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A NOTE ON THE ORDER AND TYPE OF BICOMPLEX VALUED ENTIRE FUNCTIONS

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Abstract: The main target of this paper is to find out the estimates of the order and type of a bicomplex valued entire function. Also the famous Lucas's theorem on the zeros of a polynomial is deduced in the light of bicomplex analysis. A result is proved to show that the order and type remain invariant under differentiation of an entire function in $\mathbb{C}_2$. Also we prove some results related to Hadamard composition of two entire functions in $\mathbb{C}_2$. In fact, we find out here an estimate of the type of the Hadamard composition of two bicomplex valued entire functions. Also we show that the zeros of the derivative of a polynomial $P(z)$ in $\mathbb{C}_2$ are contained within the convex hull of the zeros of $P(z)$. Some examples are provided to justify the

results obtained here.

Keywords and Phrases: Analytic function, Bicomplex valued function, Lucas's Theorem, Order, Taylor's Theorem, Type.

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1. Introduction and Preliminaries

The theory of bicomplex numbers is a matter of active research for quite a long time since seminal work as carried in [8] and [1] in search of special algebra. The algebra of bicomplex numbers are widely used in the literature as it becomes viable commutative alternative [9] to the non skew field of quaternions introduced by Hamilton [5] (both are four dimensional and generalization of complex numbers). Now we will discuss some basic definitions and preliminaries of bicomplex analysis. A bicomplex number is defined as $z = x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4 = (x_1 + i_1x_2) + i_2(x_3 + i_1x_4) = z_1 + i_2z_2$ where $x_i, i = 1, 2, 3, 4$ are all real numbers with $i_1^2 = i_2^2 = -1, i_1i_2 = i_2i_1, (i_1i_2)^2 = 1$ and z_1, z_2 are complex numbers. The set of all bicomplex numbers, complex numbers and real numbers are respectively denoted by $\mathbb{C}_2, \mathbb{C}_1$ and $\mathbb{C}_0$. i_2-conjugate bicomplex number of $z_1+i_2z_2$ is $z_1-i_2z_2$ and i_1i_2-conjugate bicomplex number of $z_1 + i_2z_2$ is $\overline{z}_1 - i_2\overline{z}_2$.
Addition is the operation on $\mathbb{C}_2$ defined by the function $\oplus : \mathbb{C}_2 \times \mathbb{C}_2 \to \mathbb{C}_2$,

$$(x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4,\ y_1 + i_1y_2 + i_2y_3 + i_1i_2y_4) =$$

$$(x_1 + y_1) + i_1(x_2 + y_2) + i_2(x_3 + y_3) + i_1i_2(x_4 + y_4).$$

Scalar multiplication is the operation on $\mathbb{C}_2$ defined by the function $\odot : \mathbb{C}_0 \times \mathbb{C}_2 \to \mathbb{C}_2$,

$$(a, x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4) = (ax_1 + i_1ax_2 + i_2ax_3 + i_1i_2ax_4)$$

where $a \in \mathbb{C}_0$ be any real number. The system $(\mathbb{C}_2, \oplus, \odot)$ is a linear space. Here the norm is defined as

$$\begin{aligned} ||\ || &: \mathbb{C}_2 \to \mathbb{R}_{\geq 0}, \\ \|x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4\| &= \left(x_1^2 + x_2^2 + x_3^2 + x_4^2\right)^{\frac{1}{2}}. \end{aligned}$$

So the system $(\mathbb{C}_2, \oplus, \odot, ||\ ||)$ is a normed linear space. Now, we will discuss the idempotent representation of bicomplex numbers. There are four idempotent elements in $\mathbb{C}_2$. They are

$$0, 1, \frac{1 + i_1i_2}{2}, \frac{1 - i_1i_2}{2}.$$

We now denote two non trivial idempotent elements by

$$e_1 = \frac{1 + i_1 i_2}{2} \quad \text{and} \quad e_2 = \frac{1 - i_1 i_2}{2} \quad \text{in } \mathbb{C}_2.$$

$$\text{where } e_1^2 = e_1, e_2^2 = e_2, e_1 e_2 = e_2 e_1 = 0, e_1 + e_2 = 1.$$

So, e_1 and e_2 are alternatively called orthogonal idempotents. Every element $\xi = (z_1 + i_2 z_2) \in \mathbb{C}_2$ has the following unique representation,

$$\begin{aligned} \xi &= (z_1 - i_1 z_2)\, e_1 + (z_1 + i_1 z_2)\, e_2 \\ &= \xi_1 e_1 + \xi_2 e_2, \ \text{where } \xi_1, \xi_2 \text{ are complex numbers.} \end{aligned}$$

This is known as idempotent representation of the element $\xi = (z_1 + i_2 z_2) \in \mathbb{C}_2$. An element $\xi = (z_1 + i_2 z_2) \in \mathbb{C}_2$ is non-singular iff $|z_1^2 + z_2^2| \neq 0$ and it is singular iff $|z_1^2 + z_2^2| = 0$. The set of all singular elements is denoted by θ_2.

