{\Delta_2} - \Big(e^{\Delta_2}_{\ominus\frac{1}{p}}(t_2,t_0')f^{\Delta_2}(t_1,t_2)\Big)\Big]\Delta t_2\right]\Delta t_1\\
&= \frac{1}{p}\int_{t_0}^{\infty} e^{\sigma_1}_{\ominus s}(t_1,t_0)\left[-f^{\Delta_2}(t_1,t_0') + \frac{1}{p}\int_{t_0'}^{\infty} e^{\sigma_2}_{\ominus\frac{1}{p}}(t_2,t_0')\ f^{\Delta_2}(t_1,t_2)\ \Delta t_2\right]\Delta t_1\\
&= -\frac{1}{p}\int_{t_0}^{\infty} e^{\sigma_1}_{\ominus s}(t_1,t_0)f^{\Delta_2}(t_1,t_0')\ \Delta t_1 - \frac{1}{p^2}\int_{t_0}^{\infty} e^{\sigma_1}_{\ominus s}(t_1,t_0)f(t_1,t_0')\ \Delta t_1\\
&\qquad + \frac{1}{p^2}\int_{t_0'}^{\infty} e^{\sigma_1\sigma_2}_{\ominus s\ominus\frac{1}{p}}(t_1,t_2,t_0,t_0')f(t_1,t_2)\ \Delta t_1\ \Delta t_2\\
&= \frac{1}{p^2}\mathscr{L}_{t_1}\mathscr{S}_{t_2}[f(t_1,t_2)] - \frac{1}{p^2}\mathscr{L}_{t_1}[f(t_1,t_0')] - \frac{1}{p}\mathscr{L}_{t_1}[f^{\Delta_2}(t_1,t_0')]
\end{aligned}$$

4. Some Important Results

Theorem 4.1. *Let* $f(t_1,t_2):\mathbb{T}_1\times\mathbb{T}_2\to\mathbb{C}$, *is regulated and*

$$F(t_1,t_2) = \int_{t_0}^{\infty}\int_{t_0'}^{\infty} f(\gamma_1,\gamma_2)\ \Delta\gamma_1\ \Delta\gamma_2 \qquad for \quad (t_1,t_2)\in\mathbb{T}_1\times\mathbb{T}_2$$

$$\text{then} \qquad \mathscr{L}_{t_1}\mathscr{S}_{t_2}\big[f(t_1,t_2)\big] = \frac{p}{s}\ \mathscr{L}_{t_1}\mathscr{S}_{t_2}\big[f(t_1,t_2)\big]$$

Proof.

$$\begin{aligned}
&\mathscr{L}_{t_1}\mathscr{S}_{t_2}\big[F(t_1,t_2)\big]\\
&= \frac{1}{p}\int_{t_0}^{\infty}\int_{t_0'}^{\infty} e^{\sigma_1\sigma_2}_{\ominus s\ominus\frac{1}{p}}(t_1,t_2,t_0,t_0')\ F(t_1,t_2)\ \Delta t_1\ \Delta t_2\\
&= \frac{1}{p}\int_{t_0}^{\infty}\int_{t_0'}^{\infty} \frac{e_{\ominus s\ominus\frac{1}{p}}(t_1,t_2,t_0,t_0')}{\Big(1+\mu_1 s\Big)\Big(1+\mu_2\frac{1}{p}\Big)}\ F(t_1,t_2)\ \Delta t_1\ \Delta t_2\\
&= \frac{p}{s}\cdot\frac{1}{p}\int_{t_0}^{\infty}\int_{t_0'}^{\infty} \ominus s\ominus\frac{1}{p}\ e_{\ominus s\ominus\frac{1}{p}(t_1,t_2,t_0,t_0')}\ F(t_1,t_2)\ \Delta t_1\ \Delta t_2
\end{aligned}$$

