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A NOTE ON q-ANALOGUE OF CATALAN NUMBERS ASSOCIATED WITH q-CHANGHEE NUMBERS

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Abstract: In this paper, we study q-analogue of Catalan numbers and polynomials by using p-adic q-integral on $\mathbb{Z}_p$. We investigate some properties of these numbers and polynomials. In addition, we define q-analogue of $\frac{1}{2}$-Changhee numbers by using p-adic q-integral on $\mathbb{Z}_p$ and derive their explicit expressions and some identities involving them.

Keywords and Phrases: Catalan numbers, $\frac{1}{2}$-Changhee numbers, q-Catalan numbers, q-analogue of $\frac{1}{2}$-Changhee numbers, q-Euler numbers.

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1. Introduction

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the filed of p-adic rational numbers and the completion of an algebraic closure of $\mathbb{Q}_p$. The p-adic norm $|\ .\ |_p$ is normalized by $|\ p\ |_p = \frac{1}{p}$. Let $C(\mathbb{Z}_p)$ be the space of continuous function on $\mathbb{Z}_p$. Let q be an indeterminate in $\mathbb{C}_p$ with $|\ 1-q\ |_p < 1$ and q-extension of x is defined by $[x]_q = \frac{1-q^x}{1-q}$.

Then the fermionic p-adic q-integral of f on $\mathbb{Z}_p$ is defined by Kim as follows

$$I_{-q}(f)=\int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x)=\lim_{N\to\infty}\sum_{x=0}^{p^N-1} f(x)\mu_{-q}(x+p^N\mathbb{Z}_p),$$

$$=\lim_{N\to\infty}\frac{1}{[p^N]_{-q}}\sum_{x=0}^{p^N-1} f(x)(-q)^x, \quad \text{(see [4, 10, 14, 15, 26])}. \tag{1.1}$$

Let $f_1(x)=f(x+1)$. Then, by (1.1), we get

$$qI_{-q}(f_1)+I_{-q}(f)=[2]_q f(0). \tag{1.2}$$

It is well known that the Euler numbers are defined by

$$\frac{2}{e^t+1}=\sum_{n=0}^{\infty} E_n\frac{t^n}{n!}, \quad \text{(see [12, 17, 19])}. \tag{1.3}$$

Let q be an indeterminate in $\mathbb{C}_p$ with $\mid 1-q \mid_p < 1$. The q-analogues of Euler numbers are given by

$$\frac{[2]_q}{qe^t+1}=\sum_{n=0}^{\infty} E_{n,q}\frac{t^n}{n!}, \quad \text{(see [7, 23])}. \tag{1.4}$$

Note that $\lim_{q\to 1} E_{n,q}=E_n, (n\geq 0)$.
The q-analogues of Changhee numbers are given by

$$\frac{[2]_q}{[2]_q+t}=\sum_{n=0}^{\infty} Ch_{n,q}\frac{t^n}{n!}, \quad \text{(see [6-10, 13, 14])}. \tag{1.5}$$

Kim *et al.* [8] introduced the λ-Changhee polynomials defined by

$$\frac{2}{(1+t)^\lambda+1}(1+t)^{\lambda x}=\sum_{n=0}^{\infty} Ch_{n,\lambda}(x)\frac{t^n}{n!}, \tag{1.6}$$

where $\lambda\in\mathbb{Z}_p$.
When $x=0$, $Ch_{n,\lambda}=Ch_{n,\lambda}(0)$ are called the λ-Changhee numbers.
For $n\geq 0$, the Stirling numbers of the first kind are defined by

$$(x)_n=\sum_{l=0}^{n} S_1(n,l)x^l, \quad \text{(see [1-15])} \tag{1.7}$$

where $(x)_0 = 1$, and $(x)_n = x(x-1)\cdots(x-n+1), (n \geq 1)$. From (1.7), it is easy to see that

$$\frac{1}{r!}(\log(1+t))^r = \sum_{n=r}^{\infty} S_1(n,r)\frac{t^n}{n!}, \quad (r \geq 0), \qquad \text{(see [11-20])}. \tag{1.8}$$