If $f(z)$ is a bicomplex valued function, then f can be represented as $f(z) = f_1(z_1)\, e_1 + f_2(z_2)\, e_2$ where $f_1(z_1), f_2(z_2) \in \mathbb{C}_1$ and f_1, f_2 are both functions in $\mathbb{C}_1$. This type of decomposition is known as Ringleb decomposition [6] in $\mathbb{C}_2$. Let $a = (a_1 + i_1 a_2 + i_2 a_3 + i_1 i_2 a_4)$ be a fixed point in $\mathbb{C}_2$. Set $\alpha = a_1 + i_1 a_2$ and $\beta = a_3 + i_1 a_4$. Then $a = (a_1 + i_1 a_2 + i_2 a_3 + i_1 i_2 a_4) = \alpha + i_2 \beta$. Let r, r_1, r_2 denote numbers in $\mathbb{C}_0$ such that $r > 0, \quad r_1 > 0, \quad r_2 > 0$. Let $A_1 = \{z_1 - i_1 z_2 : z_1, z_2 \text{ in } \mathbb{C}_1\}$ and $A_2 = \{z_1 + i_1 z_2 : z_1, z_2 \text{ in } \mathbb{C}_1\}$. Let w_1 and w_2 denote the numbers in A_1 and A_2, repectively. Observe that w_1 and w_2 are in fact complex numbers in $\mathbb{C}_1$. We should recall here that the open ball with centre a and radius r is denoted by $B(a, r)$ and the closed ball is denoted by $\overline{B}(a, r)$. They are defined respectively as follows:

$$\begin{aligned} B(a,r) &= \{z_1 + i_2 z_2 \in \mathbb{C}_2 : \|(z_1 + i_2 z_2 - (\alpha + i_2 \beta))\| < r\} \; \textit{and} \\ \overline{B}(a,r) &= \{z_1 + i_2 z_2 \in \mathbb{C}_2 : \|(z_1 + i_2 z_2 - (\alpha + i_2 \beta))\| \leq r\}. \end{aligned}$$

Then the open and closed discus with centre a and radii r_1, r_2 respectively denoted by $D(a; r_1, r_2)$ and $\overline{D}(a; r_1, r_2)$ are defined as

$$D(a; r_1, r_2) = \left\{ \begin{array}{c} z_1 + i_2 z_2 \in \mathbb{C}_2 : z_1 + i_2 z_2 = w_1 e_1 + w_2 e_2, \\ |w_1 - (\alpha - i_1 \beta)| < r_1, |w_2 - (\alpha - i_1 \beta)| < r_2 \end{array} \right\} \textit{and}$$

$$\overline{D}(a; r_1, r_2) = \left\{ \begin{array}{c} z_1 + i_2 z_2 \in \mathbb{C}_2 : z_1 + i_2 z_2 = w_1 e_1 + w_2 e_2, \\ |w_1 - (\alpha - i_1 \beta)| \leq r_1, |w_2 - (\alpha - i_1 \beta)| \leq r_2 \end{array} \right\}.$$

If w is any bicomplex number then the sequence $\left(1 + \frac{w}{n}\right)^n$ converges to a bicomplex number denoted by $\exp(w)$ or e^w, called the bicomplex exponential function. That is

$$e^w = \lim_{n \to \infty} \left(1 + \frac{w}{n}\right)^n.$$

If $w = (z_1 + i_2 z_2)$, then we get that,

$$e^w = e^{z_1}(\cos z_2 + i_2 \sin z_2) = e^{|w|_{i_1}}\left(\cos \arg_{i_1} w + \sin \arg_{i_1} w\right) \quad \text{where } e^w \notin \theta_2.$$

Let $f : \Omega \subset \mathbb{C}_2 \to \mathbb{C}_2$ be a bicomplex valued function. The derivative of f at a point $\omega_0 \in \Omega$ is defined by $f'(\omega_0) = \lim_{h\to 0} \frac{f(\omega_0+h)-f(\omega_0)}{h}$, provided the limit exists and the domain is so chosen that $h = h_0 + i_1 h_1 + i_2 h_2 + i_1 i_2 h_3$ is invertible. It is easy to prove that h is not invertible only for $h_0 = -h_3$, $h_1 = h_2$ or $h_0 = h_3$, $h_1 = -h_2$.

If the bicomplex derivative of f exists at each point of its domain then in similar to complex function, f will be a bicomplex holomorphic function in Ω. Indeed if f can be expressed as $f(\omega) = g_1(z_1, z_2) + i_2 g_2(z_1, z_2)$, $\omega = z_1 + i_2 z_2 \in \Omega$ then f will be holomorphic if and only if g_1, g_2 are both complex holomorphic in z_1, z_2 and

$$\frac{\partial g_1}{\partial z_1} = \frac{\partial g_2}{\partial z_2}, \frac{\partial g_1}{\partial z_2} = -\frac{\partial g_2}{\partial z_1}.$$

Moreover, $f'(\omega) = \frac{\partial g_1}{\partial z_1} + i_2 \frac{\partial g_2}{\partial z_1}$. A function f is said to be a bicomplex entire function if f is bicomplex holomorphic in the whole bicomplex plane $\mathbb{C}_2$. A function f is said to be bicomplex meromorphic function in an open set Ω if f is a quotient $\frac{g}{h}$ of two functions which are bicomplex holomorphic in Ω where $h \notin \theta_2$. If f is a bicomplex meromorphic function, then f can be represented as $f(z) = f_1(z_1) e_1 + f_2(z_2) e_2$ where $f_1(z_1), f_2(z_2) \in \mathbb{C}_1$ and $f_1(z_1), f_2(z_2)$ are both meromorphic functions in $\mathbb{C}_1$. A bicomplex entire function $f(w)$ can be expressed as $f(w) = f_{e_1}(z_1 - i_1 z_2)e_1 + f_{e_2}(z_1 + i_1 z_2)e_2$. For more details, one can refer [7].

