South East Asian J. of Mathematics and Mathematical Sciences
Vol. 19, No. 1 (2023), pp. 01-16

DOI: 10.56827/SEAJMMS.2023.1901.1 **ISSN (Online): 2582-0850**

ISSN (Print): 0972-7752

ON SOME COMBINATORIAL INTERPRETATIONS FOR ROGERS-RAMANUJAN TYPE IDENTITIES

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(**Received:** *Nov. 24, 2022* **Accepted:** *Mar. 12, 2023* **Published:** *Apr. 30, 2023)*

Abstract: We implement an advanced technique to provide combinatorial interpretations of some Rogers–Ramanujan type identities, also known as sum–product identities. Specifically, we elaborate on the notion of modular Ferrers diagrams to explain these identities in terms of n–color overpartitions. Additionally, we reveal the interdependence between split part n–color partitions, 2–color F–partitions, and n–color overpartitions.

Keywords and Phrases: Rogers–Ramanujan type identities; n–color overpartitions; Split part n–color partition; 2–color F–partition; Modular Ferrers diagram; Combinatorial interpretation.

2020 Mathematics Subject Classification: 05A17, 19, 11P81, 84.

1. n–color Overpartition

A partition of a positive integer n is a weakly decreasing sequence of positive integers $(\lambda_1, \lambda_2, \cdots, \lambda_r)$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_r = n$. We use $l(\lambda)$ to denote the number of parts in a partition λ and $|\lambda|$ to denote the number being partitioned. As a convention, we consider the number of partitions of 0 to be 1. Partitions can also be represented graphically by a Ferrers diagram. A Ferrers diagram of a partition $(\lambda_1, \lambda_2, ..., \lambda_r)$ of n consist of r rows of left aligned cells, with the i^{th} row having λ_i cells. For example, the partition $\lambda = (6, 4, 3, 1)$ of 14 has the following Ferrers diagram:

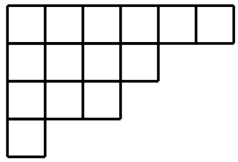

Figure 1: Ferrers diagram for partition λ=(6,4,3,1)

In this paper, we make use of Ferrers diagrams with some variations such as modular Ferrers diagram for combinatorial interpretations of some Rogers–Ramanujan type identities. We start by defining a modular Ferrers diagram.
Modular Ferrers diagrams are also called $p-$modular Ferrers diagrams. For a partition λ into parts λ_i congruent to k modulo p where $0 < k \leq p$, its $p-$modular Ferrers diagram is the diagram in which the i^{th} row has $\lceil \frac{\lambda_i}{p} \rceil$ cells, the cells in the last column have k and others have p. The sum of the numbers in all the cells equals $|\lambda|$. For example, 2$-$modular Ferrers diagram for the partition $(9, 7, 5, 3, 1)$ of 25 is shown in Figure 1.

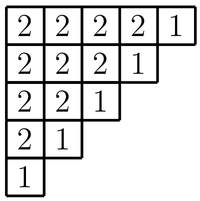

Figure 2: 2$-$modular Ferrers diagram for the partition λ=(9,7,5,3,1)

Motivated by [6], we make use of $p-$modular Ferrers diagrams to visualise the $n-$color overpartitions of the following Rogers–Ramanujan type identity that appears in [7] as Identity No. 195.

$$f(q) = \sum_{\nu=1}^{\infty} \frac{(-q^2; q^2)_{\nu-1} q^{\nu^2}}{(q; q)_{2\nu}} = \frac{[q^{16}, q^2, q^{14}; q^{16}]_\infty [q^{20}, q^{12}; q^{32}]_\infty}{(q; q)_\infty} \tag{1.1}$$

and employ the standard notations as:

$$(a_1; q)_0 = 1, \quad (a_1; q)_n = (1 - a_1)(1 - a_1 q)(1 - a_1 q^2) \cdots (1 - a_1 q^{n-1}),$$

$$(a_1; q)_\infty = \lim_{n \to \infty} (a_1; q)_n,$$

$$\text{and} \qquad [a_1, a_2, a_3; q]_\infty = (a_1; q)_\infty (a_2; q)_\infty (a_3; q)_\infty,$$

where a_1, a_2, a_3, q are complex numbers with $|q| < 1$.

Before proceeding we recall n–color partitions and n–color overpartitions. The $n-$color partition was introduced by Agarwal and Andrews [2], and its overpartition analogue was launched by Lovejoy and Mallet [10].

An $n-$color partition of a positive integer ν is a partition in which each part of size n may appear with up to n different colors denoted by subscripts from 1 to n, and parts are ordered first by their size and then according to the color. The parts satisfy the order,

$$1_1 < 2_1 < 2_2 < 3_1 < 3_2 < 3_3 < 4_1 < 4_2 < 4_3 < 4_4 \cdots .$$

Let $((\lambda_1)_{x_1}, (\lambda_1)_{x_2}, \cdots, (\lambda_r)_{x_r})$ represent the $n-$color partition into r parts. Since we have n different copies of part n, we also call it a partition with "n copies of n". For example, there are six $n-$color partitions of 3:

$$3_3, 3_2, 3_1, 2_2 1_1, 2_1 1_1, 1_1 1_1 1_1.$$

We define the weighted difference of two parts $(\lambda_i)_{x_i}, (\lambda_j)_{x_j}$ denoted by $(((\lambda_i)_{x_i} - (\lambda_j)_{x_j}))$, as $\lambda_i - \lambda_j - x_i - x_j$ provided that $\lambda_i \geq \lambda_j$. For convenience, we denote $\delta_i = (((\lambda_i)_{x_i} - (\lambda_{i+1})_{x_{i+1}}))$ where $\lambda_i \geq \lambda_{i+1}$. When we plot the modular Ferrers diagram for the $n-$color partitions, we color only the last cell on the right of each row, the remaining boxes are uncolored as done in [5]. For instance, modular Ferrers diagram for the n–color partition $(8_2, 5_3, 2_2, 1_1)$ of 16 is shown in Figure 3.