For $n \geq 0$, the Stirling numbers of the second kind are defined by

$$x^n = \sum_{l=0}^{n} S_2(n,l)(x)_l, \qquad \text{(see [15-27])}. \tag{1.9}$$

From (1.9), we see that

$$\frac{1}{r!}(e^t-1)^r = \sum_{n=r}^{\infty} S_2(n,r)\frac{t^n}{n!}. \tag{1.10}$$

As is well known, the Catalan numbers are defined by the generating function as follows (see [1, 2, 3, 20, 21, 22, 24, 25, 27])

$$\frac{2}{1+\sqrt{1-4t}} = \frac{1-\sqrt{1-4t}}{2t} = \sum_{n=0}^{\infty} C_n t^n, \tag{1.11}$$

where $C_n = \binom{2n}{n}\frac{1}{n+1}, (n \geq 0)$.
The Catalan polynomials are defined by the generating function as follows (see [13])

$$\begin{aligned}\int_{\mathbb{Z}_p} (1-4t)^{\frac{x+y}{2}} d\mu_{-1}(y) &= \frac{2}{1+\sqrt{1-4t}}(1-4t)^{\frac{x}{2}} \\ &= \sum_{n=0}^{\infty} C_n(x)t^n. \end{aligned} \tag{1.12}$$

When $x = 0$, $C_n = C_n(0)$ are called the Catalan numbers.
Thus, by (1.11) and (1.12), we have

$$C_n(x) = \sum_{m=0}^{n}\sum_{j=0}^{m} \left(\frac{x}{2}\right)^j S_1(m,j)(-4)^m \frac{C_{n-m}}{m!}.$$

Kim introduced the $\frac{1}{2}$-Changhee polynomials which are given by the generating function (see [12])

$$\int_{\mathbb{Z}_p} (1+t)^{\frac{x+y}{2}} d\mu_{-1}(y) = \frac{2}{1+\sqrt{1+t}}\sqrt{(1+t)^x}$$

$$= \sum_{n=0}^{\infty} Ch_{n,\frac{1}{2}}(x)\frac{t^n}{n!}. \tag{1.13}$$

When $x = 0$, $Ch_{n,\frac{1}{2}} = Ch_{n,\frac{1}{2}}(0)$ are called the $\frac{1}{2}$-Changhee numbers.
On replacing t by $-4t$ in (1.13) and by using (1.12), we have

$$\int_{\mathbb{Z}_p} (1-4t)^{\frac{x+y}{2}} d\mu_{-1}(y) = \frac{2}{1+\sqrt{1-4t}}\sqrt{(1-4t)^x}$$

$$= \sum_{n=0}^{\infty} Ch_{n,\frac{1}{2}}(x)(-4)^n\frac{t^n}{n!}.$$

$$\sum_{n=0}^{\infty} C_n(x)t^n = \sum_{n=0}^{\infty} Ch_{n,\frac{1}{2}}(x)(-4)^n\frac{t^n}{n!}. \tag{1.14}$$

Comparing the coefficients of t, we get

$$C_n(x) = \frac{(-1)^n}{n!}Ch_{n,\frac{1}{2}}(x)2^{2n}.$$

Recently, Kim *et al.* [11] introduced the q-analogues of Catalan polynomials which are given by

$$\int_{\mathbb{Z}_p} (1-4t)^{\frac{x+y}{2}} d\mu_{-q}(y) = \frac{[2]_q}{1+q\sqrt{1-4t}}(1-4t)^{\frac{x}{2}}$$

$$= \sum_{n=0}^{\infty} C_{n,q}(x)t^n. \tag{1.15}$$

When $x = 0$, $C_{n,q} = C_{n,q}(0)$ are called the q-Catalan numbers.

The aim of the paper is to introduce the q-analogues of Catalan numbers $C_{n,q}$ with the help of a p-adic q-integral on $\mathbb{Z}_p$ and derive explicit expressions and some identities for those numbers. In more detail, we deduce explicit expressions of $C_{n,q}$, as a rational function in terms of Euler number and Stirling numbers of the first kind, as a fermionic p-adic q-integral on $\mathbb{Z}_p$ and involving q-analogue of $\frac{1}{2}$-Changhee numbers.

