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LYAPUNOV TYPE INEQUALITY FOR DISCRETE FRACTIONAL BOUNDARY VALUE PROBLEM

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Abstract: In this paper we consider the discrete fractional boundary value problem. The Green's function and it's properties are used to find maximum value of function. With the help of maximum value of the function Lyapunov type inequality is obtain for this problem.

Keywords and Phrases: Lyapunov type inequality, Fractional difference equation, discrete fractional boundary value problem.

2020 Mathematics Subject Classification: 26D15, 39A12.

1. Introduction

In 1907 Lyapunov [14] proved that if the boundary value problem

$$\begin{cases} y''(t) + q(t)y(t) = 0, \quad a < t < b, \\ y(a) = y(b) = 0, \end{cases} \tag{1.1}$$

has a nontrivial solution, where $q(t)$ is a continuous and real valued function on $[a, b]$, then

$$\int_a^b |q(u)|du > \frac{4}{b-a}. \tag{1.2}$$

Ferreira has generalized this result by replacing classical derivative y'' by a fractional order derivative D^α, in both Riemann-Liouville fractional derivative and Caputo fractional derivative sense [9, 10]. Also there are so many generalizations and extensions of the result (1.2) exist in the literature [4, 5, 6, 7, 15, 16, 20, 21].

Recently, the authors [1, 17] obtained Lyapunov type inequalities for fractional boundary value problem using generalized fractional differential and integral operators such as Prabhakar derivative, k-Prabhakar derivative, Prabhakar integral and k-Prabhakar integral.

In [8] Ferreira consider the following discrete fractional boundary value problem

$$(\triangle^\alpha y)t = -q(t+\alpha-1)y(t+\alpha-1), \quad 1 < \alpha \leq 2, \tag{1.3}$$

coupled with one of the following boundary conditions

$$y(\alpha-2) = y(t+\alpha-1) = 0 \tag{1.4}$$

$$\text{or} \quad y(\alpha-2) = \triangle y(\alpha+b) = 0 \tag{1.5}$$

where $b \in \mathbb{N}$ and obtained the discrete Lyapunov type inequalities for the above conjugate boundary value problem (1.3) and (1.4), and right focal boundary value problem (1.3) and (1.5).

Also, in [8] Chidouh and et al. proved generalization of the Lyapunov inequality of [10]. Motivated by the above results. In this paper we consider the following discrete boundary value problem of fractional difference equation of the form

$$-\triangle^\alpha y(t) = \lambda h(t+\alpha-1)f(y(t+\alpha-1)) \tag{1.6}$$

$$y(\alpha-2) = 0, \quad \triangle y(\alpha-2) = \triangle y(\alpha+b-1) \tag{1.7}$$

where $t \in [0,b]_{\mathbb{N}_0}$, $f : [0,\infty) \to [0,\infty)$ is continuous and non decreasing, $h : [\alpha-1, \alpha+b]_{\mathbb{N}_{\alpha-1}} \to [0,\infty)$, $1 < \alpha \leq 2$, λ is a positive parameter and we obtain the discrete Lyapunov type inequality for this problem.

In [18, 19] D. Pachpatte and et al. obtained the sufficient condition for existence of solutions to discrete boundary value problem (1.6) − (1.7) and also established the existence result using Krasnoselski fixed point theorem.

2. Some Basic Definitions and Preliminary Results

Definition 2.1. [2, 11] *The falling factorial function is defined as*

$$t^{\underline{\alpha}} = \frac{\Gamma(t+1)}{\Gamma(t+1-\alpha)},$$

for any t and α for which the right hand side is defined. We also appeal to convention that if $t+1-\alpha$ is pole of the Gamma function and $t+1$ is not a pole, then $t^{\underline{\alpha}} = 0$.

