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A NOTE ON HEINE'S TRANSFORMATION

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Abstract: In this paper, making use of $q-$binomial theorem different generalizations of Heine's first transformation have been discussed.

Keywords and Phrases: Heine transformation, bi-basic and multi-basic series, Ramanujan's theta functions.

2020 Mathematics Subject Classification: 33D15, 33D65.

1. Introduction, Notations and Definitions

The $q-$ rising factorial is defined as,

$$(a;q)_0 = 1, \quad (a;q)_n = (1-a)(1-aq)...(1-aq^{n-1}), \quad n \in (1,2,3,...),$$

where the parameter q is called the base and $|q| < 1$.
The infinite $q-$rising factorial is defined as,

$$(a;q)_\infty = \prod_{r=0}^{\infty}(1-aq^r) = \lim_{n\to\infty}(a;q)_n.$$

When k is complex number, we write

$$(a;q)_k = \frac{(a;q)_\infty}{(aq^k;q)_\infty}.$$

Analogues to Gauss' series, Heine used the series

$$ {}_2\Phi_1\left[\begin{array}{l} a, b; q; z \\ c \end{array}\right] = \sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(q;q)_n (c;q)_n} z^n, \qquad |z| < 1. \tag{1.1} $$

Further generalization of Heine series is given as,

$$ {}_r\Phi_s\left[\begin{array}{l} a_1, a_2, ..., a_r; q; z \\ b_1, b_2, ..., b_s \end{array}\right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, ..., a_r; q)_n}{(q, b_1, b_2, ..., b_s; q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n, \tag{1.2} $$

where $(a_1, a_2, ..., a_r; q)_n = (a_1; q)_n (a_2; q)_n ... (a_r; q)_n$ and $\binom{n}{2} = n(n-1)/2$. The series (1.2) converges for $|z| < \infty$ if $r \leq s$ and for $|z| < 1$ if $r = s + 1$. If $r > s + 1$, it diverges everywhere except z=0.

In this paper, we shall make use of $q-$binomial theorem,

$$ \sum_{n=0}^{\infty} \frac{(a;q)_n z^n}{(q;q)_n} = \frac{(az;q)_\infty}{(z;q)_\infty}, \quad |z| < 1, \; |q| < 1. \tag{1.3} $$

[Gasper, G. and Rahman, M. (6) eq. (1.3.2), pp. 8]

Heine in 1878 established the following transformation formulas,

$$ {}_2\Phi_1\left[\begin{array}{l} a, b; q; z \\ c \end{array}\right] = \frac{(b, az; q)_\infty}{(c, z; q)_\infty} \; {}_2\Phi_1\left[\begin{array}{l} c/b, z; q; b \\ az \end{array}\right]. \tag{1.4} $$

One can prove this transformation by making use (1.3) as,

$$ \begin{aligned} {}_2\Phi_1\left[\begin{array}{l} a, b; q; z \\ c \end{array}\right] &= \sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(q;q)_n (c;q)_n} z^n \\ &= \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{n=0}^{\infty} \frac{(a;q)_n (cq^n;q)_\infty}{(q;q)_n (bq^n;q)_\infty} z^n, \\ &= \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n \sum_{r=0}^{\infty} \frac{(c/b;q)_r}{(q;q)_r} b^r q^{nr}, \end{aligned} $$

If $|z| < 1$, $|b| < 1$ then

$$ = \frac{(b;q)_n}{(c;q)_n} \sum_{r=0}^{\infty} \frac{(c/b;q)_r}{(q;q)_r} b^r \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} (zq^r)^n, $$

$$= \frac{(b;q)_n}{(c;q)_n} \sum_{r=0}^{\infty} \frac{(c/b;q)_r}{(q;q)_r} b^r \frac{(azq^r;q)_\infty}{(zq^r;q)_\infty},$$

$$= \frac{(b;q)_n (az;q)_\infty}{(c;q)_n (z;q)_\infty} \sum_{r=0}^{\infty} \frac{(c/b;q)_r (z;q)_r}{(q;q)_r (az;q)_r} b^r,$$

$$= \frac{(b;q)_n (az;q)_\infty}{(c;q)_n (z;q)_\infty} \, {}_2\Phi_1 \left[\begin{matrix} c/b, z; q; b \\ az \end{matrix} \right],$$

which is the complete proof of (1.4).
Ramanujan generalized the Heine's transformation (1.4) as