2. q-analogue Catalan Numbers Associated with q-Changhee Numbers

In this section, we assume that $q, t \in \mathbb{C}_p$ with $\mid 1-q \mid< 1$ and $\mid t \mid< p^{-\frac{1}{p-1}}$. Let us apply (1.2) with $f(x) = (1+t)^{\frac{1}{2}}$. Then, we have

$$\int_{\mathbb{Z}_p} (1+t)^{\frac{x}{2}} d\mu_{-q}(x) = \frac{[2]_q}{1+q\sqrt{1+t}} = \frac{[2]_q}{1-q^2-q^2t}(1-q\sqrt{1+t}). \tag{2.1}$$

Now, we consider the q-analogues of $\frac{1}{2}$-Changhee numbers which are defined by

$$\frac{[2]_q}{1-q^2-q^2t}(1-q\sqrt{1+t})=\sum_{n=0}^{\infty}Ch_{n,q,\frac{1}{2}}\frac{t^n}{n!}. \tag{2.2}$$

Note that

$$\lim_{q\to 1}Ch_{n,q,\frac{1}{2}}=Ch_{n,\frac{1}{2}} \quad (n\geq 0).$$

From (1.2), we note that

$$\int_{\mathbb{Z}_p}e^{xt}d\mu_{-q}(x)=\frac{[2]_q}{qe^t+1}=\sum_{n=0}^{\infty}E_{n,q}\frac{t^n}{n!}. \tag{2.3}$$

Thus, by (2.3), we get

$$\int_{\mathbb{Z}_p}x^n d\mu_{-q}(x)=E_{n,q}, (n\geq 0). \tag{2.4}$$

On the other hand, by using (2.4) we also have

$$\begin{aligned}\int_{\mathbb{Z}_p}(1+t)^{\frac{x}{2}}d\mu_{-q}(x)&=\sum_{m=0}^{\infty}\int_{\mathbb{Z}_p}x^m d\mu_{-q}(x)\frac{1}{m!}(\log(1+t))^m\\ &=\sum_{m=0}^{\infty}E_{m,q}2^{-m}\sum_{n=m}^{\infty}S_1(n,m)\frac{t^n}{n!}\\ &=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}E_{m,q}2^{-m}S_1(n,m)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.5}$$

Therefore, by (2.2) and (2.5), we obtain the following theorem.

Theorem 2.1. *For $n\geq 0$, we have*

$$Ch_{n,q,\frac{1}{2}}=\sum_{m=0}^{n}E_{m,q}2^{-m}S_1(n,m).$$

By replacing t by $-4t$ in (2.2), we have

$$\int_{\mathbb{Z}_p}(1-4t)^{\frac{x}{2}}d\mu_{-q}(x)=\frac{[2]_q}{q\sqrt{1-4t}+1}=\sum_{n=0}^{\infty}C_{n,q}t^n. \tag{2.6}$$

On the other hand,

$$\int_{\mathbb{Z}_p} (1-4t)^{\frac{x}{2}} d\mu_{-q}(x) = \sum_{n=0}^{\infty} Ch_{n,q,\frac{1}{2}} \frac{(-4)^n t^n}{n!}. \tag{2.7}$$

Therefore, by (2.6) *and* (2.7)*, we obtain the following theorem.*

Theorem 2.2. *For $n \geq 0$, we have*

$$C_{n,q} = (-1)^n \frac{4^n Ch_{n,q,\frac{1}{2}}}{n!}.$$

From (2.1)*, we observe that*

$$\begin{aligned}
\int_{\mathbb{Z}_p} (1+t)^{\frac{x}{2}} d\mu_{-q}(x) &= \sum_{m=0}^{\infty} \int_{\mathbb{Z}_p} \binom{\frac{x}{2}}{m} d\mu_{-q}(x) \frac{[\log(1+t)]^m}{m!} \\
&= \sum_{m=0}^{\infty} C_{m,q} \sum_{n=m}^{\infty} S_1(n,m) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} C_{m,q} S_1(n,m) \right) \frac{t^n}{n!}.
\end{aligned} \tag{2.8}$$