Remark 2.1. *Using the properties of the Gamma function it is clear that $t^{\underline{\alpha}} \geq 0$, for $t \geq \alpha \geq 0$.*

Definition 2.2. [11, 13] *The α-th order fractional sum of a function f defined on $\mathbb{N}_a := \{a, a+1, a+2, ...\}$, $a \in \mathbb{R}$ and for $\alpha > 0$, is defined as*

$$\triangle_a^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t-\alpha} (t-s-1)^{\underline{\alpha-1}} f(s), \quad where\, t \in \mathbb{N}_{a+\alpha}.$$

Definition 2.3. [11, 13] *The α-th order fractional difference of the function $f : \mathbb{N}_a \to \mathbb{R}$, where $\mathbb{N}_a := \{a, a+1, a+2, ...\}$, $a \in \mathbb{R}$ and for $\alpha > 0$ ($N-1 \leq \alpha \leq N$, where $N \in \mathbb{N}$) is defined by*

$$\triangle^{\alpha} f(t) = \triangle^n \triangle^{-(n-\alpha)} f(t), \quad t \in \mathbb{N}_{a+n-\alpha},$$

where $\triangle^n$ is the standard forward difference of order n.

Lemma 2.1. [2, 12] *Let t and α be any numbers for which $t^{\underline{\alpha}}$ and $t^{\underline{\alpha-1}}$ are defined. Then*

$$\triangle t^{\underline{\alpha}} = \alpha t^{\underline{\alpha-1}}.$$

Lemma 2.2. [3] *Assume $\alpha, \mu > 0$ and $f : \mathbb{N}_a \to \mathbb{R}$ be a real-valued function. Moreover, let $\alpha, \mu > 0$. Then we have*

$$\triangle^{-\alpha}[\triangle^{-\mu} f(t)] = \triangle^{-(\mu+\alpha)} f(t) = \triangle^{-\mu}[\triangle^{-\alpha} f(t)], \text{ where } t \in \mathbb{N}_{\mu+\alpha+a}.$$

Lemma 2.3. [2, 12] *Let $0 \leq N-1 < \alpha \leq N$. Then*

$$\triangle^{-\alpha}\triangle^{\alpha} y(t) = y(t) + C_1 t^{\underline{\alpha-1}} + C_2 t^{\underline{\alpha-2}} + ... + C_N t^{\underline{\alpha-N}},$$

for some $C_i \in \mathbb{R}$, with $1 \leq i \leq N$.

Lemma 2.4. [22] *Let* $h : [\alpha-1, \alpha+b]_{\mathbb{N}_{\alpha-1}} \to [0, \infty)$, *be given. Then the unique solution of discrete fractional boundary value problem*

$$-\triangle_0^\alpha y(t) = h(t+\alpha-1),$$
$$y(\alpha-2) = 0, \triangle y(\alpha-2) = \triangle y(\alpha+b-1),$$

where, $t \in [0, b]_{\mathbb{N}_0}$, *is*

$$y(t) = -\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{b} G(t,s)h(s+\alpha-1), \tag{2.1}$$

where, $G : [\alpha-2, \alpha+b]_{\mathbb{N}_{\alpha-2}} \times [0, b]_{\mathbb{N}_0} \to \mathbb{R}$ *is defined as*

$$G(t,s) = \begin{cases} \frac{t^{\underline{\alpha-1}}(\alpha+b-s-2)^{\underline{\alpha-2}}}{\Gamma(\alpha-1)-(\alpha+b-1)^{\underline{\alpha-2}}} + (t-s-1)^{\underline{\alpha-1}}, & 0 \le s < t-\alpha \le b \\ \frac{t^{\underline{\alpha-1}}(\alpha+b-s-2)^{\underline{\alpha-2}}}{\Gamma(\alpha-1)-(\alpha+b-1)^{\underline{\alpha-2}}}, & 0 \le t-\alpha < s \le b. \end{cases} \tag{2.2}$$