A series of the form $\sum_{k=0}^{\infty} \xi_k$, $\xi_k \in \mathbb{C}_2$ is called an infinite series in $\mathbb{C}_2$. Let $\{S_n\}$ be the sequence of partial sum of the above series. Then $S_n = \sum_{k=0}^{n} \xi_k$, for all $n \in \mathbb{N}$. Then the infinite sum converges iff $\lim_{n\to\infty} S_n$ exists and diverges iff the limit does not exist. If $\lim_{n\to\infty} S_n = \xi^*$ then ξ^* is called the sum of the series and we write $\sum_{k=0}^{\infty} \xi_k = \xi^*$. The infinite series $\sum_{k=0}^{\infty} \xi_k$ has the sum $\xi^* = z_1^* + i_2 z_2^*$ iff the following two infinite series converge and have the sums

$$\begin{aligned}\sum_{k=0}^{\infty}(z_{1k} - i_1 z_{2k}) &= z_1^* - i_1 z_2^*,\\ \sum_{k=0}^{\infty}(z_{1k} + i_1 z_{2k}) &= z_1^* + i_1 z_2^*.\end{aligned}$$

For better understanding of series of bicomplex numbers, one can see [3].
Now let us define the order of a bicomplex entire function. The order ρ_f of an entire function f in $\mathbb{C}_1$ is defined ([4], [2]) in the following way:

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log r} \ \textit{where} \ \ M_f(r) = \max\{|f(z)| : |z| = r\}.$$

The order ρ_f of a bicomplex entire function $f(w) = f_{e_1}(z_1 - i_1 z_2)e_1 + f_{e_2}(z_1 + i_1 z_2)e_2$ is defined by $\rho_f = \max\{\rho_{f_{e_1}}, \rho_{f_{e_2}}\}$

$$\text{where } \rho_{f_{e_i}} = \limsup_{r\to\infty} \frac{\log^{[2]} M_{f_{e_i}}(r_i)}{\log r_i} \text{ for } i = 1, 2.$$

Now, the type σ_f of an entire function f in $\mathbb{C}_1$ is defined ([4], [2]) as follows:

$$\sigma_f = \limsup_{r\to\infty} \frac{\log M_f(r)}{r^{\rho_f}} \ \textit{where} \ \ M_f(r) = \max\{|f(z)| : |z| = r\}.$$

The type σ_f of a bicomplex entire function, $f(w) = f_{e_1}(z_1 - i_1 z_2)e_1 + f_{e_2}(z_1 + i_1 z_2)e_2$ in $\mathbb{C}_2$ is defined as $\sigma_f = \max\{\sigma_{f_{e_1}}, \sigma_{f_{e_2}}\}$

$$\text{where } \sigma_{f_{e_i}} = \limsup_{r\to\infty} \frac{\log M_{f_{e_i}}(r_i)}{r^{\rho_{f_{e_i}}}} \text{ for } i = 1, 2.$$

In this paper our prime concern is to estimate the order and type of a bicomplex valued entire function and also to derive the well known Lucas's theorem on the zeros of a polynomial in the bicomplexial context. We do not explain the standard definitions and notatios of the theories of bicomplex valued entire functions as those are available in [2], [6], [7] and [9].

2. Lemmas

In this section, we present some relevant lemmas which will be needed in the sequel.

Lemma 2.1. ([7]) *Let X be a domain in $\mathbb{C}_2$ and $f : X \to \mathbb{C}_2$ be a differentiable function on X. Then for each a in X there is a discus $D(a; R_1, R_2)$ such that $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$ where $a_n = \frac{f^n(a)}{n!}$ for $n = 0, 1, 2, \ldots$ and for all $z \in D(a; R_1, R_2)$.*

Remark 2.1. *Lemma 2.1 is known as Taylor's theorem in $\mathbb{C}_2$.*

Lemma 2.2. *Let a bicomplex valued function $f(z)$ has Taylor's series expansion*

$\sum\limits_{n=0}^{\infty} a_n z^n$ *on a discuss* $D(0; R_1, R_2)$. *Suppose there exists numbers* $\mu > 0$, $\lambda > 0$ *and an integer* $N = N(\mu; \lambda) > 0$ *with* $\|a_n\| < \left(\frac{e\mu\lambda}{n}\right)^{\frac{n}{\mu}}$ *for all* $n > N$. *Then* $f(z)$ *is entire and also given any* $\varepsilon > 0$, *there are numbers* $R_0' > 0$ *and* $R_0'' > 0$ *such that*

$$M_f(R_1) = \max\{|f(z)| : |z| = R_1\} < \exp\{(\lambda + \varepsilon) R_1^{\mu}\} \text{ for all } R_1 > R_0',$$

$$M_f(R_2) = \max\{|f(z)| : |z| = R_2\} < \exp\{(\lambda + \varepsilon) R_2^{\mu}\} \text{ for all } R_2 > R_0''.$$