Applying integration by parts and Fundamental theorem of Calculus and using $F(t_0,t_0')=0$ we obtain

$$\mathscr{L}_{t_1}\mathscr{S}_{t_2}\big[f(t_1,t_2)\big] = \frac{p}{s}\ \mathscr{L}_{t_1}\mathscr{S}_{t_2}\big[f(t_1,t_2)\big]$$

Theorem 4.2. *If* $\alpha_1 \in \mathbb{T}_1$ *and* $\alpha_2 \in \mathbb{T}_2$ *with* $\alpha_1, \alpha_2 > 0$ *we have*

$$H_{\alpha_1,\alpha_2}(t_1,t_2) = \begin{cases} 0 & t_1 \in \mathbb{T}_1, t_2 \in \mathbb{T}_2 \ \ \& \ \ t_1 < \alpha_1, t_2 < \alpha_2 \\ 1 & t_1 \in \mathbb{T}_1, t_2 \in \mathbb{T}_2 \ \ \& \ \ t_1 \geq \alpha_1, t_2 \geq \alpha_2 \end{cases}$$

with $H_{\alpha_1,\alpha_2}(t_1,t_2) = H_{\alpha_1} \otimes H_{\alpha_2}$ *where* $\otimes$ *denotes tensor product then*

$$\mathscr{L}_{t_1}\mathscr{S}_{t_2}\big[H_{\alpha_1,\alpha_2}(t_1,t_2)\big] \ = \ \frac{1}{s} e_{\ominus s \ominus \frac{1}{p}}(\alpha_1,\alpha_2,t_0,t_0')$$

Proof.

$$\begin{aligned}
\mathscr{L}_{t_1}\mathscr{S}_{t_2}\big[H_{\alpha_1,\alpha_2}(t_1,t_2)\big] &= \frac{1}{p}\int_{t_0}^{\infty}\int_{t_0'}^{\infty} e^{\sigma_1\sigma_2}_{\ominus s \ominus \frac{1}{p}}(t_1,t_2,t_0,t_0') \ H_{\alpha_1,\alpha_2}(t_1,t_2) \ \Delta t_1 \ \Delta t_2 \\
&= \frac{1}{p}\int_{t_0}^{\infty}\int_{t_0'}^{\infty} \frac{e_{\ominus s \ominus \frac{1}{p}}(t_1,t_2,t_0,t_0')}{\Big(1+\mu_1 s\Big)\Big(1+\mu_2 \frac{1}{p}\Big)} \ H_{\alpha_1,\alpha_2}(t_1,t_2) \ \Delta t_1 \ \Delta t_2 \\
&= \frac{1}{p}\int_{\alpha_1}^{\infty}\int_{\alpha_2}^{\infty} \frac{e_{\ominus s \ominus \frac{1}{p}}(t_1,t_2,t_0,t_0')}{\Big(1+\mu_1 s\Big)\Big(1+\mu_2 \frac{1}{p}\Big)} \ \Delta t_1 \ \Delta t_2 \\
&= \frac{p}{s}\int_{\alpha_1}^{\infty} e^{\Delta t_1}_{\ominus s}(t_1,t_0)\left[\frac{1}{p}\int_{\alpha_2}^{\infty} e^{\Delta_2}_{\ominus \frac{1}{p}}(t_2,t_0') \ \Delta t_2\right] \Delta t_1 \\
&= \frac{1}{s} e_{\ominus s}(\alpha_1,t_0) \ e_{\ominus \frac{1}{p}}(\alpha_2,t_0') \\
&= \frac{1}{s} e_{\ominus s \ominus \frac{1}{p}}(\alpha_1,\alpha_2,t_0,t_0')
\end{aligned}$$

Theorem 4.3. *If* $F(s,p) = \mathscr{L}_{t_1}\mathscr{S}_{t_2}\big[f(t_1,t_2)\big]$ *then*

$$\begin{aligned}
\mathscr{L}_{t_1}\mathscr{S}_{t_2}\big[H_{\alpha_1,\alpha_2}(t_1,t_2)f(t_1,t_2)\big] &= e_{\ominus s \ominus \frac{1}{p}}(\alpha_1,\alpha_2,t_1,t_2)\mathscr{L}_{t_1}\mathscr{S}_{t_2}\big[f(t_1,t_2)\big] \\
&= e_{\ominus s \ominus \frac{1}{p}}(\alpha_1,\alpha_2,t_1,t_2) \ F(s,p)
\end{aligned}$$