Therefore, by (2.2) *and* (2.8)*, we get the following theorem.*

Theorem 2.3. *For $n \geq 0$, we have*

$$Ch_{n,q,\frac{1}{2}} = \sum_{m=0}^{n} C_{m,q} S_1(n,m).$$

First, we note that

$$\begin{aligned}
(1+t)^{\frac{1}{2}} &= \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} t^n = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n}{n!} t^n \\
&= \sum_{n=0}^{\infty} \frac{1(\frac{1}{2}-1)(\frac{1}{2}-2)\cdots(\frac{1}{2}-n+1)}{n!} t^n \\
&= \sum_{n=0}^{\infty} \frac{(-1)^{n-1} 1.3.5 \cdots (2n-3)}{n! 2^n} t^n \\
&= \sum_{n=0}^{\infty} \frac{(-1)^{n-1} 1.3.4 \cdots (2n-3)(2n-2)(2n-1)(2n)}{n! 2^n 2.4.6 \cdots (2n-2)(2n-1)(2n)} t^n
\end{aligned}$$

$$= \sum_{n=0}^{\infty} (-1)^{n-1}(-1)^{n-1} \frac{(2n)!}{n!4^n(2n-1)n!} t^n = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{(-1)^{n-1}}{4^n(2n-1)} t^n. \tag{2.9}$$

By (2.1) *and* (2.9), *we get*

$$[2]_q = \left(\sum_{n=0}^{\infty} Ch_{n,q,\frac{1}{2}} \frac{t^n}{n!} \right) \left((1 + q\sqrt{1+t}) \right)$$

$$= \sum_{n=0}^{\infty} Ch_{n,q,\frac{1}{2}} \frac{t^n}{n!} + \left(\sum_{n=0}^{\infty} Ch_{n,q,\frac{1}{2}} \frac{t^n}{n!} \right) \left(q \sum_{m=0}^{\infty} \binom{2m}{m} \frac{(-1)^{m-1}}{4^m(2m-1)} t^m \right)$$

$$= \sum_{n=0}^{\infty} Ch_{n,q,\frac{1}{2}} \frac{t^n}{n!} + \sum_{n=0}^{\infty} \left(q \sum_{m=0}^{n} C_m \frac{(m+1)(-1)^{m-1}}{4^m(2m-1)} \frac{m!n!}{(n-m)!m!} Ch_{n-m,q,\frac{1}{2}} \right) \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} Ch_{n,q,\frac{1}{2}} \frac{t^n}{n!} + \sum_{n=0}^{\infty} \left(q \sum_{m=0}^{n} C_m \frac{(m+1)(-1)^{m-1}}{4^m(2m-1)} \binom{n}{m} Ch_{n-m,q,\frac{1}{2}} \right) \frac{t^n}{n!}. \tag{2.10}$$

By comparing the coefficients of t *on both sides, we obtain the following theorem.*

Theorem 2.4. *For* $n \geq 0$, *we have*

$$Ch_{n,q,\frac{1}{2}} + q \sum_{m=0}^{n} C_m \frac{(m+1)(-1)^{m-1}}{4^m(2m-1)} \binom{n}{m} Ch_{n-m,q,\frac{1}{2}} = \begin{cases} [2]_q, & if n = 0 \\ 0, & if n > 1. \end{cases}$$

By replacing t *by* $-\frac{t}{4}$ *in* (1.4), *we get*

$$\frac{[2]_q}{1 + q\sqrt{1+t}} = \sum_{n=0}^{\infty} C_{n,q} \left(-\frac{t}{4} \right)^n. \tag{2.11}$$

$$= \sum_{n=0}^{\infty} C_{n,q} (-1)^n 4^{-n} t^n$$

On the other hand, we have

$$\frac{[2]_q}{1 + q\sqrt{1+t}} = \sum_{n=0}^{\infty} Ch_{n,q,\frac{1}{2}} \frac{t^n}{n!}. \tag{2.12}$$

