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SYMMETRIC IDENTITIES FOR DEGENERATE q-POLY-GENOCCHI NUMBERS AND POLYNOMIALS

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Abstract: In the present article, we introduce a new class of degenerate q-poly-Genocchi polynomials and numbers including q-logarithm function. We derive some relations with this polynomials and the Stirling numbers of the second kind and investigate some symmetric identities using special functions that are involving these polynomials.

Keywords and Phrases: Degenerate q-poly-Genocchi polynomials, Stirling numbers, q-logarithm function, Symmetric identities.

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1. Introduction

Throughout this presentation, we use the following standard notions $\mathbb{N} = \{1, 2, \cdots\}$, $\mathbb{N}_0 = \{0, 1, 2, \cdots\} = \mathrm{N} \cup \{0\}$, $\mathbb{Z}^- = \{-1, -2, \cdots\}$. Also as usual $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the set of complex numbers. For any $n \in \mathbb{N}$, the q-number can be defined as follows

$$[n]_q = \frac{1 - q^n}{1 - q}.$$

Note that $\lim_{q\longrightarrow 1}[n]_q = n$.

The classical Genocchi numbers G_n, the classical Genocchi polynomials $G_n(x)$ and the generalized Genocchi polynomials $G_n^{(\alpha)}(x)$ of (real or complex) order α are usually defined by means of the following generating functions (see [11, 12, 15-24]):

$$\frac{2t}{e^t+1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!} \qquad (|\, t \,|< \pi), \tag{1.1}$$

$$\frac{2t}{e^t+1} e^{xt} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!} \qquad (|\, t \,|< \pi), \tag{1.2}$$

and

$$\left(\frac{2t}{e^t+1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} G_n^{(\alpha)}(x) \frac{t^n}{n!} \qquad (|\, t \,|< \pi; 1^{\alpha}=1), \tag{1.3}$$

with

$$G_n^1(0) \;=\; G_n.$$

The degenerate exponential function [4, 8, 9] is defined by

$$e_{\lambda}^x(t) = (1+\lambda t)^{\frac{x}{\lambda}} \quad and \quad e_{\lambda}^1(t) = e_{\lambda}(t), (\lambda \in \mathbb{R}). \tag{1.4}$$

Note that

$$\lim_{\lambda\to 0}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \frac{x^n t^n}{n!} = e^{xt}.$$

In [2, 3], Carlitz introduced the degenerate Bernoulli and degenerate Euler polynomials defined by

$$\frac{t}{e_{\lambda}(t)-1} e_{\lambda}^x(t) = \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_n(x;\lambda) \frac{t^n}{n!}, \tag{1.5}$$

and

$$\frac{2}{e_{\lambda}(t)+1} e_{\lambda}^x(t) = \frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \mathfrak{E}_n(x;\lambda) \frac{t^n}{n!}. \tag{1.6}$$

In the case when $x=0$, $B_{n,\lambda}(0) := B_{n,\lambda}$ are called the degenerate Bernoulli numbers and $E_{n,\lambda}(0) := E_{n,\lambda}$ are called the degenerate Euler numbers.

Let $(x)_{n,\lambda}$ be the degenerate falling factorial sequence given by

$$(x)_{n,\lambda} := x(x-\lambda)\cdots(x-(n-1)\lambda), (n \geq 1),$$

with the assumption $(x)_{0,\lambda} = 1$.

The the degenerate Genocchi polynomials are defined by (see [14])

$$\frac{2t}{e_\lambda(t)+1} e_\lambda^x(t) = \sum_{n=0}^{\infty} G_{n,\lambda}(x) \frac{t^n}{n!}. \tag{1.7}$$

In the case when $x = 0$, $G_{n,\lambda} := G_{n,\lambda}(0)$ are called the degenerate Genocchi numbers.