Proof.

$$\mathscr{L}_{t_1}\mathscr{S}_{t_2}\big[H_{\alpha_1,\alpha_2}(t_1,t_2)f(t_1,t_2)\big]$$

$$\begin{aligned}
&= \frac{1}{p}\int_{t_0}^{\infty}\int_{t_0'}^{\infty} e^{\sigma_1\sigma_2}_{\ominus s\ominus\frac{1}{p}}(t_1,t_2,t_0,t_0')H_{\alpha_1,\alpha_2}(t_1,t_2)f(t_1,t_2)\ \Delta t_1\ \Delta t_2\\
&= \frac{1}{p}\int_{t_0}^{\infty}\int_{t_0'}^{\infty} \frac{e_{\ominus s\ominus\frac{1}{p}}(t_1,t_2,t_0,t_0')}{\big(1+\mu_1 s\big)\big(1+\mu_2\frac{1}{p}\big)}\ H_{\alpha_1,\alpha_2}(t_1,t_2)f(t_1,t_2)\ \Delta t_1\ \Delta t_2\\
&= \frac{1}{p}\int_{\alpha_1}^{\infty}\int_{\alpha_2}^{\infty} \frac{e_{\ominus s\ominus\frac{1}{p}}(t_1,t_2,\alpha_1,\alpha_2)}{\big(1+\mu_1 s\big)\big(1+\mu_2\frac{1}{p}\big)}\ e_{\ominus s\ominus\frac{1}{p}}(\alpha_1,\alpha_2,t_0,t_0')\ f(t_1,t_2)\ \Delta t_1\ \Delta t_2\\
&= e_{\ominus s\ominus\frac{1}{p}}(\alpha_1,\alpha_2,t_0,t_0')\ \frac{1}{p}\ \int_{\alpha_1}^{\infty}\int_{\alpha_2}^{\infty} e^{\sigma_1\sigma_2}_{\ominus s\ominus\frac{1}{p}}(t_1,t_2,\alpha_1,\alpha_2)\ f(t_1,t_2)\ \Delta t_1\ \Delta t_2\\
&= e_{\ominus s\ominus\frac{1}{p}}(\alpha_1,\alpha_2,t_0,t_0')\ \mathscr{L}_{t_1}\mathscr{S}_{t_2}\big[f(t_1,t_2)\big]
\end{aligned}$$

Next we prove convolution theorem for Laplace-Sumudu transform.

Definition 4.1. [6] *Let $f_1(t_1,t_2):\mathbb{T}_1\times\mathbb{T}_2\to\mathbb{C}$ and $f_2(t_1,t_2):\mathbb{T}_1\times\mathbb{T}_2\to\mathbb{C}$ are $\Delta-$ integrable functions then the double convolution of $f_1(t_1,t_2)$ and $f_2(t_1,t_2)$ is given by*

$$\big(f_1 ** f_2\ \big)(t_1,t_2)\ =\ \int_{t_0}^{t_1}\int_{t_0'}^{t_2} f_1\big(t_1,t_2,\sigma_1(\tau_1),\sigma_2(\tau_2)\big)f_2\big(\tau_1,\tau_2\big)\Delta\tau_1\ \Delta\tau_2$$