If h is a positive integer and $|z|,\ |b| < 1$ then

$$\sum_{n=0}^{\infty} \frac{(a;q^h)_n (b;q)_{nh}}{(q^h;q^h)_n (c;q)_{nh}} z^n = \frac{(b;q)_\infty (az;q^h)_\infty}{(c;q)_\infty (z;q^h)_\infty} \sum_{r=0}^{\infty} \frac{(c/b;q)_r (z;q^h)_r b^r}{(q;q)_r (az;q^h)_r}. \tag{1.5}$$

Proof of (1.5).

$$\sum_{n=0}^{\infty} \frac{(a;q^h)_n (b;q)_{nh}}{(q^h;q^h)_n (c;q)_{nh}} z^n = \sum_{n=0}^{\infty} \frac{(a;q^h)_n (b;q)_\infty (cq^{nh};q)_\infty}{(q^h;q^h)_n (bq^{nh};q)_\infty (c;q)_\infty} z^n,$$

$$= \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{n=0}^{\infty} \frac{(a;q^h)_n}{(q^h;q^h)_n} z^n \sum_{r=0}^{\infty} \frac{(c/b;q)_r}{(q;q)_r} b^r q^{nhr},$$

Now, under the above given conditions we have,

$$= \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{r=0}^{\infty} \frac{(c/b;q)_r}{(q;q)_r} b^r \sum_{n=0}^{\infty} \frac{(a;q^h)_n}{(q^h;q^h)_n} (zq^{hr})^n,$$

$$= \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{r=0}^{\infty} \frac{(c/b;q)_r}{(q;q)_r} b^r \frac{(azq^{hr};q^h)_\infty}{(zq^{hr};q^h)_\infty},$$

$$= \frac{(b;q)_\infty (az;q^h)_\infty}{(c;q)_\infty (z;q^h)_\infty} \sum_{r=0}^{\infty} \frac{(c/b;q)_r (z;q^h)_r b^r}{(q;q)_r (az;q^h)_r},$$

which is precisely the right hand side of (1.5).

Gaurav Bhatnagar [5] further generalized (1.5) as,

$$\sum_{n=0}^{\infty} \frac{(a;q^h)_n (b;q^t)_{nh}}{(q^h;q^h)_n (c;q^t)_{nh}} z^n = \frac{(b;q^t)_\infty (az;q^h)_\infty}{(c;q^t)_\infty (z;q^h)_\infty} \sum_{r=0}^{\infty} \frac{(c/b;q^t)_r (z;q^h)_{rt}}{(q^t;q^t)_r (az;q^h)_{rt}} b^r, \tag{1.6}$$

where $\max(z,b)<1$, h and t are positive integers such that $|q^h|<1$, $|q^t|<1$ and $|q^{ht}|<1$.

Proof of (1.6).

Left hand side of (1.6)

$$=\sum_{n=0}^{\infty}\frac{(a;q^h)_n(b;q^t)_{nh}}{(q^h;q^h)_n(c;q^t)_{nh}}z^n,$$

$$=\frac{(b;q^t)_\infty}{(c;q^t)_\infty}\sum_{n=0}^{\infty}\frac{(a;q^h)_n(cq^{nht};q^t)_\infty}{(q^h;q^h)_n(bq^{nht};q^t)_\infty}z^n,$$

Applying (1.3)

$$=\frac{(b;q^t)_\infty}{(c;q^t)_\infty}\sum_{n=0}^{\infty}\frac{(a;q^h)_n}{(q^h;q^h)_n}z^n\sum_{r=0}^{\infty}\frac{(c/b;q^t)_r}{(q^t;q^t)_r}b^r(q^{nht})^r,$$

Under the given conditions

$$=\frac{(b;q^t)_\infty}{(c;q^t)_\infty}\sum_{r=0}^{\infty}\frac{(c/b;q^t)_r}{(q^t;q^t)_r}b^r\sum_{n=0}^{\infty}\frac{(a;q^h)_n}{(q^h;q^h)_n}(zq^{htr})^n,$$

Again, applying (1.3)

$$=\frac{(b;q^t)_\infty}{(c;q^t)_\infty}\sum_{r=0}^{\infty}\frac{(c/b;q^t)_r}{(q^t;q^t)_r}b^r\frac{(azq^{htr};q^h)_\infty}{(zq^{htr};q^h)_\infty},$$

which on simplifications

$$=\frac{(b;q^t)_\infty(az;q^h)_\infty}{(c;q^t)_\infty(z;q^h)_\infty}\sum_{r=0}^{\infty}\frac{(c/b;q^t)_r(z;q^h)_{tr}}{(q^t;q^t)_r(az;q^h)_{tr}}b^r,$$

which is precisely the right hand side of (1.6).