Proof. By idempotent decomposition, $f(z)$ can be written as $f(z) = f_1(z_1) e_1 + f_2(z_2) e_2$, where $f_1(z_1), f_2(z_2) \in \mathbb{C}_1$. On the discuss $D(0; R_1, R_2)$, the maximum modulus function corresponding to f_i denoted by $M_{f_i}(R_i)$ can be written as $M_{f_i}(R_i) = \max\limits_{|z_i|=R_i} |f_i(z_i)|$ for $i = 1, 2$. Since $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n = \sum\limits_{n=0}^{\infty} (a_n' e_1 + a_n'' e_2)(z_1 e_1 + z_2 e_2)^n = \left(\sum\limits_{n=0}^{\infty} a_n' z_1^n\right) e_1 + \left(\sum\limits_{n=0}^{\infty} a_n'' z_2^n\right) e_2$, therefore we can write $f_1(z_1) = \sum a_n' z_1^n$ on $|z_1| = R_1$ and $f_2(z_2) = \sum a_n'' z_2^n$ on $|z_2| = R_2$.

Since $\|a_n\| < \left(\frac{e\mu\lambda}{n}\right)^{\frac{n}{\mu}}$, $\forall\ n > N$ we have $\frac{|a_n'|}{\sqrt{2}} \le \|a_n\| < \left(\frac{e\mu\lambda}{n}\right)^{\frac{n}{\mu}}, \forall\ n > N$ i.e., $|a_n'|^{\frac{1}{n}} < \left(\frac{e\mu\lambda}{n}\right)^{\frac{1}{\mu}} 2^{\frac{1}{2n}}, \forall\ n > N$. Thus $|a_n'|^{\frac{1}{n}} \to 0$ as $n \to \infty$ and hence $f_1(z_1)$ is entire. Similarly, $f_2(z_2)$ is so. Hence f is entire in $\mathbb{C}_2$. Further, $\sqrt[n]{\frac{|a_n'|}{2} R_1^n} < \sqrt[n]{\|a_n\| R_1^n} < \left(\frac{e\mu\lambda}{n}\right)^{\frac{1}{\mu}} R_1 < \frac{1}{2}$ if $n > n_0 = n_0(R_1) = \{e\mu\lambda 2^{\mu} R_1^{\mu}\}$. Now choose $R' = R'(\mu, \lambda) > 1$ so large that $n_0(R_1) > N$ if $R_1 > R'$, then we have

$$\sqrt[n]{\frac{|a_n'|}{2} R_1^n} < \frac{1}{2}.$$

That is

$$|a_n'| R_1^n < \frac{\sqrt{2}}{2^n} \text{ provided } n > n_0.$$

We now deduce an upper bound for $M_f(R_1)$.

$$\begin{aligned} \text{Now, } M_f(R_1) &= \max_{|z_1|=R_1} \left|\sum_{n=0}^{\infty} a_n' z_1^n\right| \le \sum_{n=0}^{\infty} |a_n'| R_1^n \\ &= \sum_{n=0}^{n_0} |a_n'| R_1^n + \sum_{n=n_0+1}^{\infty} |a_n'| R_1^n \\ &< \sum_{n=0}^{n_0} |a_n'| R_1^n + \sum_{n=n_0+1}^{\infty} \frac{\sqrt{2}}{2^n} \end{aligned}$$

$$< \sum_{n=0}^{n_0} |a'_n| R_1^n + \sqrt{2} \text{ if } R_1 > R'.$$

$$\text{Also, } \sum_{n=0}^{n_0} |a'_n| R_1^n = \sum_{n=0}^{N} |a'_n| R_1^n + \sum_{n=N+1}^{n_0} |a'_n| R_1^n$$

$$< R_1^N \sum_{n=0}^{N} |a'_n| + (n_0 - N) \max_{N+1 \le n \le n_0} |a'_n| R_1^n.$$

Now, $\max_{N+1\le n\le n_0} |a'_n| R_1^n \le \max_{N+1\le n} |a'_n| R_1^n \le \sqrt{2} \max_{N+1\le n} \|a'_n\| R_1^n < \sqrt{2} \max_{N+1\le n} \left(\frac{e\mu\lambda}{n}\right)^{\frac{n}{\mu}} R_1^n$ $= \sqrt{2}\exp\left(\lambda R_1^{\mu}\right)$. The maximum of $\left(\frac{e\mu\lambda}{n}\right)^{\frac{n}{\mu}}$ is achieved for $n = \mu\lambda R_1^{\mu}$, thus we have $\max_{N+1\le n\le n_0} |a_n| R_1^n < \sqrt{2}\exp\left(\lambda R_1^{\mu}\right)$. Hence if $R_1 > R'$, it follows that