Theorem 4.4. [**Convolution Theorem**] *If $f_1(t_1,t_2):\mathbb{T}_1\times\mathbb{T}_2\to\mathbb{C}$ and $f_2(t_1,t_2):\mathbb{T}_1\times\mathbb{T}_2\to\mathbb{C}$ are rd-continuous functions of exponential type II having double Laplace-Sumudu transform $\mathscr{L}_{t_1}\mathscr{S}_{t_2}\big[f_1(t_1,t_2)\big]$ and $\mathscr{L}_{t_1}\mathscr{S}_{t_2}\big[f_2(t_1,t_2)\big]$ respectively then*

$$\mathscr{L}_{t_1}\mathscr{S}_{t_2}\big[\big(f_1 ** f_2\big)(t_1,t_2)\big]\ =\ p\ \mathscr{L}_{t_1}\mathscr{S}_{t_2}\big[f_1(t_1,t_2)\big]\cdot\mathscr{L}_{t_1}\mathscr{S}_{t_2}\big[f_2(t_1,t_2)\big]$$

Proof.

$$\begin{aligned}
&\mathscr{L}_{t_1}\mathscr{S}_{t_2}\big[\big(f_1 ** f_2\big)(t_1,t_2)\big]\\
&= \frac{1}{p}\int_{t_0}^{\infty}\int_{t_0'}^{\infty} e^{\sigma_1\sigma_2}_{\ominus s\ominus\frac{1}{p}}(t_1,t_2,t_0,t_0')\ \big[\big(f_1 ** f_2\big)(t_1,t_2)\big]\\
&= \frac{1}{p}\int_{t_0}^{\infty}\int_{t_0'}^{\infty} e^{\sigma_1\sigma_2}_{\ominus s\ominus\frac{1}{p}}(t_1,t_2,t_0,t_0')\left[\int_{t_0}^{t_1}\int_{t_0'}^{t_2} f_1\big(t_1,t_2,\sigma_1(\tau_1),\sigma_2(\tau_2)\big)f_2\big(\tau_1,\tau_2\big)\Delta\tau_1\Delta\tau_2\right]\Delta t_1\ \Delta t_2\\
&= \int_{t_0}^{\infty}\int_{t_0'}^{\infty} f_2(\tau_1,\tau_2)\left[\frac{1}{p}\int_{\sigma_1(\tau_1)}^{\infty}\int_{\sigma_1(\tau_2)}^{\infty} f_1\big(t_1,t_2,\sigma_1(\tau_1),\sigma_2(\tau_2)\big)\cdot e^{\sigma_1\sigma_2}_{\ominus s\ominus\frac{1}{p}}(t_1,t_2,t_0,t_0')\Delta t_1\Delta t_2\right]\Delta\tau_1\Delta\tau_2
\end{aligned}$$

$$= \int_{t_0}^{\infty}\int_{t_0'}^{\infty} f_2(\tau_1,\tau_2)\Big[\frac{1}{p}\int_{t_0}^{\infty}\int_{t_0'}^{\infty} f_1\big(t_1,t_2,\sigma_1(\tau_1),\sigma_2(\tau_2)\big)$$

$$H_{\sigma_1(\tau_1),\sigma_2(\tau_2)}(t_1,t_2)\ e^{\sigma_1\sigma_2}_{\ominus s\ominus\frac{1}{p}}(t_1,t_2,t_0,t_0')\ \Delta t_1\ \Delta t_2\Big]\ \Delta\tau_1\ \Delta\tau_2$$

$$= \int_{t_0}^{\infty}\int_{t_0'}^{\infty} f_2(\tau_1,\tau_2)\ \mathscr{L}_{t_1}\mathscr{S}_{t_2}\Big[H_{\sigma_1(\tau_1),\sigma_2(\tau_2)}(t_1,t_2)\ f_1\big(t_1,t_2,\sigma_1(\tau_1),\sigma_2(\tau_2)\big)\Big]\ \Delta\tau_1\ \Delta\tau_2$$

$$= \int_{t_0}^{\infty}\int_{t_0'}^{\infty} f_2(\tau_1,\tau_2)\Big[e_{\ominus s\ominus\frac{1}{p}}\big(\sigma_1(\tau_1),\sigma_2(\tau_2),t_0,t_0'\big)\ \mathscr{L}_{t_1}\mathscr{S}_{t_2}\big[f_1(t_1,t_2)\big]\Big]\ \Delta\tau_1\ \Delta\tau_2$$