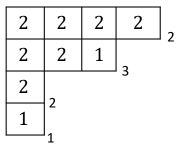

Figure 3: Modular Ferrers diagram for n–color partition $(8_2, 5_3, 2_2, 1_1)$.

An $n-$color overpartition of a positive integer ν is an $n-$color partition of ν in which we may overline the final occurrence of each part $(\lambda_i)_{x_i}$. For example, the $n-$color overpartitions of 3 are:

$$3_3, \overline{3}_3, 3_2, \overline{3}_2, 3_1, \overline{3}_1, 2_2 1_1, \overline{2}_2 1_1, 2_2 \overline{1}_1, \overline{2}_2 \overline{1}_1, 2_1 1_1, \overline{2}_1 1_1, 2_1 \overline{1}_1, \overline{2}_1 \overline{1}_1, 1_1 1_1 1_1, 1_1 1_1 \overline{1}_1.$$

The overlined part in n–color overpartitions is shown by shading the last cell in the modular Ferrers diagram of corresponding part. For example, modular Ferrers diagram for n–color overpartition $(\overline{8}_2, 5_3, \overline{4}_2, 1_1)$ of 18 is shown in Figure 4.

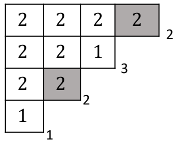

Figure 4: Modular Ferrers diagram for n–color overpartition $(\overline{8}_2, 5_3, \overline{4}_2, 1_1)$.

Now we provide the n–color overpartition theoretic interpretation of (1.1). Consider $f(q)$ and letting

$$\sum_{\nu=1}^{\infty} \overline{A}(\nu)q^{\nu} = \sum_{\nu=1}^{\infty} \frac{(-q^2;q^2)_{\nu-1}q^{\nu^2}}{(q;q)_{2\nu}} = \sum_{\nu=1}^{\infty} \frac{q^{\nu^2}(-q^2;q^2)_{\nu-1}}{(q;q^2)_{\nu}(q^2;q^2)_{\nu}}. \tag{1.2}$$

Theorem 1.1. *For $\nu \geq 1$, let $\overline{A}(\nu)$ represent the number of n–color overpartitions satisfying the following conditions*

(1.1.a) $\lambda_i \equiv x_i \pmod 2$ $\forall$ i,

(1.1.b) λ_r *is not overlined,*

(1.1.c) $\delta_i \geq 0$, $\forall$ $i < r$. *For $\delta_i = 0$, λ_i is not overlined.*

Let $B(\nu)$ represent the count of partitions of ν in which the parts are $\equiv \pm 1$, ± 3, ± 5, ± 6 (mod 32). *Then*

$$f(q) = \sum_{\nu=1}^{\infty} \overline{A}(\nu)q^{\nu} = \sum_{\nu=1}^{\infty} B(\nu)q^{\nu}.$$

Proof. The term q^{ν^2} generates the partition $\lambda^{(1)}$ into parts $1, 3, \cdots, (2\nu - 1)$. Assign color 1 to each part so δ_i between two consecutive parts is 0 for $1 \leq i < r$. The term $(q^2;q^2)_{\nu}^{-1}$ generates partition $\lambda^{(2)}$ into even parts $\leq 2\nu$. From the largest part of $\lambda^{(2)}$, we attach each part $\lambda_j^{(2)}$ in the following manner: We join 2 starting from the first row to $((\lambda_j^{(2)})/2)^{th}$ row, so δ_j remains unchanged. The factor $(q;q^2)_{\nu}^{-1}$ give rise to partition $\lambda^{(3)}$ that generates odd parts $\leq 2\nu - 1$. Now, we attach these parts to the earlier partition and append 2 starting from the first row upto $((\lambda_j^{(3)} - 1)/2)^{th}$ row and 1 to $((\lambda_j^{(3)} + 1)/2)^{th}$ row. Also, we increase the color of $((\lambda_j^{(3)} + 1)/2)^{th}$ part by 1. Here, δ_i between the parts where different enteries are attached increases by 1 and for rest of the cases it remains same. At

the end, $(-q^2;q^2)_{\nu-1}$ generates partition $\lambda^{(4)}$ into distinct even parts with largest part $\leq\ 2\nu-2$. We fix $\lambda_j^{(4)}$ in a similar way as done for $\lambda_j^{(2)}$. But now, we overline $(\lambda_j^{(4)}/2)^{th}$ part. This step does not make any changes in δ_i but give rise to overpartition.
For a better understanding, consider the following example:

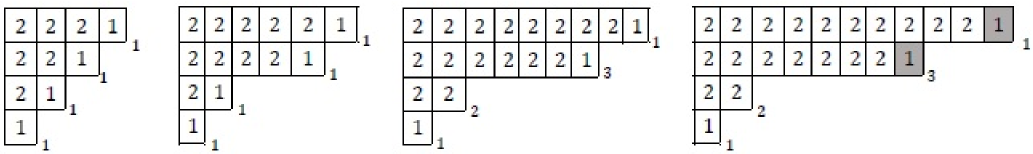

Figure 5: Insertion of $\lambda^{(2)}=(4,4)$ into $\lambda^{(1)}=(7,5,3,1)$ followed by the insertion of $\lambda^{(3)}=(5,3,3)$ and $\lambda^{(4)}=(4,2)$.