Lemma 2.5. [18] *The function* y *is a solution to the boundary value problem* $(1.6)-(1.7)$ *if and only if* y *satisfies*

$$y(t) = -\frac{\lambda}{\Gamma(\alpha)}\sum_{s=0}^{b} G(t,s)h(s+\alpha-1)f(y(s+\alpha-1)), \tag{2.3}$$

where

$$G(t,s) = \begin{cases} \frac{t^{\underline{\alpha-1}}(\alpha+b-s-2)^{\underline{\alpha-2}}}{\Gamma(\alpha-1)-(\alpha+b-1)^{\underline{\alpha-2}}} + (t-s-1)^{\underline{\alpha-1}}, & 0 \le s < t-\alpha \le b \\ \frac{t^{\underline{\alpha-1}}(\alpha+b-s-2)^{\underline{\alpha-2}}}{\Gamma(\alpha-1)-(\alpha+b-1)^{\underline{\alpha-2}}}, & 0 \le t-\alpha < s \le b. \end{cases}$$

Theorem 2.1. [18] *Assume that the function* $h : [\alpha-1, \alpha+b]_{\mathbb{N}_{\alpha-1}} \to [0, \infty)$ *is continuous and the conditions*
$\mathbb{H}_1 = \lim_{y\to 0} \frac{f(y)}{y} = \infty$ *and*
$\mathbb{H}_2 = \lim_{y\to b} \frac{f(y)}{y} = 0$ *holds then the problem* $(1.6)-(1.7)$ *has at least one solution.*

Theorem 2.2. *The Green's function G(t,s) is given by* (2.2) *satisfies the following conditions:*
1. $G(t,s) > 0$ *for all* $t \in [\alpha-2, \alpha+b]_{\mathbb{N}_{\alpha-2}}$ *and* $s \in [0, b]_{\mathbb{N}_0}$;
2. $\max_{t\in[\alpha-2,\alpha+b]} G(t,s) = G(s+\alpha-2, s), \quad s \in [0, b]_{\mathbb{N}_0},\ t \in [\alpha-2, \alpha+b]_{\mathbb{N}_{\alpha-2}}$.

Proof. Similar to the one found in [10].

3. Main Results

Theorem 3.1. *The function $G(s+\alpha-2,s)$ has a unique maximum given by*

$$\max_{s\in[0,b]_{\mathbb{N}_0}} G(s+\alpha-2,s)=\frac{\Gamma(b+\alpha-1)\Gamma(\alpha-1)\Gamma(b+2)}{\Gamma(b)[\Gamma(\alpha-1)\Gamma(b+2)-\Gamma(\alpha+b)]}$$

Proof. To find the maximum of $G(s+\alpha-2,s)$ over s we first apply the difference operator to $G(s+\alpha-2,s)$. Specifically:

$$\begin{aligned}
&\triangle G(s+\alpha-2,s)\\
&=\triangle\left(\frac{(s+\alpha-2)^{\underline{\alpha-1}}(\alpha+b-s-2)^{\underline{\alpha-2}}}{\Gamma(\alpha-1)-(\alpha+b-1)^{\underline{\alpha-2}}}\right)\\
&=\frac{1}{\Gamma(\alpha-1)-(\alpha+b-1)^{\underline{\alpha-2}}}\triangle\left((s+\alpha-2)^{\underline{\alpha-1}}(\alpha+b-s-2)^{\underline{\alpha-2}}\right)\\
&=\frac{1}{\Gamma(\alpha-1)-(\alpha+b-1)^{\underline{\alpha-2}}}\triangle\left(\frac{\Gamma(s+\alpha-1)}{\Gamma(s)}\frac{\Gamma(\alpha+b-s-1)}{\Gamma(b-s+1)}\right)\\
&=\frac{1}{\Gamma(\alpha-1)-(\alpha+b-1)^{\underline{\alpha-2}}}\\
&\quad\times\left(\frac{\Gamma(s+\alpha)}{\Gamma(s+1)}\frac{\Gamma(\alpha+b-s-2)}{\Gamma(b-s)}-\frac{\Gamma(s+\alpha-1)}{\Gamma(s)}\frac{\Gamma(\alpha+b-s-1)}{\Gamma(b-s+1)}\right)\\
&=\frac{1}{\Gamma(\alpha-1)-(\alpha+b-1)^{\underline{\alpha-2}}}\\
&\quad\times\left(\frac{\Gamma(s+\alpha)}{\Gamma(s+1)}\frac{\Gamma(\alpha+b-s-2)}{\Gamma(b-s)}\frac{(b-s)}{(b-s)}-\frac{s}{s}\frac{\Gamma(s+\alpha-1)}{\Gamma(s)}\frac{\Gamma(\alpha+b-s-1)}{\Gamma(b-s+1)}\right)\\
&=\frac{1}{\Gamma(\alpha-1)-(\alpha+b-1)^{\underline{\alpha-2}}}\\
&\quad\times\left(\frac{(b-s)\Gamma(s+\alpha)\Gamma(\alpha+b-s-2)}{\Gamma(s+1)\Gamma(b-s+1)}-\frac{s\Gamma(s+\alpha-1)\Gamma(\alpha+b-s-1)}{\Gamma(s+1)\Gamma(b-s+1)}\right)\\
&=\frac{1}{\Gamma(\alpha-1)-(\alpha+b-1)^{\underline{\alpha-2}}}\left(\frac{(b-s)\Gamma(s+\alpha-1+1)\Gamma(\alpha+b-s-2)}{\Gamma(s+1)\Gamma(b-s+1)}\right.\\
&\qquad\left.-\frac{s\Gamma(s+\alpha-1)\Gamma(\alpha+b-s-1-1+1)}{\Gamma(s+1)\Gamma(b-s+1)}\right)\\
&=\frac{1}{\Gamma(\alpha-1)-(\alpha+b-1)^{\underline{\alpha-2}}}
\end{aligned}$$

$$\times\Bigg(\frac{(b-s)(s+\alpha-1)\Gamma(s+\alpha-1)\Gamma(\alpha+b-s-2)}{\Gamma(s+1)\Gamma(b-s+1)} - \frac{s\Gamma(s+\alpha-1)\Gamma(\alpha+b-s-2+1)}{\Gamma(s+1)\Gamma(b-s+1)}\Bigg)$$

$$= \frac{1}{\Gamma(\alpha-1)-(\alpha+b-1)^{\underline{\alpha-2}}} \times\Bigg(\frac{(b-s)(s+\alpha-1)\Gamma(s+\alpha-1)\Gamma(\alpha+b-s-2)}{\Gamma(s+1)\Gamma(b-s+1)} - \frac{s(\alpha+b-s-2)\Gamma(s+\alpha-1)\Gamma(\alpha+b-s-2)}{\Gamma(s+1)\Gamma(b-s+1)}\Bigg)$$

$$= \frac{(b-s)(s+\alpha-1)-s(\alpha+b-s-2)}{\Gamma(\alpha-1)-(\alpha+b-1)^{\underline{\alpha-2}}}\frac{\Gamma(s+\alpha-1)\Gamma(\alpha+b-s-2)}{\Gamma(s+1)\Gamma(b-s+1)}$$

$$= \frac{(b\alpha-b+3s-2s\alpha)}{\Gamma(\alpha-1)-(\alpha+b-1)^{\underline{\alpha-2}}}\frac{\Gamma(s+\alpha-1)\Gamma(\alpha+b-s-2)}{\Gamma(s+1)\Gamma(b-s+1)}$$

$$= \frac{b(\alpha-1)+s(3-2\alpha)}{\Gamma(\alpha-1)-(\alpha+b-1)^{\underline{\alpha-2}}}\frac{\Gamma(s+\alpha-1)\Gamma(\alpha+b-s-2)}{\Gamma(s+1)\Gamma(b-s+1)}$$

$$= \frac{b(\alpha-1)+s(3-2\alpha)}{\Gamma(\alpha-1)\Gamma(b+2)-\Gamma(\alpha+b)}\frac{\Gamma(b+2)\Gamma(s+\alpha-1)\Gamma(\alpha+b-s-2)}{\Gamma(s+1)\Gamma(b-s+1)}.$$