$$= p\ \mathscr{L}_{t_1}\mathscr{S}_{t_2}\big[f_2(t_1,t_2)\big]\ \frac{1}{p}\int_{t_0}^{\infty}\int_{t_0'}^{\infty} e^{\sigma_1,\sigma_2}_{\ominus s\ominus\frac{1}{p}}\big(\tau_1,\tau_2,t_0,t_0'\big)\ f_2\big(\tau_1,\tau_2\big)\ \Delta\tau_1\ \Delta\tau_2$$

$$= p\ \mathscr{L}_{t_1}\mathscr{S}_{t_2}\big[f_2(t_1,t_2)\big]\ \mathscr{L}_{t_1}\mathscr{S}_{t_2}\big[f_1(t_1,t_2)\big]$$

5. Applications

In this section we will find solution of partial-integro dynamic equation and partial dynamic equation using our discussed theory.

Example 5.1. Consider following partial-integro dynamic equation

$$\frac{\partial f(t_1,t_2)}{\Delta_1 t_1} + \frac{\partial f(t_1,t_2)}{\Delta_2 t_2} = -1 + e_1(t_1,0) + e_2(t_2,0) + e_{1\oplus 1}(t_1,t_2,0,0) + \int_0^{\infty}\int_0^{\infty} f(t_1,t_2,\sigma_2(\tau_1),\sigma_2(\tau_2))\ \tau_1\ \tau_2$$

with initial conditions

$$f(t_1,0) = e_1(t_1,0) \qquad f(0,t_2) = e_1(t_2,0)$$

Taking Laplace -Sumudu transform of given equation

$$s\ \mathscr{L}_{t_1}\mathscr{S}_{t_2}\big[f(t_1,t_2)\big] - \mathscr{S}_{t_2}\big[f(0,t_2)\big] + \frac{1}{p}\mathscr{L}_{t_1}\mathscr{S}_{t_2}\big[f(t_1,t_2)\big] - \frac{1}{p}\mathscr{L}_{t_1}\big[f(t_1,0)\big]$$

$$= -\frac{1}{s} + \frac{1}{(s-1)} + \frac{1}{s(1-p)} + \frac{1}{(s-1)(1-p)} + \frac{p}{s}\ \mathscr{L}_{t_1}\mathscr{S}_{t_2}\big[f(t_1,t_2)\big]$$

Taking Laplace transform of $f(t_1,0) = e_1(t_1,0)$ and Sumudu transform of $f(0,t_2) = e_1(t_2,0)$ we get

$$\mathscr{L}_{t_1}\big[f(t_1,0)\big] = \frac{1}{(s-1)}$$

$$\mathscr{S}_{t_2}\big[f(0,t_2)\big] = \frac{1}{(1-p)}$$

Now substituting it into above equation,

$$s\,\mathscr{L}_{t_1}\mathscr{S}_{t_2}\big[f(t_1,t_2)\big] \;-\; \frac{1}{(1-p)} \;+\; \frac{1}{p}\mathscr{L}_{t_1}\mathscr{S}_{t_2}\big[f(t_1,t_2)\big] \;-\; \frac{1}{p}\frac{1}{(s-1)}$$
$$= -\frac{1}{s} \;+\; \frac{1}{(s-1)} \;+\; \frac{1}{s(1-p)} \;+\; \frac{1}{(s-1)(1-p)} \;+\; \frac{p}{s}\,\mathscr{L}_{t_1}\mathscr{S}_{t_2}\big[f(t_1,t_2)\big]$$
$$\therefore\;\; \Big(s+\frac{1}{p}-\frac{p}{s}\Big)\mathscr{L}_{t_1}\mathscr{S}_{t_2}\big[f(t_1,t_2)\big] = \frac{1}{(1-p)} \;+\; \frac{1}{p(s-1)} \;-\; \frac{1}{s}$$
$$+\; \frac{1}{s(1-p)} \;+\; \frac{1}{(s-1)(1-p)}$$

On simplifying we get

$$\mathscr{L}_{t_1}\mathscr{S}_{t_2}\big[f(t_1,t_2)\big] = \frac{1}{(s-1)(1-p)}$$

On taking inverse transform we get

$$f(t_1,t_2) = e_{1\oplus 1}(t_1,t_2,t_0,t_0')$$

is required solution.