Remark 1.1. *Figure 5 represents the* $n-$*color overpartition* $(\overline{21}_1,\overline{15}_3,4_2,1_1)$ *with the steps described above.*

2. Split part $(n+t)$–color Partitions

Recently in [3] Agarwal and Sood interpreted two eighth order, 'mock theta functions' of Gordon and McIntosh using split $(n+t)$–colored partitions. Inspired from their work we introduce split part $(n+t)$–color partition, defined as:

Definition 2.1. *The split part* $(n+t)-$*color partition is* $(n+t)-$*color partition in which the part splits into two parts. Consider a part* $(\lambda_i)_{x_i}$ *from the partition and we split* λ_i *into two parts as* $(\lambda_i)_{x_i}=(\lambda_i^{'}+\lambda_i^{''})_{x_i}$, $1\leq\lambda_i^{'}\leq\lambda_i$ *and* $0\leq\lambda_i^{''}\leq\lambda_i-1$.

Example 2.1. The split part n–color partitions of 3 are:

$$3_1,(2+1)_1,(1+2)_1,3_2,(2+1)_2,(1+2)_2,3_3,(2+1)_3,(1+2)_3$$

$$2_11_1,(1+1)_11_1,2_21_1,(1+1)_21_1,1_11_11_1.$$

In this section we provide the combinatorial interpretations in terms of split part n–color partition for $f(q)=\sum_{\nu=1}^{\infty}A(\nu)q^{\nu}=\sum_{\nu=1}^{\infty}\frac{q^{\nu^2}(-q^2;q^2)_{\nu-1}}{(q;q^2)_{\nu}(q^2;q^2)_{\nu}}$, given in Theorem 2.1.

Theorem 2.1. *Let* $A(\nu)$ *enumerate the number of split part* $n-$*color partitions of* ν *such that*

(2.1.a) $\lambda_i\equiv x_i \pmod 2\ \forall\ i$, (2.1.$b$) λ_r *should not be splitted,*

(2.1.c) $\delta_i \geq 0$, *and* $\delta_i \equiv 0 \pmod 2$, *if* $\delta_i = 0$ *then the part* λ_i *should not be splitted otherwise* $(\lambda_i)_{x_i} = (\lambda_i^{'} + \lambda_i^{''})_{x_i}$ *where* $\lambda_i^{'} = \lambda_{i+1} + x_i + x_{i+1}$ *and* $\lambda_i^{''} = \lambda_i - \lambda_i^{'} \equiv 0 \pmod 2$. *Then*

$$f(q) = \sum_{\nu=1}^{\infty} A(\nu) q^{\nu} = \sum_{\nu=1}^{\infty} \overline{A}(\nu) q^{\nu} = \sum_{\nu=1}^{\infty} B(\nu) q^{\nu}.$$

Classical Proof. The proof of $A(\nu)$ has same technique as presented in [1, 9, 11]. Here we mention only classes and the remaining proof can be elaborated easily. To obtain the proof of $A(\nu)$ we first enumerate $M(\nu)$, in terms of split part n–color partitions, where

$$\sum_{\nu=0}^{\infty} M(\nu) q^{\nu} = \sum_{\nu=0}^{\infty} \frac{(-q^2; q^2)_{\nu} q^{\nu^2}}{(q; q)_{2\nu}}. \tag{2.1}$$

Lemma 2.1. *Let* $M(\nu)$ *enumerate the number of split part n–color partitions of* ν *satisfying* (2.1.a), (2.1.c) *and for* $\lambda_r \neq x_r$ *then it should be splitted as* $(\lambda_r^{'} + \lambda_r^{''})_{x_r}$ *where* $\lambda_r^{'} = x_r$ *and* $\lambda_r^{''} \equiv 0 \pmod 2$, *Then* (2.1) *holds.*

We illustrate the idea with the help of following example:

Example 2.2. Obtaining first few terms in the expansion for

$$\sum_{\nu=0}^{\infty} \frac{(-q^2; q^2)_{\nu} q^{\nu^2}}{(q; q)_{2\nu}} = 1 + q + q^2 + 3q^3 + 4q^4 + 6q^5 + 8q^6 + 11q^7 + 15q^8 + 20q^9 + 27q^{10} + \cdots .$$

We see that, for $\nu = 8$, fifteen partitions satisfying the conditions of $M(\nu)$ are

$$8_2, (2+6)_2, 8_4, (4+4)_4, 8_6, (6+2)_6, 8_8, 7_1 1_1, (3+4)_1 1_1, 7_3 1_1,$$
$$(5+2)_3 1_1, 7_5 1_1, 6_2 2_2, 5_1 3_1, 5_1 (1+2)_1.$$

Sketch proof of Lemma 2.1. Divide the partitions into four classes:

(i) $(\lambda_r)_{x_r}$ is not of the form $(\lambda_i)_{\lambda_i}$ or $(\lambda_i^{'} + 2)_{\lambda_i^{'}}$,

(ii) $(\lambda_r)_{x_r}$ is of the form 1_1,

(iii) $(\lambda_r)_{x_r}$ is of the form $(1+2)_1$,

(iv) $(\lambda_r)_{x_r}$ is of the form $(\lambda_r)_{\lambda_r}$ $(\lambda_r \geq 2)$ or $(\lambda_r^{'} + 2)_{\lambda_r^{'}}$ $(\lambda_r^{'} \geq 2)$.

The remaining proof can be elaborated as in [9]. The proof of Theorem 2.1 can proceed in the same manner as in the proof of Lemma 2.1, we get the desired result.