$$\begin{aligned} M_f(R_1) &< R_1^N \sum_{n=0}^{N} \left|a'_n\right| + \sqrt{2}\,(n_0 - N)\exp\left(\lambda R_1^{\mu}\right) + \sqrt{2} \\ &= R_1^N \sum_{n=0}^{N} \left|a'_n\right| + \left(2^{\mu} e\mu\lambda R_1^{\mu} - N\right)\sqrt{2}\exp\left(\lambda R_1^{\mu}\right) + \sqrt{2} \\ &= \exp\left(\lambda R_1^{\mu}\right)\left\{2^{\mu+\frac{1}{2}} e\mu\lambda R_1^{\mu} - \sqrt{2}N\exp\left(-\lambda R_1^{\mu}\right)\sum_{n=0}^{N}\left|a'_n\right| + \sqrt{2}\exp\left(-\lambda R_1^{\mu}\right)\right\}. \end{aligned}$$

Given any $\varepsilon > 0$, there exists a number $R'_0 = R'_0(\varepsilon) > R'$ such that the expression above within brackets is less than $\exp\left(\varepsilon R_1^{\mu}\right)$, provided $R_1 > R_0$. Therefore, $M_f(R_1) < \exp\left\{(\lambda+\varepsilon) R_1^{\mu}\right\}$ for all $R_1 > R'_0$. Analogously, we can write for $\varepsilon > 0$, there exists $R''_0 > 0$ such that $M_f(R_2) < \exp\left\{(\lambda+\varepsilon) R_2^{\mu}\right\}$ for all $R_2 > R''_0$. This completes the proof of the lemma.

Lemma 2.3. *If* $\frac{1}{z-k} = 0$ *then* $\frac{1}{\overline{z}-\overline{k}} = 0$ *where* $z, k \in \mathbb{C}_2$ *and* $\overline{z}$, $\overline{k}$ *are the respective* $i_1 i_2-$*conjugate of* z *and* k.

Proof. Let us write as $z = z_1 + i_2 z_2 = (z_1 - i_1 z_2)\, e_1 + (z_1 + i_1 z_2)\, e_2$ and $k = k_1 + i_2 k_2 = (k_1 - i_1 k_2)\, e_1 + (k_1 + i_1 k_2)\, e_2$ where $z_1, z_2, k_1, k_2 \in \mathbb{C}_1$. Now,
$\frac{1}{z-k} = \frac{1}{(z_1-k_1)+i_2(z_2-k_2)} = \frac{\left(\overline{z_1-k_1}\right)-i_2\left(\overline{z_2-k_2}\right)}{\{(z_1-k_1)+i_2(z_2-k_2)\}\left\{\left(\overline{z_1-k_1}\right)-i_2\left(\overline{z_2-k_2}\right)\right\}}$
$= \frac{\left(\overline{z_1}-\overline{k_1}\right)-i_2\left(\overline{z_2}-\overline{k_2}\right)}{|(z_1-k_1)-i_1(z_2-k_2)|^2 e_1+|(z_1-k_1)+i_1(z_2-k_2)|^2 e_2}$.
Since, $\frac{1}{z-k} = 0$, we have $\left(\overline{z_1} - \overline{k_1}\right) - i_2\left(\overline{z_2} - \overline{k_2}\right)$
$= 0$. i.e., $\left[\left(\overline{z_1} - \overline{k_1}\right) + i_1\left(\overline{z_2} - \overline{k_2}\right)\right] e_1 + \left[\left(\overline{z_1} - \overline{k_1}\right) - i_1\left(\overline{z_2} - \overline{k_2}\right)\right] e_2 = 0$.
So it follows that

$\left(\overline{z_1}-\overline{k_1}\right)+i_1\left(\overline{z_2}-\overline{k_2}\right)=0$ and $\left(\overline{z_1}-\overline{k_1}\right)-i_1\left(\overline{z_2}-\overline{k_2}\right)=0.$
Now from above two equations, we obtain that $\overline{z_1}=\overline{k_1}$ and $\overline{z_2}=\overline{k_2}$.
In view of above equations, now it follows that
$\frac{1}{\overline{z}-\overline{k}}=\frac{1}{(z_1-i_2z_2)-(k_1-i_2k_2)}$
$=\frac{\left(\overline{z_1-k_1}\right)+i_2\left(\overline{z_2-k_2}\right)}{\{(z_1-k_1)-i_2(z_2-k_2)\}\left\{\left(\overline{z_1-k_1}\right)+i_2\left(\overline{z_2-k_2}\right)\right\}}$
$=\frac{\left(\overline{z_1}-\overline{k_1}\right)+i_2\left(\overline{z_2}-\overline{k_2}\right)}{|(z_1-k_1)-i_1(z_2-k_2)|^2e_1+|(z_1-k_1)+i_1(z_2-k_2)|^2e_2}=0$. This proves the lemma.

The following lemma [10] says that the order and type remain invariant under differentiation of an entire function in $\mathbb{C}_1$.

Lemma 2.4. *The order and type of the derivative of an entire function in $\mathbb{C}_1$ is equal to the order and type of the function.*

3. Theorems

In this section we prove the main results of our paper. We prove that the result stated in Lemma 2.4 is also true if we change the domain from $\mathbb{C}_1$ to $\mathbb{C}_2$.