The classical polylogarithm function $\mathrm{Li}_k(z)$ is defined by (see [5])

$$\mathrm{Li}_k(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^k}, (k \in \mathbb{Z}) \tag{1.8}$$

so for $k \leq 1$,

$$\mathrm{Li}_1(z) = -\ln(1-z), \quad \mathrm{Li}_0(z) = \frac{z}{1-z}, \quad \mathrm{Li}_{-1}(z) = \frac{z}{(1-z)^2}, \ldots$$

The poly-Bernoulli polynomials are given by (see [13])

$$\frac{\mathrm{Li}_k(1-e^{-t})}{e^t-1} e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!}, \quad \text{(see [11])} \tag{1.9}$$

For $k = 1$ in (1.9), we have

$$\frac{\mathrm{Li}_1(1-e^{-t})}{e^t-1} e^{xt} = \frac{t}{e^t-1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}. \tag{1.10}$$

From (1.9) and (1.10), we have

$$B_n^{(1)}(x) = B_n(x).$$

Very recently, Jung and Ryoo [6] introduced the degenerate q-poly-Bernoulli polynomials $B_{n,q}^{(k)}(x;\lambda)$ defined by

$$\frac{\mathrm{Li}_{k,q}(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}-1} (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} B_{n,q}^{(k)}(x;\lambda) \frac{t^n}{n!}, \tag{1.11}$$

where

$$\mathrm{Li}_{k,q}(t) = \sum_{n=0}^{\infty} \frac{t^n}{[n]_q^k}$$

is the k-th q-polylogarithm function.

Note that

$$\lim_{\lambda\to 0}\frac{\mathrm{Li}_{k,q}(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}\lim_{\lambda\to 0}B_{n,\lambda,q}^{(k)}(x)\frac{t^n}{n!}$$

$$=\frac{\mathrm{Li}_{k,q}(1-e^{-t})}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}B_{n,q}^{(k)}(x)\frac{t^n}{n!}. \tag{1.12}$$

We recall the following definition as:
The Stirling numbers of the first kind are defined by

$$(x)_n=x(x-1)\cdots(x-n+1)=\sum_{l=0}^{n}S_1(n,l)x^l,\,(n\geq 0). \tag{1.13}$$

and the Stirling numbers of the second kind are defined by (see [1-12])

$$(e^t-1)^n=n!\sum_{l=n}^{\infty}S_2(l,n)\frac{t^l}{l!}. \tag{1.14}$$

A generalized falling factorial sum $\tau_k(n;\lambda)$ can be defined by the generating function [25]:

$$\sum_{k=0}^{\infty}\tau_k(n;\lambda)\frac{t^k}{k!}=\frac{1-(-(1+\lambda t))^{\frac{(n+1)}{\lambda}}}{1+(1+\lambda t)^{\frac{1}{\lambda}}}. \tag{1.15}$$

where $\lim_{\lambda\longrightarrow 0}\tau_k(n;\lambda)=T_k(n)$.

In this paper, we consider a new class of degenerate q poly-Genocchi polynomials $G_{n,q}^{(k)}(x;\lambda)$ and develop some elementary properties and derive some implicit formulae and symmetric identities for the degenerate q poly-Genocchi polynomials by using different analytical means of their respective generating functions.

2. Degenerate q-poly-Genocchi Numbers and Polynomials

In this section, we introduce degenerate q-poly-Genocchi numbers and polynomials and investigate some basic properties of these polynomials. We start with the following definition as.

Definition 2.1. *Let $\lambda\in\mathbb{C},k\in\mathbb{Z}$, $n\geq 0$ and $0\leq q<1$. We consider the degenerate q-poly-Genocchi polynomials by means of the following generating function*

$$\frac{2\mathrm{Li}_{k,q}(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}G_{n,q}^{(k)}(x;\lambda)\frac{t^n}{n!}. \tag{2.1}$$

When $x = 0$ in (2.1), $G_{n,q}^{(k)}(\lambda) = G_{n,q}^{(k)}(0;\lambda)$ are called the degenerate q-poly-Genocchi numbers.

Note that

$$G_{n,q\to 1}^{(1)}(x;\lambda) = G_n(x;\lambda),$$

and

$$\lim_{\lambda\longrightarrow 0} G_{n,q}^{(k)}(x;\lambda) = G_{n,q}^{(k)}(x) \quad (k \in \mathbb{Z}), \tag{2.2}$$

where $G_{n,q}^{(k)}(x)$ are called the q-poly-Genocchi polynomials.