2. Generalization of (1.6)

We shall generalize (1.6) as,

$$\sum_{n=0}^{\infty}\frac{(a;q^h)_n(bx;q^t)_{nh}}{(q^h;q^h)_n(cy;q^t)_{nh}}z^n=\frac{(az;q^h)_\infty(bx;q^t)_\infty}{(z;q^h)_\infty(cy;q^t)_\infty}\times\sum_{r=0}^{\infty}\frac{(cy/bx;q^t)_r(z;q^h)_{tr}}{(q^t;q^t)_r(az;q^h)_{tr}}(bx)^r. \qquad (2.1)$$

Proof of (2.1).

Left hand side of (2.1)

$$= \sum_{n=0}^{\infty} \frac{(a;q^h)_n (bx;q^t)_{nh}}{(q^h;q^h)_n (cy;q^t)_{nh}} z^n$$

$$= \sum_{n=0}^{\infty} \frac{(a;q^h)_n (cyq^{nht};q^t)_\infty (bx;q^t)_\infty}{(q^h;q^h)_n (cy;q^t)_\infty (bxq^{nht};q^t)_\infty} z^n,$$

$$= \frac{(bx;q^t)_\infty}{(cy;q^t)_\infty} \sum_{n=0}^{\infty} \frac{(a;q^h)_n}{(q^h;q^h)_n} z^n \frac{(cyq^{nht};q^t)_\infty}{(bxq^{nht};q^t)_\infty},$$

Applying (1.3)

$$= \frac{(bx;q^t)_\infty}{(cy;q^t)_\infty} \sum_{n=0}^{\infty} \frac{(a;q^h)_n}{(q^h;q^h)_n} z^n \sum_{r=0}^{\infty} \frac{(cy/bx;q^t)_r}{(q^t;q^t)_r} b^r x^r q^{nhtr},$$

$|z| < 1$, $|bx| < 1$, $|q^h| < 1$, $|q^t| < 1$ and $|q^{ht}| < 1$

$$= \frac{(bx;q^t)_\infty}{(cy;q^t)_\infty} \sum_{r=0}^{\infty} \frac{(cy/bx;q^t)_r}{(q^t;q^t)_r} (bx)^r \sum_{n=0}^{\infty} \frac{(a;q^h)_n}{(q^h;q^h)_n} (zq^{htr})^n,$$

Again, apply (1.3)

$$= \frac{(bx;q^t)_\infty}{(cy;q^t)_\infty} \sum_{r=0}^{\infty} \frac{(cy/bx;q^t)_r}{(q^t;q^t)_r} (bx)^r \frac{(azq^{htr};q^h)_\infty}{(zq^{htr};q^h)_\infty},$$

$$= \frac{(bx;q^t)_\infty}{(cy;q^t)_\infty} \sum_{r=0}^{\infty} \frac{(cy/bx;q^t)_r}{(q^t;q^t)_r} (bx)^r \frac{(az;q^h)_\infty (z;q^h)_{tr}}{(az;q^h)_{tr} (z;q^h)_\infty},$$

$$= \frac{(bx;q^t)_\infty (az;q^h)_\infty}{(cy;q^t)_\infty (z;q^h)_\infty} \sum_{r=0}^{\infty} \frac{(cy/bx;q^t)_r (z;q^h)_{tr}}{(q^t;q^t)_r (az;q^h)_{tr}} (bx)^r,$$

which is the right hand side of (2.1).
Taking $y = x, h = t = 1$ in (2.1) we get

$$\frac{(cx;q)_\infty}{(bx;q)_\infty} \sum_{n=0}^{\infty} \frac{(a;q)_n (bx;q)_n}{(q;q)_n (cx;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty} \times \sum_{r=0}^{\infty} \frac{(c/b;q)_r (z;q)_r}{(q;q)_r (az;q)_r} (bx)^r, \tag{2.2}$$

which is Heine's transformation formula [7; eq. 78].