Bijective Proof. We can naturally connect the interpretations of $f(q)$ in terms of n–color overpartitions and split part n–color partitions. Let $(\lambda_i)_{x_i}$ and $(\lambda_{i+1})_{x_{i+1}}$ be i^{th} and $(i+1)^{th}$ parts of a n–color overpartition, then the corresponding i^{th} part of split part n–color partition be $(\lambda_i^{'}+\lambda_i^{''})_{x_i}$, that is given by

$$\phi : (\lambda_i)_{x_i} \to \begin{cases} (\lambda_i)_{x_i} & \text{if } \lambda_i \text{ is not overlined,} \\ \\ (\lambda_i^{'}+\lambda_i^{''})_{x_i} & \text{if } \lambda_i \text{ is overlined,} \end{cases} \tag{2.2}$$

where $\lambda_i^{'} = \lambda_{i+1} + x_{i+1} + x_i$ and $\lambda_i^{''} = \lambda_i - \lambda_i^{'}$.

In the reverse implication, let $(\lambda_i^{'}+\lambda_i^{''})_{x_i}$ be any part of split part n–color partitions then the corresponding part in an n–color overpartition $(\lambda_i)_{x_i}$, is given by

$$\phi^{-1} : (\lambda_i^{'}+\lambda_i^{''})_{x_i} \to \begin{cases} (\lambda_i^{'})_{x_i} & \text{if } \lambda_i^{''} = 0, \\ \\ (\overline{\lambda_i^{'}+\lambda_i^{''}})_{x_i} & \text{if } \lambda_i^{''} \neq 0. \end{cases} \tag{2.3}$$

In a sequel, we give a connection for $f(q)$ in terms of overpartitions and 2–color F–partition. Before procceding ahead we recall some definitions.

Definition 2.2. [8] *The F–partitions of a positive integer ν, which is a two-rowed array of distinct non–negative integers*

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix}$$

such that integers are arranged in a non-increasing order in each row and

$$\nu = \sum_{i=1}^{r} (a_i + b_i + 1).$$

Definition 2.3. [4] *The m–color F–partitions is the color F–partitions with m copies of the non negative integer 'x' with color 'l', so*

$$x_l : 0 \leq x \leq l-1,\ 1 \leq l \leq x,$$

and $x_l \neq x'_{l'}$, unless $x = x'$ and $l = l'$. There is a strict decrease among the parts along the rows and the parts follow the order

$$0_1 < 0_2 < \cdots < 1_1 < 1_2 < \cdots < 2_1 < 2_2 < \cdots < 3_1 < 3_2 < \cdots .$$

Consider colored F–partitions of ν in which the parts in either row appear from m copies and are distinct. Let $cF_m(\nu)$ denote the number of all such partitions.

Example 2.3. For $\nu = 2$, the 2–color F–partitions enumerated by $cF_2(2)$ are

$$\begin{pmatrix}1_1\\0_1\end{pmatrix}, \begin{pmatrix}1_2\\0_1\end{pmatrix}, \begin{pmatrix}1_1\\0_2\end{pmatrix}, \begin{pmatrix}1_2\\0_2\end{pmatrix}, \begin{pmatrix}0_1\\1_1\end{pmatrix}, \begin{pmatrix}0_1\\1_2\end{pmatrix}, \begin{pmatrix}0_2\\1_1\end{pmatrix}, \begin{pmatrix}0_2\\1_2\end{pmatrix}, \begin{pmatrix}0_2 & 0_1\\0_2 & 0_1\end{pmatrix}.$$

Remark 2.1. *In the main results, we use 2–color F–partitions in which top and bottom row entries of each column appear with same subscripts. For the notation purpose 2–color F–partitions with each column having same subscript array as follows:*

$$\begin{pmatrix}(a_1)_{l_1} & (a_2)_{l_2} & \cdots & (a_r)_{l_r}\\(b_1)_{l_1} & (b_2)_{l_2} & \cdots & (b_r)_{l_r}\end{pmatrix}$$

where $a_1 > a_2 > \cdots > a_r$; $b_1 > b_2 > \cdots > b_r$ and $l_i = 1$ or 2, for $1 \leq i \leq r$.

Theorem 2.2. *For $\nu \geq 1$, let $cF_2(\nu)$ enumerate the number of 2–color F–partitions of ν such that*

(2.2.a) *for each array $\begin{pmatrix}(a_i)_{l_i}\\(b_i)_{l_i}\end{pmatrix}$, $a_i - b_i \geq 0$,*

(2.2.b) *$l_r = 1$,*

(2.2.c) *for two consecutive arrays $\begin{pmatrix}(a_i)_{l_i} & (a_{i+1})_{l_{i+1}}\\(b_i)_{l_i} & (b_{i+1})_{l_{i+1}}\end{pmatrix}$, $b_i - a_{i+1} \geq 1$ and for $b_i - a_{i+1} = 1$, $l_i = 1$. Then*

$$cF_2(\nu) = \overline{A}(\nu) \quad \forall\, \nu \geq 1.$$

Example 2.4. For $\nu = 6$,

$$\overline{A}(6) = cF_2(6) = 7.$$

The partitions enumerated by $\overline{A}(6)$ are

$$6_2, 6_4, 6_6, 5_3 1_1, 5_1 1_1, \overline{5}_1 1_1, 4_2 1_1.$$

And corresponding to $cF_2(6)$, the relevant partitions are

$$\begin{pmatrix}3_1\\2_1\end{pmatrix}, \begin{pmatrix}4_1\\1_1\end{pmatrix}, \begin{pmatrix}5_1\\0_1\end{pmatrix}, \begin{pmatrix}3_1 & 0_1\\1_1 & 0_1\end{pmatrix}, \begin{pmatrix}2_1 & 0_1\\2_1 & 0_1\end{pmatrix}, \begin{pmatrix}2_2 & 0_1\\2_2 & 0_1\end{pmatrix}, \begin{pmatrix}2_1 & 0_1\\1_1 & 0_1\end{pmatrix}.$$