Theorem 2.1. *For $n \geq 0$, we have*

$$G_{n,q}^{(k)}(x;\lambda) = \sum_{m=0}^{n} \binom{n}{m} G_{m,q}^{(k)}(x)_{n-m,\lambda}. \tag{2.3}$$

Proof. Using definition (2.1), we have

$$\sum_{n=0}^{\infty} G_{n,q}^{(k)}(x;\lambda)\frac{t^n}{n!} = \sum_{m=0}^{\infty} G_{m,q}^{(k)}\frac{t^m}{m!}\sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} G_{m,q}^{(k)}(x)_{n-m,\lambda}\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ in both sides, we get (2.3).

Theorem 2.2. *For $n \geq 0$, we have*

$$G_{n,1}^{(2)}(x;\lambda) = \sum_{m=0}^{n} \binom{n}{m} \frac{B_m m!}{m+1} E_{n-m}(x;\lambda). \tag{2.4}$$

Proof. Applying Definition (2.1), we have

$$\sum_{n=0}^{\infty} G_{n,1}^{(k)}(x;\lambda)\frac{t^n}{n!} = \frac{2\mathrm{Li}_k(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}}$$

$$= \frac{2(1+\lambda t)^{\frac{x}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}}+1}\int_0^t \underbrace{\frac{1}{e^z-1}\int_0^t \frac{1}{e^z-1}\cdots\frac{1}{e^z-1}\int_0^t \frac{z}{e^z-1}}_{(k-2)-\text{times}}dz\cdots dz. \tag{2.5}$$

For $k = 2$ in (2.5), we have

$$\sum_{n=0}^{\infty} G_{n,1}^{(2)}(x;\lambda)\frac{t^n}{n!} = \frac{2(1+\lambda t)^{\frac{x}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}}+1}\int_0^t \frac{z}{e^z-1}dz$$

$$= \left(\sum_{m=0}^{\infty} \frac{B_m t^m}{m+1}\right) \frac{2(1+\lambda t)^{\frac{x}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}}+1}$$

$$= \left(\sum_{m=0}^{\infty} \frac{B_m m!}{m+1} \frac{t^m}{m!}\right) \left(\sum_{n=0}^{\infty} E_n(x;\lambda) \frac{t^n}{n!}\right).$$

Replacing n by $n-m$ in above equation, we have

$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} \frac{B_m m!}{m+1} E_{n-m}(x;\lambda)\right) \frac{t^n}{n!}.$$

On equating the coefficients of the like powers of $\frac{t^n}{n!}$ in the above equation, we get the result (2.4).

Theorem 2.3. *For $n \geq 0$, we have*

$$G_{n,q}^{(k)}(x;\lambda) = \sum_{p=0}^{n} \binom{n}{p} \left(\sum_{l=1}^{p+1} \frac{(-1)^{l+p+1} l! S_2(p+1,l)}{[l]_q^k (p+1)}\right) G_{n-p}(x;\lambda). \tag{2.6}$$

Proof. From equation (2.1), we have

$$\sum_{n=0}^{\infty} G_{n,q}^{(k)}(x;\lambda) \frac{t^n}{n!} = \left(\frac{\mathrm{Li}_{k,q}(1-e^{-t})}{t}\right) \left(\frac{2t(1+\lambda t)^{\frac{x}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}}+1}\right). \tag{2.7}$$

Now

$$\frac{1}{t}\mathrm{Li}_{k,q}(1-e^{-t}) = \frac{1}{t}\sum_{l=1}^{\infty} \frac{(1-e^{-t})^l}{[l]_q^k} = \frac{1}{t}\sum_{l=1}^{\infty} \frac{(-1)^l}{[l]_q^k}(1-e^{-t})^l$$

$$= \frac{1}{t}\sum_{l=1}^{\infty} \frac{(-1)^l}{[l]_q^k} l! \sum_{p=l}^{\infty} (-1)^p S_2(p,l) \frac{t^p}{p!}$$

$$= \frac{1}{t}\sum_{p=1}^{\infty} \sum_{l=1}^{p} \frac{(-1)^{l+p}}{[l]_q^k} l! S_2(p,l) \frac{t^p}{p!}$$

$$= \sum_{p=0}^{\infty} \left(\sum_{l=1}^{p+1} \frac{(-1)^{l+p+1}}{[l]_q^k} l! \frac{S_2(p+1,l)}{p+1}\right) \frac{t^p}{p!}. \tag{2.8}$$