3. Further Generalization of (2.1)

Let us consider

$$\sum_{n=0}^{\infty}\frac{(a;q^h)_n(bx;q^t)_{nh}(b_1x_1;q^{t_1})_{nh}}{(q^h;q^h)_n(cy;q^t)_{nh}(c_1y_1;q^{t_1})_{nh}}z^n$$

$$=\sum_{n=0}^{\infty}\frac{(a;q^h)_n(bx;q^t)_\infty(cyq^{nth};q^t)_\infty(c_1y_1q^{t_1nh};q^{t_1})_\infty(b_1x_1;q^{t_1})_\infty}{(q^h;q^h)_n(cy;q^t)_\infty(bxq^{nht};q^t)_\infty(b_1x_1q^{nht_1};q^{t_1})_\infty(c_1y_1;q^{t_1})_\infty}z^n$$

$$=\frac{(bx;q^t)_\infty(b_1x_1;q^{t_1})_\infty}{(cy;q^t)_\infty(c_1y_1;q^{t_1})_\infty}\sum_{n=0}^{\infty}\frac{(a;q^h)_n}{(q^h;q^h)_n}z^n\sum_{r=0}^{\infty}\frac{(cy/bx;q^t)_r}{(q^t;q^t)_r}(bx)^rq^{nhtr}$$

$$\times\sum_{r_1=0}^{\infty}\frac{(c_1y_1/b_1x_1;q^{t_1})_{r_1}}{(q^{t_1};q^{t_1})_{r_1}}(b_1x_1)^{r_1}q^{nht_1r_1}$$

If $|z|<1$, $|bx|<1$, $|b_1x_1|<1$, $|q^{ht}|<1$, $|q^{ht_1}|<1$ then we have

$$=\frac{(bx;q^t)_\infty(b_1x_1;q^{t_1})_\infty}{(cy;q^t)_\infty(c_1y_1;q^{t_1})_\infty}\sum_{r,r_1=0}^{\infty}\frac{(cy/bx;q^t)_r}{(q^t;q^t)_r}(bx)^r\frac{(c_1y_1/b_1x_1;q^{t_1})_{r_1}}{(q^{t_1};q^{t_1})_{r_1}}(b_1x_1)^{r_1}$$

$$\times\sum_{n=0}^{\infty}\frac{(a;q^h)_n}{(q^h;q^h)_n}(zq^{htr+ht_1r_1})^n$$

$$=\frac{(bx;q^t)_\infty(b_1x_1;q^{t_1})_\infty}{(cy;q^t)_\infty(c_1y_1;q^{t_1})_\infty}\sum_{r,r_1=0}^{\infty}\frac{(cy/bx;q^t)_r(c_1y_1/b_1x_1;q^{t_1})_{r_1}}{(q^t;q^t)_r(q^{t_1};q^{t_1})_{r_1}}(bx)^r(b_1x_1)^{r_1}$$

$$\times\sum_{n=0}^{\infty}\frac{(a;q^h)_n}{(q^h;q^h)_n}\left\{zq^{h(tr+t_1r_1)}\right\}^n$$

$$=\frac{(az;q^h)_\infty(bx;q^t)_\infty(b_1x_1;q^{t_1})_\infty}{(z;q^h)_\infty(cy;q^t)_\infty(c_1y_1;q^{t_1})_\infty}$$

$$\times\sum_{r,r_1=0}^{\infty}\frac{(z;q^h)_{tr+t_1r_1}(cy/bx;q^t)_r(c_1y_1/b_1x_1;q^{t_1})_{r_1}}{(az;q^h)_{tr+t_1r_1}(q^t;q^t)_r(q^{t_1};q^{t_1})_{r_1}}(bx)^r(b_1x_1)^{r_1}$$

Finally we have the following result,

$$\sum_{n=0}^{\infty}\frac{(a;q^h)_n(bx;q^t)_{nh}(b_1x_1;q^{t_1})_{nh}}{(q^h;q^h)_n(cy;q^t)_{nh}(c_1y_1;q^{t_1})_{nh}}z^n=\frac{(az;q^h)_\infty(bx;q^t)_\infty(b_1x_1;q^{t_1})_\infty}{(z;q^h)_\infty(cy;q^t)_\infty(c_1y_1;q^{t_1})_\infty}$$

$$\times\sum_{r,r_1=0}^{\infty}\frac{(z;q^h)_{tr+t_1r_1}(cy/bx;q^t)_r(c_1y_1/b_1x_1;q^{t_1})_{r_1}}{(az;q^h)_{tr+t_1r_1}(q^t;q^t)_r(q^{t_1};q^{t_1})_{r_1}}(bx)^r(b_1x_1)^{r_1}. \tag{3.1}$$