Theorem 3.1. *Let $f(z)\in\mathbb{C}_2$ be entire. Then the order and type of $f'(z)$ are same as those of $f(z)$.*

Proof. Since $f(z)\in\mathbb{C}_2$, we have by its idempotent representation $f(z)=f_1(z_1-i_1z_2)e_1+f_2(z_1+i_1z_2)e_2$ where $f_1(z_1-i_1z_2)$ and $f_2(z_1+i_1z_2)\in\mathbb{C}_1$. Now $f'(z)=f_1'(z_1-i_1z_2)e_1+f_2'(z_1+i_1z_2)e_2$. In view of Lemma 2.4, $\rho_f=\max\{\rho_{f_1},\rho_{f_2}\}=\max\left\{\rho_{f_1'},\rho_{f_2'}\right\}=\rho_{f'}$. Similarly, in view of Lemma 2.4, $\sigma_f=\max\{\sigma_{f_1},\sigma_{f_2}\}=\max\left\{\sigma_{f_1'},\sigma_{f_2'}\right\}=\sigma_{f'}$. Thus the theorem is established.

Theorem 3.2. *If $f(z)\in\mathbb{C}_2$ is entire of finite order ρ ($0<\rho<\infty$) and type σ then $\sigma=\frac{1}{e\rho}\limsup_{n\to\infty} n\,\|a_n\|^{\frac{\rho}{n}}$.*

Proof. Let us suppose that σ is finite. As $\sigma=\max\{\sigma_1,\sigma_2\}$ then both σ_1 and σ_2 are finite. Also $\rho=\max\{\rho_1,\rho_2\}$ with $0<\rho_i<\infty$ for $i=1,2$. Now, $a_n=a_n'e_1+a_n''e_2$ where a_n', $a_n''\in\mathbb{C}_1$.

Case I. Let $|a_n'|>|a_n''|$.

As $\|a_n\|<\max\{|a_n'|,|a_n''|\}=|a_n'|$, using Cauchy's inequality in $\mathbb{C}_1$ we may write

$$\|a_n\|<|a_n'|\leq\frac{M_f(R_1)}{r_1^n}<\frac{\exp\left(kr_1^{\rho_1}\right)}{r_1^n}<\frac{\exp\left(kr_1^{\rho}\right)}{r_1^n}\text{ for all } r_1>R_1.$$

Now the minimum value of $\frac{\exp\left(kr_1^{\rho}\right)}{r_1^n}$ occurs for $r_1=\left(\frac{n}{k\rho}\right)^{\frac{1}{\rho}}$. Thus $\|a_n\|<\left(\frac{e\rho k}{n}\right)^{\frac{n}{\rho}}$ if $n>N$ and $r_1=\left(\frac{n}{k\rho}\right)^{\frac{1}{\rho}}>R_1(k)$. Rewriting $k>\frac{1}{e\rho}n\,\|a_n\|^{\frac{\rho}{n}}$, we have

$k \geq \frac{1}{e\rho} \limsup_{n\to\infty} n \|a_n\|^{\frac{\rho}{n}}$. Since k is an arbitrary number exceeding σ, $\sigma \geq \frac{1}{e\rho} \limsup_{n\to\infty} n \|a_n\|^{\frac{\rho}{n}}$.

Now, as the right hand side of the above inequality is finite, let k' be any number exceeding the same. Then there exists a number $N' = N'(k') > 0$ such that $\|a_n\| < \left(\frac{e\rho_1 k'}{n}\right)^{\frac{n}{\rho}}$ for all $n > N'$. Applying Lemma 2.2 with $\lambda = k'$ and $\mu = \rho_1$, given any $\varepsilon > 0$, there exists $R' > 0$ such that $M_f(r_1) < \exp\{(k' + \varepsilon) r_1^{\rho_1}\}$ for all $r_1 > R'$. Thus we have $\sigma_1 \leq k'$ and because of the choice of k', $\sigma \leq \frac{1}{e\rho} \limsup_{n\to\infty} n \|a_n\|^{\frac{\rho}{n}}$.

Case II. Let $|a_n'| < |a_n''|$.

Now we can write, $\|a_n\| < \max\{|a_n'|, |a_n''|\} = |a_n''| \leq \frac{M_f(R_2)}{r_2^n} < \frac{\exp\left(k r_2^{\rho_1}\right)}{r_2^n} < \frac{\exp\left(k r_2^{\rho}\right)}{r_2^n}$ for all $r_2 > R_2$. Now the minimum value of $\frac{\exp\left(k r_2^{\rho}\right)}{r_2^n}$ occurs for $r_2 = \left(\frac{n}{k\rho}\right)^{\frac{1}{\rho}}$.