From equations (2.7) and (2.8), we have

$$\sum_{n=0}^{\infty} G_{n,q}^{(k)}(x;\lambda) \frac{t^n}{n!} = \sum_{p=0}^{\infty} \left(\sum_{l=1}^{p+1} \frac{(-1)^{l+p+1}}{[l]_q^k} l! \frac{S_2(p+1,l)}{p+1}\right) \frac{t^p}{p!} \left(\sum_{n=0}^{\infty} G_n(x;\lambda) \frac{t^n}{n!}\right).$$

Replacing n by $n-p$ in the r.h.s of above equation and comparing the coefficients of $\frac{t^n}{n!}$, we get the result (2.6).

Theorem 2.4. *For $n \geq 1$, we have*

$$G_{n,q}^{(k)}(x+1;\lambda) + G_{n,q}^{(k)}(x;\lambda)$$

$$= 2\sum_{p=1}^{n} \left(\begin{array}{c} n \\ p \end{array} \right) \left(\sum_{l=0}^{p-1} \frac{(-1)^{l+p+1}}{[l+1]_q{}^k} (l+1)! S_2(p,l+1) \right) (x)_{n-p,\lambda}. \tag{2.9}$$

Proof. Using the definition (2.1), we have

$$\sum_{n=0}^{\infty} G_{n,q}^{(k)}(x+1,\lambda)\frac{t^n}{n!} + \sum_{n=0}^{\infty} G_{n,q}^{(k)}(x;\lambda)\frac{t^n}{n!}$$

$$= \frac{2\mathrm{Li}_{k,q}(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x+1}{\lambda}} + \frac{2\mathrm{Li}_{k,q}(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}}$$

$$= 2\mathrm{Li}_{k,q}(1-e^{-t})(1+\lambda t)^{\frac{x}{\lambda}}$$

$$= 2\sum_{l=0}^{\infty} \frac{(1-e^{-t})^{l+1}}{[l+1]_q{}^k}(1+\lambda t)^{\frac{x}{\lambda}}$$

$$= 2\sum_{p=1}^{\infty} \left(\sum_{l=0}^{p-1} \frac{(-1)^{l+p+1}}{[l+1]_q^k} (l+1)! S_2(p,l+1) \right) \frac{t^p}{p!} (1+\lambda t)^{\frac{x}{\lambda}} (1+\lambda t^2)^{\frac{y}{\lambda}}$$

$$= 2\left(\sum_{p=1}^{\infty} \left(\sum_{l=0}^{p-1} \frac{(-1)^{l+p+1}}{[l+1]_q{}^k} (l+1)! S_2(p,l+1) \right) \frac{t^p}{p!} \right) \left(\sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!} \right).$$

Replacing n by $n-p$ in the above equation and comparing the coefficients of $\frac{t^n}{n!}$, we get the result (2.9).

Theorem 2.5. *For $n \geq 0$, $d \in \mathbb{N}$ and $k \in \mathbb{Z}$, we have*

$$G_{n,q}^{(k)}(x+y;\lambda) = \sum_{m=0}^{n} \binom{n}{m} G_{n-m,q}^{(k)}(x)(y)_{m,\lambda}. \tag{2.10}$$

Proof. From equation (2.1), we have

$$\sum_{n=0}^{\infty} G_{n,q}^{(k)}(x+y;\lambda)\frac{t^n}{n!} = \frac{2\mathrm{Li}_{k,q}(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x+y}{\lambda}}$$

$$= \sum G_{n,q}^{(k)}(x)\frac{t^n}{n!}\sum_{m=0}^{\infty}(y)_{m,\lambda}\frac{t^m}{m!}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}G_{n-m,q}^{(k)}(x)(y)_{m,\lambda}\right)\frac{t^n}{n!}.$$

On comparing the coefficient of $\frac{t^n}{n!}$, we get the result (2.10).