Example 5.2. Consider the following partial dynamic equation

$$\frac{\partial^2 g(t_1,t_2)}{\Delta_1^2 t_1^2} - \frac{\partial^2 g(t_1,t_2)}{\Delta_2^2 t_2^2} - \frac{\partial g(t_1,t_2)}{\Delta_2 t_2} - g(t_1,t_2) + h_2(t_1,0) + h_1(t_2,0) = 1$$

with initial conditions

$$g(t_1,0) = h_2(t_1,0), \quad , g(0,t_2) = h_1(t_2,0), \quad \frac{\partial g(t_1,0)}{\Delta_2 t_2} = 1, \quad \frac{\partial g(0,t_2)}{\Delta_1 t_1} = 0$$

Taking Laplace-Sumudu transform of given equation

$$s^2\mathscr{L}_{t_1}\mathscr{S}_{t_2}[g(t_1,t_2)] - s\mathscr{S}_{t_2}[g(0,t_2)] - \mathscr{S}_{t_2}[g^{\Delta_1}(0,t_2)] - \frac{1}{p^2}\mathscr{L}_{t_1}\mathscr{S}_{t_2}[g(t_1,t_2)] + \frac{1}{p^2}\mathscr{L}_{t_1}[g(t_1,0)]$$
$$+\frac{1}{p}\mathscr{L}_{t_1}[g^{\Delta_2}(t_1,0)] - \frac{1}{p}\mathscr{L}_{t_1}\mathscr{S}_{t_2}[g(t_1,t_2)] + \frac{1}{p}\mathscr{L}_{t_1}[g(t_1,0)] - \mathscr{L}_{t_1}\mathscr{S}_{t_2}[g(t_1,t_2)] + \frac{1}{s^3} + \frac{p}{s} = \frac{1}{s}$$

Taking Laplace and Sumudu transforms of initial conditions as appropriate

$$\mathscr{S}_{t_1}[g(0,t_2)] = \mathscr{S}_{t_2}[h_1(t_2,0)] = p, \quad \mathscr{S}_{t_2}[g^{\Delta_1}(0,t_2)] = \mathscr{S}_{t_2}[0] = 0$$

$$\mathscr{L}_{t_1}[g(t_1,0)] = \mathscr{L}_{t_1}[h_2(t_1,0)] = \frac{1}{s^3}, \quad \mathscr{L}_{t_1}[g^{\Delta_2}(t_1,0)] = \mathscr{L}_{t_1}[1] = \frac{1}{s}$$

After substitution and simplification we get,

$$\left[s^2 - \frac{1}{p^2} - \frac{1}{p} - 1\right]\mathscr{L}_{t_1}\mathscr{S}_{t_2}[g(t_1,t_2)] = sp - \frac{1}{p^2s^3} - \frac{1}{ps} - \frac{1}{ps^3} - \frac{1}{s^3} - \frac{p}{s} + \frac{1}{s}$$

$$\mathscr{L}_{t_1}\mathscr{S}_{t_2}[g(t_1,t_2)] = \frac{1}{s^2} + \frac{p^2}{s}$$

On taking inverse transform we get,

$$g(t_1,t_2) = h_1(t_1,0) + h_2(t_2,0)$$

6. Conclusion

In this paper, the Laplace-Sumudu integral transform on time scales is studied. Existence theorem and some important properties including convolution theorem are proved. Using Laplace-Sumudu integral transform partial dynamic and partial-integro dynamic equations can be solved efficiently. Further we try to study linear and non-linear partial dynamic equations and partial-integro dynamic equations in our future work.

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