Thus $\|a_n\| < \left(\frac{e\rho k}{n}\right)^{\frac{n}{\rho}}$ if $n > N_0$ and $r_2 = \left(\frac{n}{k\rho}\right)^{\frac{1}{\rho}} > R_2(k)$. Rewriting $k > \frac{1}{e\rho} n \|a_n\|^{\frac{\rho}{n}}$, we have $k \geq \frac{1}{e\rho} \limsup_{n\to\infty} n \|a_n\|^{\frac{\rho}{n}}$. Since k is an arbitrary number exceeding σ, $\sigma \geq \frac{1}{e\rho} \limsup_{n\to\infty} n \|a_n\|^{\frac{\rho}{n}}$. Now as the right hand side of the above inequality is finite, let k'' be any number exceeding the same. Then there exists a number $N'' = N''(k'') > 0$ such that $\|a_n\| < \left(\frac{e\rho_1 k''}{n}\right)^{\frac{n}{\rho_2}}$ for all $n > N''$. Applying Lemma 2.2 with $\lambda = k''$ and $\mu = \rho_2$, given any $\varepsilon > 0$ there exists $R'' > 0$ such that

$M_f(r_2) < \exp\{(k'' + \varepsilon) r_2^{\rho_2}\}$ for all $r_2 > R''$. Thus $\sigma_2 \leq k''$ and because of the choice of k', $\sigma_2 \leq \frac{1}{e\rho} \limsup_{n\to\infty} n \|a_n\|^{\frac{\rho}{n}}$.

Case III. Let $|a_n'| = |a_n''|$ .Then the case is trivial. Combining Cases I, II and III, we obtain that $\max\{\sigma_1, \sigma_2\} \leq \frac{1}{e\rho} \limsup_{n\to\infty} n \|a_n\|^{\frac{\rho}{n}}$. i.e., $\sigma \leq \frac{1}{e\rho} \limsup_{n\to\infty} n \|a_n\|^{\frac{\rho}{n}}$. Thus we can say that $\sigma = \frac{1}{e\rho} \limsup_{n\to\infty} n \|a_n\|^{\frac{\rho}{n}}$. Thus the theorem is established.

Remark 3.1. *The following examples ensure the validity of the above theorem.*

Example 3.1. The function $f(z) = \sum_{n=1}^{\infty} \left(\frac{e\varrho\sigma}{n}\right)^{\frac{n}{\varrho}} z^n$ is of order ϱ and type σ.

Example 3.2. Since $\lim_{n\to\infty} \frac{\log n}{\log\left(\frac{1}{\sqrt[n]{|a_n|}}\right)} = 0$ characterizes an entire function of order zero, any function with coefficients $\|a_n\| = \frac{1}{n^{\frac{n}{\varepsilon_n}}}$ where $\{\varepsilon_n\}$ is a sequence of positive numbers converging to zero is of order zero.

Example 3.3. As the condition $\limsup\limits_{n\to\infty}\frac{\log n}{\log\left(\frac{1}{\sqrt[n]{|a_n|}}\right)} = \infty$ characterizes an entire function of infinite order, considering $\|a_n\| = \frac{1}{n^{n\varepsilon_n}}$, $\{\varepsilon_n\}$ to a sequence of positive numbers converging to zero slowly enough with $\lim\limits_{n\to\infty}\varepsilon_n \log n = \infty$.

We see that the sequence $\varepsilon_n = \frac{1}{(\log n)^{1-\delta}}$ $(n = 1, 2, ...)$ meets these requirements if $0 < \delta < 1$, as because $\varepsilon_n \to 0$ but $\lim\limits_{n\to\infty}\varepsilon_n \log n \to \infty$. Thus the series $f(z) = \sum\limits_{n=0}^{\infty}\frac{z^n}{\exp\left(n^{\delta}\log n\right)}$, $0 < \delta < 1$ represents an entire function of infinite order.

Let $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum\limits_{n=0}^{\infty} b_n z^n$ be two bicomplex valued entire functions. Then the Hadamard composition [11] of $f(z)$ and $g(z)$ denoted by $f(z) \circ g(z)$ is defined by $f(z) \circ g(z) = \sum\limits_{n=0}^{\infty} a_n b_n z^n = \sum\limits_{n=0}^{\infty} c_n z^n$ where $c_n = a_n b_n$.

As a consequence of Theorem 3.2, we may prove the following result related to the Hadamard composition of two entire functions in $\mathbb{C}_2$. In fact we will find out here an estimate of the type of the Hadamard composition of two bicomplex valued entire functions.

Theorem 3.3. *Let* $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ *and* $g(z) = \sum\limits_{n=0}^{\infty} b_n z^n$ *be entire in* $\mathbb{C}_2$ *with respective orders and types* ρ_1, ρ_2 *and* σ_1, σ_2. *Also let* ρ, σ *denote the order and type of* $f(z) \circ g(z)$ *respectively. Then* $\left(\frac{\sigma}{\rho_1^{-1}+\rho_2^{-1}}\right)^{\rho_1^{-1}+\rho_2^{-1}} \le (\sigma_1\rho_1)^{\rho_1^{-1}}(\sigma_2\rho_2)^{\rho_2^{-1}}$, *if* $\frac{1}{\rho} = \frac{1}{\rho_1} + \frac{1}{\rho_2}$.