3. Symmetry Identities for Degenerate q-poly-Genocchi Polynomials

In this section, we introduce general symmetry identities for the degenerate q-poly-Genocchi polynomials $G_{n,q}^{(k)}(x;\lambda)$ by applying the generating function (2.1). We begin following identities as.

Theorem 3.1. *Let $a, b > 0$ and $a \neq b$. For $x \in \mathbb{R}$ and $n \geq 0$, the following identity holds true:*

$$\sum_{m=0}^{n}\binom{n}{m}a^{n-m}b^{m}G_{n-m,q}^{(k)}\left(bx,\frac{\lambda}{a}\right)G_{m,q}^{(k)}\left(ax,\frac{\lambda}{b}\right)$$

$$= \sum_{m=0}^{n}\binom{n}{m}b^{n-m}a^{m}G_{n-m,q}^{(k)}\left(ax,\frac{\lambda}{b}\right)G_{m,q}^{(k)}\left(bx,\frac{\lambda}{a}\right). \tag{3.1}$$

Proof. Let

$$G(t) = \left(\frac{2\mathrm{Li}_{k,q}(1-e^{-at})2\mathrm{Li}_{k,q}(1-e^{-bt})}{((1+\lambda t)^{\frac{a}{\lambda}}+1)((1+\lambda t)^{\frac{b}{\lambda}}+1)}\right)(1+\lambda t)^{\frac{2abx}{\lambda}}. \tag{3.2}$$

Then $G(t)$ is symmetric in a and b and we can written

$$G(t) = \sum_{n=0}^{\infty}G_{n,q}^{(k)}\left(bx,\frac{\lambda}{a}\right)\frac{(at)^n}{n!}\sum_{m=0}^{\infty}G_{m,q}^{(k)}\left(ax,\frac{\lambda}{b}\right)\frac{(bt)^m}{m!}$$

$$G(t) = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}a^{n-m}b^{m}G_{n-m,q}^{(k)}\left(bx,\frac{\lambda}{a}\right)G_{m,q}^{(k)}\left(ax,\frac{\lambda}{b}\right)\right)\frac{t^n}{n!}.$$

Similarly, we can show that

$$G(t) = \sum_{n=0}^{\infty}G_{n,q}^{(k)}\left(ax,\frac{\lambda}{b}\right)\frac{(bt)^n}{n!}\sum_{m=0}^{\infty}G_{m,q}^{(k)}\left(bx,\frac{\lambda}{a}\right)\frac{(at)^m}{m!}$$

$$G(t) = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}b^{n-m}a^{m}G_{n-m,q}^{(k)}\left(ax,\frac{\lambda}{b}\right)G_{m,q}^{(k)}\left(bx,\frac{\lambda}{a}\right)\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on the right hand sides of the last two equations, we arrive the desired result.

Corollary 3.1. *On setting $b = 1$ in Theorem 3.1, we get*

$$\sum_{m=0}^{n}\binom{n}{m}a^{n-m}G_{n-m,q}^{(k)}\left(x,\frac{\lambda}{a}\right)G_{m,q}^{(k)}(ax,\lambda)$$

$$=\sum_{m=0}^{n}\binom{n}{m}a^{m}G_{n-m,q}^{(k)}(ax,\lambda)\,G_{m,q}^{(k)}\left(x,\frac{\lambda}{a}\right). \tag{3.3}$$

Theorem 3.2. *For all integers $a > 0, b > 0$, and $n \geq 0$, the following identity holds true:*

$$\sum_{m=0}^{n}\binom{n}{m}a^{n-m}b^{m}G_{n-m,q}^{(k)}\left(bx,\frac{\lambda}{a}\right)\sum_{i=0}^{m}\binom{m}{i}\tau_i(a-1;\lambda)G_{m-i,q}^{(k)}\left(ay,\frac{\lambda}{b}\right)$$

$$=\sum_{m=0}^{n}\binom{n}{m}b^{n-m}a^{m}G_{n-m,q}^{(k)}\left(ax,\frac{\lambda}{b}\right)\sum_{i=0}^{m}\binom{m}{i}\tau_i(b-1;\lambda)G_{m-i,q}^{(k)}\left(by,\frac{\lambda}{a}\right), \tag{3.4}$$

where generalized falling factorial sum $\tau_k(n;\lambda)$ is given by (1.15).