Proof. We establish a bijection between the 2–color F–partitions enumerated by $cF_2(\nu)$ and n–color overpartitions enumerated by $\overline{A}(\nu)$. We do this by defining a map ϕ from each column $\begin{pmatrix}(a_i)_{l_i}\\(b_i)_{l_i}\end{pmatrix}$ of the 2–color F–partition enumerated by $cF_2(\nu)$ to a single part $(\lambda_i)_{x_i}$ or $(\overline{\lambda_i})_{x_i}$ of n–color overpartition enumerated by $\overline{A}(\nu)$. The mapping ϕ is

$$\phi:\begin{pmatrix}(a_i)_{l_i}\\(b_i)_{l_i}\end{pmatrix}\to\begin{cases}(a_i+b_i+1)_{a_i-b_i+1} & \text{if } l_i=1,\\ \overline{(a_i+b_i+1)}_{a_i-b_i+1} & \text{if } l_i=2.\end{cases}\tag{2.4}$$

Now suppose we have

$$\phi:\begin{pmatrix}(a_i)_{l_i}\\(b_i)_{l_i}\end{pmatrix}\to\begin{cases}(\lambda_i)_{x_i} & \text{if } l_i=1,\\ (\overline{\lambda_i})_{x_i} & \text{if } l_i=2,\end{cases}$$

and

$$\phi:\begin{pmatrix}(a_{i+1})_{l_{i+1}}\\(b_{i+1})_{l_{i+1}}\end{pmatrix}\to\begin{cases}(\lambda_{i+1})_{x_{i+1}} & \text{if } l_i=1,\\ (\overline{\lambda_{i+1}})_{x_{i+1}} & \text{if } l_i=2.\end{cases}$$

Then the weighted difference for two parts $(\lambda_i)_{x_i}$ and $(\lambda_{i+1})_{x_{i+1}}$ is given by

$$\begin{aligned}\delta_i=(((\lambda_i)_{x_i}-(\lambda_{i+1})_{x_{i+1}}))&=\lambda_i-\lambda_{i+1}-x_i-x_{i+1},\\ &=2(b_i-a_{i+1}-1).\end{aligned}\tag{2.5}$$

Also,

$$\lambda_i-x_i=(a_i+b_i+1)-(a_i-b_i+1)=2b_i,\tag{2.6}$$

which imply $\lambda_i-x_i\equiv 0 \pmod 2$.
Using (2.5), (2.6) and the given conditions (1.1.a)–(1.1.c) we get the desired conditions (2.2.a)–(2.2.c). To see the reverse implications, we consider the inverse images of two consecutive parts $(\lambda_i)_{x_i}$ or $(\overline{\lambda}_i)_{x_i}$, $(\lambda_{i+1})_{x_{i+1}}$ or $(\overline{\lambda}_{i+1})_{x_{i+1}}$ of n–color overpartition enumerated by $\overline{A}(\nu)$ as:
$\phi^{-1}:(\lambda_i)_{x_i}=\begin{pmatrix}(\frac{\lambda_i+x_i-2}{2})_1\\(\frac{\lambda_i-x_i}{2})_1\end{pmatrix}$ or $\phi^{-1}:(\overline{\lambda}_i)_{x_i}=\begin{pmatrix}(\frac{\lambda_i+x_i-2}{2})_2\\(\frac{\lambda_i-x_i}{2})_2\end{pmatrix}$, and
$\phi^{-1}:(\lambda_{i+1})_{x_{i+1}}=\begin{pmatrix}(\frac{\lambda_{i+1}+x_{i+1}-2}{2})_1\\(\frac{\lambda_{i+1}-x_{i+1}}{2})_1\end{pmatrix}$ or $\phi^{-1}:(\overline{\lambda}_{i+1})_{x_{i+1}}=\begin{pmatrix}(\frac{\lambda_{i+1}+x_{i+1}-2}{2})_2\\(\frac{\lambda_{i+1}-x_{i+1}}{2})_2\end{pmatrix}$.
So,

$$\lambda_i-x_i=2b_i,\tag{2.7}$$

$$\lambda_{i+1}+x_{i+1}=2a_{i+1}+2,\tag{2.8}$$

hence

$$\begin{aligned}\delta_i &= (a_i + b_i + 1) - (a_{i+1} + b_{i+1} + 1) - (a_i - b_i + 1) - (a_{i+1} - b_{i+1} + 1),\\ &= 2(b_i - a_{i+1} - 1). \end{aligned} \tag{2.9}$$

$$\begin{aligned}b_i - a_{i+1} &= \frac{\lambda_i - \lambda_{i+1} - x_i - x_{i+1} + 2}{2} - \frac{\lambda_i - x_i}{2},\\ &= x_i - 1. \end{aligned} \tag{2.10}$$

From (2.9), (2.10) and the conditions (2.2.a)–(2.2.c), we easily get (1.1.a)–(1.1.c).

3. Some More Combinatorial Interpretations

In the spirit of results in Section 1 and 2, and in our endeavor to contribute further towards the legacy for studying Rogers–Ramanujan Identities, here, we interpret additional Rogers–Ramanujan type identities given in [7, 12]. The following identities appear in [7] with Identity No. 45, 46, 11, 12, 37, 106, 40, respectively, given below:

$$\begin{aligned}
f_1(q) &= \sum_{\nu=0}^{\infty} \frac{(-1;q^2)_\nu q^{\nu(\nu+1)}}{(q;q)_{2\nu}} = \frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}[q^6,-q^3,-q^3;q^6]_\infty,\\
f_2(q) &= \sum_{\nu=0}^{\infty} \frac{(-q^2;q^2)_\nu q^{\nu(\nu+1)}}{(q;q)_{2\nu+1}} = \frac{(-q^2,q^2)_\infty}{(q^2;q^2)_\infty}[q^6,-q,-q^5;q^6]_\infty,\\
f_3(q) &= \sum_{\nu=0}^{\infty} \frac{(-1;q^4)_\nu q^{\nu^2}}{(q;q^2)_\nu (q^4;q^4)_\nu} = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}[-q^4,-q,q^3;-q^4]_\infty,\\
f_4(q) &= \sum_{\nu=0}^{\infty} \frac{(-1;q^4)_\nu q^{\nu(\nu+2)}}{(q;q^2)_\nu (q^4;q^4)_\nu} = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}[-q^4,q,-q^3;-q^4]_\infty,\\
f_5(q) &= \sum_{\nu=0}^{\infty} \frac{(-1;q)_\nu q^{\nu^2}}{(q;q^2)_\nu (q;q)_\nu} = \frac{(-q;q)_\infty}{(q;q)_\infty}[q^6,q^3,q^3;q^6]_\infty,\\
f_6(q) &= \sum_{\nu=1}^{\infty} \frac{(-q;q)_{\nu-1} q^{\nu^2}}{(q;q^2)_\nu (q;q)_\nu} = \frac{[q^{12},-q^5,-q^7;q^{12}]_\infty}{(q;q)_\infty},\\
f_7(q) &= \sum_{\nu=0}^{\infty} \frac{(-q;q)_\nu q^{\nu(\nu+1)}}{(q;q^2)_{\nu+1} (q;q)_\nu} = \frac{(-q;q)_\infty}{(q;q)_\infty}[q^6,q,q^5;q^6]_\infty.
\end{aligned}$$

Throughout this section, sums in Rogers–Ramanujan type identities represent generating functions $f_i(q)$ for either $A_i(\nu)$, which count partitions in terms of split parts n–color partitions, or $\overline{A}_i(\nu)$, which count n–color overpartitions, where $1 \leq i \leq 7$. Additionally, $cF_i(\nu)$, counts the number of 2–colored F–partitions for $1 \leq$

$i \leq 4$. The generating function for $B_i(\nu)$, which counts ordinary partitions, is expressed without a sum and instead only uses products from the q–series notation described above. These lead to 4-way combinatorial interpretations that satisfy:

$$f_i(q) = \sum_{\nu=0}^{\infty} A_i(\nu)q^\nu = \sum_{\nu=0}^{\infty} \overline{A}_i(\nu)q^\nu = \sum_{\nu=0}^{\infty} cF_i(\nu)q^\nu = \sum_{\nu=0}^{\infty} B_i(\nu)q^\nu,\ 1 \leq i \leq 4, \quad (3.1)$$

and 3-way combinatorial interpretations for $f_i(q)$ where $5 \leq i \leq 7$.

$$f_i(q) = \sum_{\nu=0}^{\infty} A_i(\nu)q^\nu = \sum_{\nu=0}^{\infty} \overline{A}_i(\nu)q^\nu = \sum_{\nu=0}^{\infty} B_i(\nu)q^\nu, \quad 5 \leq i \leq 7. \quad (3.2)$$

Now, we will summarize the combinatorial interpretations that involve all the combinatorial tools discussed in this paper. To do this, let's consider the left-hand side of $f_1(q)$,

$$\begin{aligned}\sum_{\nu=0}^{\infty} \frac{(-1;q^2)_\nu q^{\nu(\nu+1)}}{(q;q)_{2\nu}} &= 1 + 2\sum_{\nu=1}^{\infty} \frac{(-q^2;q^2)_{\nu-1} q^{\nu(\nu+1)}}{(q;q)_{2\nu}} \\ &= 1 + 2\sum_{\nu=1}^{\infty} \hat{A}_1(\nu)q^\nu, \end{aligned} \quad (3.3)$$

where $\sum_{\nu=1}^{\infty} \hat{A}_1(\nu)q^\nu = \sum_{\nu=1}^{\infty} \frac{(-q^2;q^2)_{\nu-1} q^{\nu(\nu+1)}}{(q;q)_{2\nu}}$. Now we give the combinatorial interpretation of $f_1(q)$ in the following theorem.

Theorem 3.1. *Let $\overline{\hat{A}}_1(\nu)$ count the number of n–color overpartitions of ν satisfying*

(3.1.a) $\lambda_i \equiv x_i$ (*mod* 2) $\forall\ i$,

(3.1.b) $\lambda_r, x_r > 1$,

(3.1.c) λ_r *is not overlined,*

(3.1.d) $\delta_i \geq -2$, *and* $\delta_i \equiv 0 \pmod 2$ $\forall\ i < r$, *for* $\delta_i = -2$, λ_i *is not overlined.*

Let $\hat{A}_1(\nu)$ enumerate the number of split part n–color partitions of ν satisfying (3.1.a) *along with*

(3.1.e) λ_r *should not be splitted,*

(3.1.f) $\delta_i \geq -2$ *and* $\delta_i \equiv 0 \pmod 2$ $\forall\ i < r$, *if* $\delta_i = -2$ *then the part* λ_i *should not be splitted otherwise* $(\lambda_i)_{x_i} = (\lambda_i^{'} + \lambda_i^{''})_{x_i}$ *where* $\lambda_i^{'} = \lambda_{i+1} + x_i + x_{i+1} - 2$ *and* $\lambda_i^{''} = \lambda_i - \lambda_i^{'} \equiv 0 \pmod 2$ *for* $\delta_i = 0$, $\lambda_i^{'} = \lambda_{i+1} + x_i + x_{i+1}$ *and* $\lambda_i^{''} = \lambda_i - \lambda_i^{'} \equiv 0 \pmod 2$ *for* $\delta_i > 0$.

Let $c\hat{F}_1(\nu)$ *enumerate the number of* 2*–color* F*–partitions of* ν *such that*

(3.1.g) *for each array* $\begin{pmatrix}(a_i)_{l_i}\\(b_i)_{l_i}\end{pmatrix}$, $a_i - b_i \geq 1$,

(3.1.h) $l_r = 1$,

(3.1.i) *for two consecutive arrays* $\begin{pmatrix}(a_i)_{l_i} & (a_{i+1})_{l_{i+1}}\\(b_i)_{l_i} & (b_{i+1})_{l_{i+1}}\end{pmatrix}$, $b_i - a_{i+1} \geq 0$ *and for* $b_i - a_{i+1} = 0$, $l_i = 1$.