Proof. In view of Theorem 3.2, we have $\sigma = \frac{1}{e\rho}\limsup\limits_{n\to\infty} n\|c_n\|^{\frac{\rho}{n}}$. Therefore,

$$
\begin{aligned}
(e\rho\sigma)^{1/\rho} = \limsup_{n\to\infty} n^{1/\rho}\|c_n\|^{\frac{1}{n}} &= \limsup_{n\to\infty} n^{\frac{1}{\rho_1}+\frac{1}{\rho_2}}\|a_n b_n\|^{\frac{1}{n}} \\
&\le \limsup_{n\to\infty} n^{\frac{1}{\rho_1}}\|a_n\|^{\frac{1}{n}} \limsup_{n\to\infty} n^{\frac{1}{\rho_2}}\|b_n\|^{\frac{1}{n}} \\
&= (e\rho_1\sigma_1)^{1/\rho_1}(e\rho_2\sigma_2)^{1/\rho_2} \\
&= e^{1/\rho}(\rho_1\sigma_1)^{1/\rho_1}(\rho_2\sigma_2)^{1/\rho_2} \\
i.e., (\rho\sigma)^{1/\rho} &\le (\rho_1\sigma_1)^{1/\rho_1}(\rho_2\sigma_2)^{1/\rho_2} \\
i.e., \left(\frac{\sigma}{1/\rho}\right)^{1/\rho} &\le (\rho_1\sigma_1)^{1/\rho_1}(\rho_2\sigma_2)^{1/\rho_2}
\end{aligned}
$$

$$
\text{Hence,}\quad \left(\frac{\sigma}{\rho_1^{-1}+\rho_2^{-1}}\right)^{\rho_1^{-1}+\rho_2^{-1}} \le (\sigma_1\rho_1)^{\rho_1^{-1}}(\sigma_2\rho_2)^{\rho_2^{-1}}.
$$

This completes the proof of the theorem.

The convex hull of a shape is the smallest convex set containing it. The convex hull may be defined either as the intersection of all convex sets containing a given subset of a Euclidean space or equivalently as the set of all convex combinations of points in the subset. A convex polygon is defined as a polygon with all its interior angles less than 180°. This means that all the vertices of the polygon will point outwards away from the interior of the shape. Here, we show that the zeros of the derivative of a polynomial $P(z)$ in $\mathbb{C}_2$ are contained within the convex hull of the zeros of $P(z)$.

Theorem 3.4. *The zeros of the derivative $P'(z)$ of a polynomial $P(z)$ in $\mathbb{C}_2$ are contained within the convex hull of the zeros of $P(z)$.*

Proof. Let $P(z)$ have zeros $z_1, z_2, ..., z_n$. Let Γ be the least convex polygon containing these zeros. It is sufficient to show that $P'(z)$ cannot vanish anywhere in the exterior of Γ.

Since $P(z) = (z-z_1)(z-z_2)\cdots(z-z_n)$ then $\frac{P'(z)}{P(z)} = \frac{d}{dz}\{\log P(z)\} = \sum_{k=1}^{n} \frac{1}{z-z_k}$. If $P'(z) = 0$ then there exists $z_0 \in \mathbb{C}_2$ such that $\sum_{k=1}^{n} \frac{1}{z_0-z_k} = 0$ and therefore in view of Lemma 2.3, we have $\sum_{k=1}^{n} \frac{1}{\overline{z_0}-\overline{z_k}} = 0$. Thus, $\sum_{k=1}^{n} \frac{z_0-z_k}{(\overline{z_0}-\overline{z_k})(z_0-z_k)} = 0$. i.e., $z_0 \sum_{k=1}^{n} \frac{1}{(\overline{z_0}-\overline{z_k})(z_0-z_k)} = \sum_{k=1}^{n} \frac{z_k}{(\overline{z_0}-\overline{z_k})(z_0-z_k)}$. That is $z_0 = \frac{1}{K}\sum_{k=1}^{n} a_k z_k$, where $K = \sum_{k=1}^{n} \frac{1}{(\overline{z_0}-\overline{z_k})(z_0-z_k)}$. Since $z_0 = \sum_{r=1}^{n} b_r z_r$ with $\sum_{r=1}^{n} b_r = \sum_{k=1}^{n} \frac{a_k}{K} = \frac{1}{K}\sum_{k=1}^{n} a_k = \frac{K}{K} = 1$, $b_r \geq 0$, we have z_0 lies within the convex hull of z_r's where $r = 1, 2, ..., n$. This proves the theorem.

Remark 3.2. *Above theorem is the bicomplex version of Lucas's Theorem [4] in $\mathbb{C}_1$.*

4. Future Scope

In the line of the works as carried out in the paper one may think of the formation of the results in the light of in n-dimensional bicomplex numbers with the help of the idempotents 0, 1, $\frac{1+i_1i_2}{2}$, $\frac{1-i_1i_2}{2}$, $\frac{1+i_1i_3}{2}$, $\frac{1-i_1i_3}{2}$, $\frac{1+i_2i_3}{2}$, $\frac{1-i_2i_3}{2}$,...., $\frac{1+i_{n-1}i_n}{2}$ and $\frac{1-i_{n-1}i_n}{2}$ in $\mathbb{C}_n$. As a consequence, the derivation of relevant results in this area is still virgin and may be an active area of research to the future workers of this branch.

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