Proof. We now use

$$H(t)=\frac{2\mathrm{Li}_{k,q}(1-e^{-at})2\mathrm{Li}_{k,q}(1-e^{-bt})(1-(-(1+\lambda t))^{\frac{ab}{\lambda}})(1+\lambda t)^{\frac{ab(x+y)}{\lambda}}}{((1+\lambda t)^{\frac{a}{\lambda}}+1)((1+\lambda t)^{\frac{b}{\lambda}}+1)^2}$$

to find that

$$H(t)=\left(\frac{2\mathrm{Li}_{k,q}(1-e^{-at})}{(1+\lambda t)^{\frac{a}{\lambda}}+1}\right)(1+\lambda t)^{\frac{abx}{\lambda}}\left(\frac{1-(-(1+\lambda t))^{\frac{ab}{\lambda}}}{(1+\lambda t)^{\frac{b}{\lambda}}+1}\right)$$

$$\left(\frac{2\mathrm{Li}_{k,q}(1-e^{-bt})}{(1+\lambda t)^{\frac{b}{\lambda}}+1}\right)(1+\lambda t)^{\frac{aby}{\lambda}}$$

$$=\sum_{n=0}^{\infty}G_{n,q}^{(k)}\left(bx,\frac{\lambda}{a}\right)\frac{(at)^n}{n!}\sum_{n=0}^{\infty}\tau_n(a-1;\lambda)\frac{(bt)^n}{n!}\sum_{n=0}^{\infty}G_{n,q}^{(k)}\left(ay,\frac{\lambda}{b}\right)\frac{(bt)^n}{n!}$$

$$=\sum_{n=0}^{\infty}G_{n,q}^{(k)}\left(bx,\frac{\lambda}{a}\right)\frac{(at)^n}{n!}\sum_{m=0}^{\infty}\sum_{i=0}^{m}\binom{m}{i}b^{m}\tau_i(a-1;\lambda)G_{m-i,q}^{(k)}\left(ay,\frac{\lambda}{b}\right)\frac{t^m}{m!}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}a^{n-m}b^{m}G_{n-m,q}^{(k)}\left(bx,\frac{\lambda}{a}\right)\sum_{i=0}^{m}\binom{m}{i}\tau_i(a-1;\lambda)G_{m-i,q}^{(k)}\left(ay,\frac{\lambda}{b}\right)\right)\frac{t^n}{n!}. \tag{3.5}$$

By using a similar plan, we get

$$H(t) = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}b^{n-m}a^{m}G_{n-m,q}^{(k)}\left(ax,\frac{\lambda}{b}\right)\sum_{i=0}^{m}\binom{m}{i}\tau_i(b-1;\lambda)G_{m-i,q}^{(k)}\left(by,\frac{\lambda}{a}\right)\right)\frac{t^n}{n!}. \tag{3.6}$$

After comparing the coefficients of $\frac{t^n}{n!}$ on the right hand sides of the last two equations, we arrive at the desired result.

References

[1] Alatawi, M. S., Khan, W. A., Ryoo, C. S., Explicit properties of q-Cosine and q-Sine Array-type polynomials containing symmetric structures, Symmetry, 14(1675) (2022), 1-18.

[2] Carlitz, L., Degenerate Stirling, Bernoulli and Eulerian numbers, Utilitas Math., 15 (1979), 51-88.

[3] Carlitz, L. A degenerate Staudt-Clausen theorem, Arch. Math. (Basel), 7 (1956), 28-33.

[4] Dolgy, D. V., Khan, W. A., A note on type two degenerate poly-Changhee polynomials of the second kind, Symmetry, 13 (579) (2021), 1-12.