Let $B_1(\nu)$ *is the number of overpartitions of* ν *in which the parts are* $\equiv \pm 2, 3 \pmod 6$*. Then,*

$$2\overline{\hat{A}}_1(\nu) = \overline{A}_1(\nu) = 2\hat{A}_1(\nu) = A_1(\nu) = 2c\hat{F}_1(\nu) = B_1(\nu)\ \forall\ \nu \geq 1.$$

Theorem 3.2. *Let* $\overline{A}_2(\nu)$ *counts the number of* $(n+1)$*–color overpartitions of* ν *satisfying* (3.1.c)

(3.2.a) $x_r = \lambda_r + 1$,

(3.2.b) $\lambda_i - x_i \equiv 1 \pmod 2\ \forall\ i$,

(3.2.c) $\delta_i \geq 0$, *and* $\delta_i \equiv 0$ (*mod* 2) $\forall\ i < r$, *for* $\delta_i = 0$, λ_i *is not overlined.*

Let $A_2(\nu)$ *enumerate the number of split part* $(n+1)$*–color partitions of* ν *satisfying* (3.1.e), (3.2.a), (3.2.b), *along with* $\delta_i \geq 0$ *and* $\delta_i \equiv 0 \pmod 2\ \forall\ i < r$, *if* $\delta_i = 0$ *then the part* λ_i *should not be splitted otherwise* $(\lambda_i)_{x_i} = (\lambda_i^{'} + \lambda_i^{''})_{x_i}$ *where* $\lambda_i^{'} = \lambda_{i+1} + x_i + x_{i+1}$ *and* $\lambda_i^{''} = \lambda_i - \lambda_i^{'} \equiv 0 \pmod 2$.

Let $cF_2(\nu)$ *enumerate the number of* 2*–color* F*–partitions of* ν *such that*

(3.2.d) *for each array* $\begin{pmatrix}(a_i)_{l_i}\\(b_i)_{l_i}\end{pmatrix}$, $b_i \leq a_i + 1$,

(3.2.e) $b_r = 0$, $b_i \geq 1$ *for* $2 \leq i \leq r-1$ *and* $l_r = 1$,

(3.2.f) *for two consecutive arrays* $\begin{pmatrix}(a_i)_{l_i} & (a_{i+1})_{l_{i+1}}\\(b_i)_{l_i} & (b_{i+1})_{l_{i+1}}\end{pmatrix}$, $b_i - a_{i+1} \geq 2$ *and for* $b_i - a_{i+1} = 2$, $l_i = 1$.

Let $B_2(\nu)$ *is the number of overpartitions of* ν *in which the overlined parts are* $\equiv \pm 2 \pmod 6$ *and non-overlined parts are distinct and* $\not\equiv 3 \pmod 6$. *Then*

$$\overline{A}_2(\nu) = A_2(\nu) = cF_2(\nu) = B_2(\nu)\ \forall\ \nu \geq 0.$$

Theorem 3.3. *Let* $\overline{\hat{A}}_3(\nu)$ *count the number of* $n-$*color overpartitions of* ν *satisfying* $(3.1.c)$ *and*

$(3.3.a)$ $\lambda_r - x_r \equiv 0 \pmod 4$,

$(3.3.b)$ $\delta_i \geq 0$ *and* $\delta_i \equiv 0 \pmod 4\ \forall\ i < r$, *for* $\delta_i = 0$, λ_i *is not overlined.*

Let $\hat{A}_3(\nu)$ *enumerate the number of split part* $n-$*color partitions of* ν *satisfying* $(3.1.e)$, $(3.3.a)$ *along with*

$(3.3.c)$ $\delta_i \geq 0$ *and* $\equiv 0 \pmod 4\ \forall\ i < r$, *if* $\delta_i = 0$ *then the part* λ_i *should not be splitted otherwise* $(\lambda_i)_{x_i} = (\lambda_i^{'} + \lambda_i^{''})_{x_i}$ *where* $\lambda_i^{'} = \lambda_{i+1} + x_i + x_{i+1}$ *and* $\lambda_i^{''} = \lambda_i - \lambda_i^{'} \equiv 0 \pmod 4$.

Let $c\hat{F}_3(\nu)$ *enumerate the number of* $2-$*color* $F-$*partitions of* ν *such that*

$(3.3.d)$ *for each array* $\begin{pmatrix} (a_i)_{l_i} \\ (b_i)_{l_i} \end{pmatrix}$, $a_i - b_i \geq 0$,

$(3.3.e)$ $b_r = 0 \pmod 2$, *and* $l_r = 1$,

$(3.3.f)$ *for two consecutive arrays* $\begin{pmatrix} (a_i)_{l_i} & (a_{i+1})_{l_{i+1}} \\ (b_i)_{l_i} & (b_{i+1})_{l_{i+1}} \end{pmatrix}$, $b_i - a_{i+1} \geq 1$, $b_i \not\equiv a_{i+1} \pmod 2$ *and for* $b_i - a_{i+1} = 1$, $l_i = 1$.

Let $B_3(\nu)$ *is the number of overpartitions of* ν *in which the parts are* $\equiv \pm 1, 4 \pmod 8$. *Then,*

$$2\overline{\hat{A}}_3(\nu) = \overline{A}_3(\nu) = 2\hat{A}_3(\nu) = A_3(\nu) = 2c\hat{F}_3(\nu) = B_3(\nu)\ \forall\ \nu \geq 1.$$

Remark 3.1. *In the above theorem, we use similar argument as given in* (3.3) *and letting* $\sum_{\nu=1}^{\infty} \hat{A}_3(\nu) q^\nu = \sum_{\nu=1}^{\infty} \frac{(-q^4;q^4)_{\nu-1} q^{\nu^2}}{(q;q^2)_\nu (q^4,q^4)_\nu}$.