Therefore, by (2.11) *and* (2.12), *we get the following theorem.*

Theorem 2.5. *For $n \geq 0$, we have*

$$Ch_{n,q,\frac{1}{2}} = n!C_{n,q}(-1)^n 2^{-2n}.$$

Replacing t by $e^{2t}-1$ in (2.1), we have

$$\begin{aligned}\int_{\mathbb{Z}_p} e^{xt} d\mu_{-q}(x) &= \frac{[2]_q}{qe^t+1} = \sum_{m=0}^{\infty} Ch_{m,q,\frac{1}{2}} \frac{(e^{2t}-1)^m}{m!} \\ &= \sum_{m=0}^{\infty} Ch_{m,q,\frac{1}{2}} \sum_{n=m}^{\infty} S_2(n,m) \frac{2^n t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} 2^n Ch_{m,q,\frac{1}{2}} S_2(n,m) \right) \frac{t^n}{n!}. \end{aligned} \tag{2.13}$$

On the other hand, we have

$$\int_{\mathbb{Z}_p} e^{xt} d\mu_{-q}(x) = \sum_{n=0}^{\infty} E_{n,q} \frac{t^n}{n!}. \tag{2.14}$$

Therefore, by (2.13) and (2.14), we obtain the following theorem.

Theorem 2.6. *For $n \geq 0$, we have*

$$E_{n,q} = \sum_{m=0}^{n} 2^n Ch_{m,q,\frac{1}{2}} S_2(n,m).$$

Now, we observe that

$$\begin{aligned}(1+t)^{\frac{x}{2}} &= \sum_{m=0}^{\infty} \left(\frac{x}{2}\right)^m \frac{[\log(1+t)]^m}{m!} \\ &= \sum_{m=0}^{\infty} \left(\frac{x}{2}\right)^m \sum_{n=m}^{\infty} S_1(n,m) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \left(\frac{x}{2}\right)^m S_1(n,m) \right) \frac{t^n}{n!}. \end{aligned} \tag{2.15}$$

Now, we consider the q-analogues of $\frac{1}{2}$-Changhee polynomials which are given by the generating function to be

$$\int_{\mathbb{Z}_p} (1+t)^{\frac{x+y}{2}} d\mu_{-q}(y) = \frac{[2]_q}{1+q\sqrt{1+t}} \sqrt{(1+t)^x}$$

$$= \sum_{n=0}^{\infty} Ch_{n,q,\frac{1}{2}}(x)\frac{t^n}{n!}. \tag{2.16}$$

When $x = 0$, $Ch_{n,q,\frac{1}{2}} = Ch_{n,q,\frac{1}{2}}(0)$ *are called the q-analogues of* $\frac{1}{2}$*-Changhee numbers.*
From (2.16), *we note that*

$$\sum_{n=0}^{\infty} Ch_{n,q,\frac{1}{2}}(x)\frac{t^n}{n!} = \frac{[2]_q}{1+q\sqrt{1+t}}\sqrt{(1+t)^x}$$

$$= \left(\sum_{n=0}^{\infty} Ch_{n,q,\frac{1}{2}}\frac{t^n}{n!}\right)\left(\sum_{l=0}^{\infty}\sum_{m=0}^{l}\left(\frac{x}{2}\right)^m S_1(l,m)\frac{t^l}{l!}\right)$$

$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{l}Ch_{n-l,q,\frac{1}{2}}\sum_{m=0}^{l}\left(\frac{x}{2}\right)^m S_1(l,m)\right)\frac{t^n}{n!}. \tag{2.17}$$

By (2.16) *and* (2.17), *we obtain the following theorem.*

Theorem 2.7. *For* $n \geq 0$, *we have*

$$Ch_{n,q,\frac{1}{2}}(x) = \sum_{m=0}^{n}\binom{n}{l}Ch_{n-l,q,\frac{1}{2}}\sum_{m=0}^{l}\left(\frac{x}{2}\right)^m S_1(l,m).$$