[5] Kaneko, M., Poly-Bernoulli numbers, J. Théor Nombres Bordeaux, 9 (1) (1997), 221-228.

[6] Jung, N. S., Ryoo, C. S., Symmetric identities for degenerate q-poly-Bernoulli numbers and polynomials, J. Appl. Math. and Informatics, 36 (1-2) (2018), 29-38.

[7] Khan, W. A., Alatawi, M. S., Ryoo, C. S., Duran, U., Novel properties of q-Sine-based and q-Cosine-based q-Fubini polynomials, Symmetry, 15, 356 (2023), 1-18.

[8] Khan, W. A., A note on q-analogue of degenerate Catalan numbers associated p-adic integral on $\mathbb{Z}_p$, Symmetry, 14 (119) (2022), 1-10.

[9] Khan, W. A., A note on q-analogues of degenerate Catalan-Daehee numbers and polynomials, Journal of Mathematics, (2022), Article ID 9486880, 9 pages.

[10] Khan, W. A., Srivastava, D., A new class of q-Hermite based Apostol type polynomials and its applications, Notes on Number Theory and Discrete Mathematics, 26 (1) (2020), 75-85.

[11] Kang, J. Y., Khan, W. A., A new class of q-Hermite based Apostol-type Frobenius-Genocchi polynomials, Communication of the Korean Mathematical Society, 35 (3) (2020), 759-771.

[12] Khan, W. A., Khan, I. A., Ali, M., Degenerate Hermite poly-Bernoulli numbers and polynomials with q-parameter, Stud. Univ. Babes-Bolayi Math., 65 (1) (2020), 3-15.

[13] Kim, D. S., Kim, T., A note on degenerate poly-Bernoulli numbers and polynomials, Advan. Diff. Equat., 258 (2015), 1-8.

[14] Kim, T., Kim, D. S., Kwon, J., Kim, H. Y., A note on degenerate Genocchi and poly-Genocchi numbers and polynomials, J. Ineq. Appl., 110 (2020), 13 pages.

[15] Kim, T., New approach to q-Euler numbers and their interpolation functions, Adv. Stud. Contemp. Math. (Kyungshang), 18 (2) (2009), 105-112.

[16] Kim, T., On the multiple q-Genocchi and Euler numbers, Russ. J. Math. Phys., 15 (4) (2008), 481-486.

[17] Kim, T., Note on q-Genocchi numbers and polynomials, Adv. Stud. Contemp. Math. (Kyungshang), 17 (1) (2008), 9-15.

[18] Kim, T. Jang. L.-C., Pak, H. K., A note on q-Euler and Genocchi numbers, Proc. Japan Acad. Ser. A Math. Sci., 77 (8) (2001), 139-141.

[19] Kim, D. S., Kim, T., Kim, H. Y., Kwon J., Identities for Euler polynomials and alternating power sums, Proc. Jangjeon Math. Soc., 24 (2) (2021), 153-170.

[20] Kim, D. S., Kim, T., Some symmetric identities for the higher-order q-Euler polynomials related to symmetry group S_3 arising from p-adic q-fermionic integrals on $\mathbb{Z}_p$, Filomat, 30 (7) (2016), 1717-1721.

[21] Muhiuddin, G., Khan, W. A., Duran, U., Kadi, D.-Al., Some identities of the multi poly-Bernoulli polynomials of complex variable, J. Function Spaces, Volume 2021, Article ID 7172054, 8 pages.

[22] Nadeem, M., Khan, W. A., Shadab, M., A note on q-analogue of poly-Genocchi numbers and polynomials, International Journal of Applied Mathematics, 35 (1) (2022), 89-102.

[23] Nisar, K. S., Khan, W. A., Notes on q-Hermite based unified Apostol type polynomials, Journal of Interdisciplinary Mathematics, 22 (7) (2019), 1185-1203.

[24] Sharma, S. K., A note on degenerate poly-Genocchi polynomials, Int. J. Adv. Appl. Sci., 7 (5) (2020), 1-5.

[25] Young, P.T., Degenerate Bernoulli polynomials generalized factorials sums and their application, J. Number Theory, 128 (4) (2008), 738-758.