Therefore
$\triangle G(s+\alpha-2,s)=(b(\alpha-1)+s(3-2\alpha)+1)f(s)$, with $f(s)>0$ for all $s\in[0,b]_{\mathbb{N}_0}$. Now, if $\alpha<\frac{3}{2}$ then $q(s)=(b(\alpha-1)+s(3-2\alpha)+1)$, is increasing and since $q(0)=b(\alpha-1)+1>0$, then we conclude that $G(s+\alpha-2,s)$ is increasing. On the other hand, if $\alpha\geq\frac{3}{2}$ then q is decreasing but nevertheless positive since $q(b)=b(\alpha-1)+b(3-2\alpha)+1=b(2-\alpha)+1>0$. In conclusion, $G(s+\alpha-2,s)$ is increasing for all s. Therefore,

$$\max_{s\in[0,b]_{\mathbb{N}_0}} G(s+\alpha-2,s)=G(b+\alpha-2,b)=\frac{\Gamma(b+\alpha-1)\Gamma(\alpha-1)\Gamma(b+2)}{\Gamma(b)[\Gamma(\alpha-1)\Gamma(b+2)-\Gamma(\alpha+b)]}$$

And this completes the proof.

Theorem 3.2. *Let $h:[\alpha-1,\alpha+b]_{\mathbb{N}_{\alpha-1}}\to[0,\infty)$ be a nontrivial function. Assume that $f\in C(\mathbb{R}_+,\mathbb{R}_+)$ is a nondecreasing function. If the discrete fractional boundary value problem* (1.6) $-$ (1.7) *has a nontrivial solution given by* (2.3), *then*

$$\sum_{s=0}^{b}|h(s+\alpha-1)|\geq\frac{\Gamma(\alpha)\Gamma(b)[\Gamma(\alpha-1)\Gamma(b+2)-\Gamma(\alpha+b)]\eta}{\Gamma(\alpha-1)\Gamma(b+2)\Gamma(b+\alpha-1)f(\eta)}$$

where, $\eta = \max_{[\alpha-1,\alpha+b]_{\mathbb{N}_{\alpha-1}}} y(s+\alpha-1)$.

Proof. Since the discrete fractional boundary value problem $(1.6)-(1.7)$ has a nontrivial solution as

$$y(t) = -\frac{\lambda}{\Gamma(\alpha)}\sum_{s=0}^{b} G(t,s)h(s+\alpha-1)f(y(s+\alpha-1)),$$

$$\|y\| \leq \frac{\lambda}{\Gamma(\alpha)}\sum_{s=0}^{b} |G(t,s)||h(s+\alpha-1)||f(y(s+\alpha-1))|,$$

$$\|y\| \leq \frac{\lambda}{\Gamma(\alpha)}\sum_{s=0}^{b} G(s+\alpha-2,s)|h(s+\alpha-1)||f(y(s+\alpha-1))|,$$

where $\eta = \max_{[\alpha-1,\alpha+b]_{\mathbb{N}_{\alpha-1}}} y(s+\alpha-1)$. Taking into account that f is nondecreasing we get

$$\|y\| \leq \frac{\lambda}{\Gamma(\alpha)}\frac{\Gamma(b+\alpha-1)\Gamma(\alpha-1)\Gamma(b+2)}{\Gamma(b)[\Gamma(\alpha-1)\Gamma(b+2)-\Gamma(\alpha+b)]}\sum_{s=0}^{b}|h(s+\alpha-1)|f(\eta)$$

$$\sum_{s=0}^{b}|h(s+\alpha-1)| \geq \frac{\Gamma(\alpha)\Gamma(b)[\Gamma(\alpha-1)\Gamma(b+2)-\Gamma(\alpha+b)]\eta}{\Gamma(\alpha-1)\Gamma(b+2)\Gamma(b+\alpha-1)f(\eta)}$$

And this completes the proof.

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