Theorem 3.4. *Let* $\overline{\hat{A}}_4(\nu)$ *count the number of* $n-$*color overpartitions of* ν *satisfying* $(3.1.c)$, $(3.3.b)$, *along with*

$(3.4.a)$ $\lambda_r > 2$, $\lambda_r - x_r \equiv 2 \pmod 4$.

Let $\hat{A}_3(\nu)$ enumerate the number of split part n−color partitions of ν satisfying (3.1.*e*), (3.3.*c*) *and* (3.4.*a*).

Let $c\hat{F}_4(\nu)$ enumerate the number of 2−*color F−partitions of ν such that*

(3.4.*b*) *for each array* $\begin{pmatrix}(a_i)_{l_i}\\(b_i)_{l_i}\end{pmatrix}$, $a_i - b_i \geq 0$,

(3.4.*c*) $a_r + b_r \geq 2$, $b_r \equiv 1 (mod\ 2)$ *and* $l_r = 1$,

(3.4.*d*) *for two consecutive arrays* $\begin{pmatrix}(a_i)_{l_i} & (a_{i+1})_{l_{i+1}}\\(b_i)_{l_i} & (b_{i+1})_{l_{i+1}}\end{pmatrix}$, $b_i - a_{i+1} \geq 1$, $b_i - a_{i+1} \equiv 1 \pmod 2$ *and for* $b_i - a_{i+1} = 1$, $l_i = 1$.

Let $B_4(\nu)$ is the number of overpartitions of ν in which the parts are $\equiv \pm 3, 4 \pmod 8$. *Then,*

$$2\overline{\hat{A}}_4(\nu) = \overline{A}_4(\nu) = 2\hat{A}_4(\nu) = A_4(\nu) = 2c\hat{F}_4(\nu) = B_4(\nu)\ \forall\ \nu \geq 1.$$

Remark 3.2. *In the above theorem, we use similar argument as given in* (3.3) *and letting* $\sum_{\nu=1}^{\infty} \hat{A}_4(\nu) q^\nu = \sum_{\nu=1}^{\infty} \frac{(-q^4;q^4)_{\nu-1} q^{\nu(\nu+2)}}{(q;q^2)_\nu (q^4,q^4)_\nu}$.

Theorem 3.5. *Let $\overline{\hat{A}}_5(\nu)$ count the number of n−color overpartitions of ν satisfying* (3.1.*c*), *and*

(3.5.*a*) $\delta_i \geq 0\ \forall\ i < r$, *for* $\delta_i = 0$, λ_i *is not overlined.*

Let $\hat{A}_5(\nu)$ enumerate the number of split part n−color partitions of ν satisfying (3.1.*e*), *and*

(3.5.*b*) $\delta_i \geq 0\ \forall\ i < r$, *if* $\delta_i = 0$ *then the part* λ_i *should not be splitted otherwise* $(\lambda_i)_{x_i} = (\lambda_i' + \lambda_i'')_{x_i}$ *where* $\lambda_i' = \lambda_{i+1} + x_i + x_{i+1}$ *and* $\lambda_i'' = \lambda_i - \lambda_i'$.

Let $B_5(\nu)$ is the number of overpartitions of ν in which the parts are $\equiv \pm 1, \pm 2 \pmod 6$. *Then,*

$$2\overline{\hat{A}}_5(\nu) = \overline{A}_5(\nu) = 2\hat{A}_5(\nu) = A_5(\nu) = B_5(\nu)\ \forall\ \nu \geq 1.$$

Remark 3.3. *In the above theorem, we used similar argument as given in* (3.3) *and letting* $\sum_{\nu=1}^{\infty} \hat{A}_5(\nu) q^\nu = \sum_{\nu=1}^{\infty} \frac{(-q;q)_{\nu-1} q^{\nu^2}}{(q;q^2)_\nu (q,q)_\nu}$.

Theorem 3.6. *Let $\overline{A}_6(\nu)$ and $A_6(\nu)$ count the number of n−color overpartitions and split part n−color partitions of ν satisfying all the conditions of $\overline{\hat{A}}_5(\nu)$ and*

$A_5(\nu)$ defined in Theorem 3.5, respectively. Let $B_6(\nu)$ be the number of partitions of ν in which the overlined parts are $\equiv \pm 5 \pmod{12}$ and the non overlined parts are $\not\equiv 0 \pmod{12}$. Then,

$$\overline{A}_6(\nu) = A_6(\nu) = B_6(\nu) \ \forall \ \nu \geq 0.$$

Theorem 3.7. *Let $\overline{A}_7(\nu)$ counts the number of $(n+1)$–color overpartitions of ν satisfying (3.1.c), (3.2.a), (3.2.b) and (3.5.a). Let $A_7(\nu)$ enumerate the number of split part $(n+1)$–color partitions of ν satisfying (3.1.e), (3.2.a), (3.2.b), and (3.5.b). Let $B_7(\nu)$ is the number of overpartitions of ν in which the overlined parts are $\equiv \pm 2, 3 \pmod 6$ and the non overlined parts are distinct. Then*

$$A_7(\nu) = \overline{A}_7(\nu) = B_7(\nu) \ \forall \ \nu \geq 0.$$

Remark 3.4. *The proofs of Theorem 3.1 – 3.7 can be supplied by reader on lines of Theorem 1.1 and Theorem 2.1.*

Acknowledgement

The second author is supported by SERB-MATRICS project reference no. MTR/2019/00123.

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