By replacing t *by* $-4t$ *in* (2.16), *we have*

$$\int_{\mathbb{Z}_p}(1-4t)^{\frac{x+y}{2}}d\mu_{-q}(y) = \frac{[2]_q}{1+q\sqrt{1-4t}}\sqrt{(}1-4t)^x = \sum_{n=0}^{\infty} C_{n,q}(x)t^n. \tag{2.18}$$

On the other hand,

$$\sum_{n=0}^{\infty} Ch_{n,q,\frac{1}{2}}(x)\frac{(-4t)^n}{n!} = \sum_{n=0}^{\infty} Ch_{n,q,\frac{1}{2}}(x)(-1)^n 2^{2n}\frac{t^n}{n!}. \tag{2.19}$$

Therefore, by (2.18) *and* (2.19), *we state the following theorem.*

Theorem 2.8. *For* $n \geq 0$, *we have*

$$C_{n,q}(x) = \frac{(-1)^n}{n!}2^{2n}Ch_{n,q,\frac{1}{2}}(x).$$

From (2.16), *we note that*

$$\frac{[2]_q}{1+q\sqrt{1+t}}\sqrt{(1+t)^x} = \left(\sum_{n=0}^{\infty} Ch_{n,q,\frac{1}{2}}\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty}\binom{\frac{x}{2}}{m}(-1)^m 2^{2m} t^m\right)$$

$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{\frac{x}{2}}{m}(-1)^m 2^{2m} Ch_{n-m,q,\frac{1}{2}}\frac{1}{(n-m)!}\right)t^n. \tag{2.20}$$

Therefore, by (2.16) *and* (2.19), *we obtain the following theorem.*

Theorem 2.9. *For $n \geq 0$, we have*

$$Ch_{n,q,\frac{1}{2}}(x) = \sum_{m=0}^{n}\binom{\frac{x}{2}}{m}(-1)^m 2^{2m} Ch_{n-m,q,\frac{1}{2}}\frac{n!}{(n-m)!}.$$

From (1.2), *we see that*

$$\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}(x+y)^n d\mu_{-q}(y)\frac{t^n}{n!} = \int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_{-q}(y) = \frac{[2]_q}{qe^t+1}e^{xt}$$

$$= \sum_{n=0}^{\infty} E_{n,q}(x)\frac{t^n}{n!}, \tag{2.21}$$

where $E_{n,q}(x) = \sum_{m=0}^{n}\binom{n}{m}E_{n-m,q}x^m = \int_{\mathbb{Z}_p}(x+y)^n d\mu_{-q}(y)$ *are q-Euler polynomials.* *From* (2.16), *we have*

$$\frac{[2]_q}{1+q\sqrt{1+t}}\sqrt{(1+t)^x} = \int_{\mathbb{Z}_p}(1+t)^{\frac{x+y}{2}} d\mu_{-q}(y)$$

$$= \sum_{m=0}^{\infty} 2^{-m}\frac{1}{m!}(\log(1+t))^m = \int_{\mathbb{Z}_p}(x+y)^m d\mu_{-q}(y)$$

$$= \sum_{m=0}^{\infty} 2^{-m} E_{m,q}(x)\sum_{n=m}^{\infty} S_1(n,m)\frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} 2^{-m} E_{m,q}(x) S_1(n,m)\right)\frac{t^n}{n!}. \tag{2.22}$$

Thus, by (2.16) *and* (2.22), *we get the following theorem.*

Theorem 2.10. *For $n \geq 0$, we have*

$$Ch_{n,q,\frac{1}{2}}(x) = \sum_{m=0}^{n} 2^{-m} E_{m,q}(x) S_1(n,m).$$

3. Conclusion

The aim of the paper is to introduced q-analogue of Catalan numbers $C_{n,q}$ with the help of a p-adic q-integral on $\mathbb{Z}_p$ and derived explicit expressions and some identities for those numbers. In more detail, we deduced explicit expressions of $C_{n,q}$, as a rational function in terms of q-Euler number and Stirling numbers of the first kind, as a fermionic p-adic q-integral on $\mathbb{Z}_p$ and involving q-analogue of $\frac{1}{2}$-Changhee numbers.

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