Iterating the process k times we get,

$$\sum_{n=0}^{\infty} \frac{(a;q^h)_n(bx;q^t)_{nh}(b_1x_1;q^{t_1})_{nh}(b_2x_2;q^{t_2})_{nh}...(b_kx_k;q^{t_k})_{nh}}{(q^h;q^h)_n(cy;q^t)_{nh}(c_1y_1;q^{t_1})_{nh}(c_2y_2;q^{t_2})_{nh}...(c_ky_k;q^{t_k})_{nh}} z^n$$
$$= \frac{(az;q^h)_\infty(bx;q^t)_\infty(b_1x_1;q^{t_1})_\infty(b_2x_2;q^{t_2})_\infty...(b_kx_k;q^{t_k})_\infty}{(z;q^h)_\infty(cy;q^t)_\infty(c_1y_1;q^{t_1})_\infty(c_2y_2;q^{t_2})_\infty...(c_ky_k;q^{t_k})_\infty}$$
$$\times \sum_{r,r_1,r_2,...,r_k=0}^{\infty} \frac{(z;q^h)_{tr+t_1r_1+t_2r_2+...+t_kr_k}(cy/bx;q^t)_r(c_1y_1/b_1x_1;q^{t_1})_{r_1}}{(az;q^h)_{tr+t_1r_1+t_2r_2+...+t_kr_k}(q^t;q^t)_r(q^{t_1};q^{t_1})_{r_1}}$$
$$\times \frac{(c_2y_2/b_2x_2;q^{t_2})_{r_2}...(c_ky_k/b_kx_k;q^{t_k})_{r_1}}{(q^{t_2};q^{t_2})_{r_2}...(q^{t_k};q^{t_k})_{r_k}}(bx)^r(b_1x_1)^{r_1}...(b_kx_k)^{r_k}, \tag{3.2}$$

provided $|z| < 1$, $|bx| < 1$, $|b_1x_1| < 1$,...$|b_kx_k| < 1$ $|q^{ht}| < 1$, $|q^{ht_1}| < 1$, ...$|q^{ht_k}| < 1$. For $h = 1$, $t = t_1 = ... = t_k = 1$ (3.2) yields

$$\Phi_D\left[z; cy/bx, c_1y_1/b_1x_1, ..., c_ky_k/b_kx_k; az; q; bx, b_1x_1, ..., b_kx_k\right]$$
$$= \frac{(z, cy, c_1y_1, ..., c_ky_k; q)_\infty}{(az, bx, b_1x_1, .., b_kx_k; q)_\infty} {}_{k+2}\Phi_{k+1}\left[\begin{array}{c} a, bx, b_1x_1, ..., b_kx_k; q; z \\ cy, c_1y_1, ..., c_ky_k \end{array}\right], \tag{3.3}$$

where $|z| < 1$, $|bx| < 1$, $|b_1x_1| < 1$,...$|b_kx_k| < 1$, Φ_D is the basic Lauricella function which is the q-analogue of the fourth Lauricella function see [Andrews 1; problem 4, p. 207]. (3.3) is the theorem 5 (Andrews (1972), p. 621).
Taking $k = 1$, (3.3) yields

$$\Phi_D\left[z; \frac{cy}{bx}, \frac{c_1y_1}{b_1x_1}; az; q; bx, b_1x_1\right]$$
$$= \frac{(z, cy, c_1y_1; q)_\infty}{(az, bx, b_1x_1; q)_\infty} {}_3\Phi_2\left[\begin{array}{c} a, bx, b_1x_1; q; z \\ cy, c_1y_1 \end{array}\right]. \tag{3.4}$$

In chapter 3 of Agarwal, R. P. [4] book “Resonance of Ramanujan Mathematics, Part III” large number of results are given on the continued fractions for the ratio's of ${}_3\Phi_2$. Making use of these results one can have continued fraction representation for the ratio's of Φ_D.

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