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Fractional Integral Transformations of Mittag-Leffler Type *E*-function

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Abstract : This paper deals with various fractional integral transformations of Mittag-Leffler type *E*-function and obtain results for earlier defined Mittag-Leffler type functions.

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1. Introduction

By recent works of several authors, it has proved that Mittag-Leffler (M-L) function is the solution of fractional differential and integral equations. Many authors have denned various generalizations of M-L function. In an effort to unify results of various forms of M-L function we have defined a unified M-L type function named S-function [1]. Here we study Erdélyi-Kober, Riemann-Liouville and other fractional integral transformation of newly defined M-L type *E*-function.

Throughout this paper, we use the following definitions

- Riemann-Liouville fractional integral operator $\left(I^{\theta}_{c+}\Psi\right)(x)$ [5]

$$\left(I^{\theta}_{c+}\Psi\right)(x) = \frac{1}{\Gamma(\theta)}\int_{c}^{x}(x-t)^{\theta-1}\,\Psi(t)\,dt \tag{1}$$

where $\theta \in \mathbb{C}\; and\, \Re(\theta) > 0$.

- Erdélyi-Kober fractional integral operator $\left(\Xi^{\eta,\theta}_{0+}f\right)(x)$ [5]

$$\left(\Xi^{\eta,\theta}_{0+}f\right)(x) = \frac{x^{-\eta-\theta}}{\Gamma(\eta)}\int_{0}^{x}(x-t)^{\eta-1}\,t^{\theta}f(t)\,dt \tag{2}$$

where $\eta,\theta \in \mathbb{C}; \Re(\eta) > 0\; and\, \Re(\theta) > 0$.

- In 1903, Gösta Mittag-Leffler [4], introduced the function $E_{\alpha}(z)$, defined as

$$E_{\alpha}(z) = \sum_{n=0}^{\infty}\frac{1}{\Gamma(\alpha n+1)}\,z^{n} \tag{3}$$

where $z,\alpha \in \mathbb{C}; \Re(\alpha) \geq 0\; and\; |z| < \infty$.

- In 1905, Wiman [7], extended (3) in the form

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{1}{\Gamma(\alpha n+\beta)}z^n \tag{4}$$

where $z,\alpha,\beta\in\mathbb{C};\Re(\alpha)>0\,and\,\Re(\beta)>0$.

- In 2000, Kiryakova [3], has studied "multiindex M-L functions" defined by

$$E_{(1/\rho_i),(\mu_i)}(z)=\sum_{n=0}^{\infty}\frac{1}{\Gamma(\mu_1+n/\rho_1)\ldots\Gamma(\mu_m+n/\rho_m)}z^n \tag{5}$$

where $m>1$, is an integer, $\rho_1,\ldots,\rho_m>0$ and $\mu_1,\ldots,\mu_m$ are arbitrary real numbers.

- In 2010, Saxena and Nishimoto [6], studied an extension of M-L type function as

$$E_{\gamma,\kappa}[(\alpha_1,\beta_1),...,(\alpha_m,\beta_m);z]=\sum_{n=0}^{\infty}\frac{(\gamma)_{n\kappa}}{\prod_{j=1}^{m}\Gamma(\alpha_j n+\beta_j)}\frac{z^n}{n!} \tag{6}$$

where $z,\alpha_j,\beta_j,\gamma\in\mathbb{C},\sum_{j=1}^{m}\Re(\alpha_j)>\Re(\kappa)-1,j=1,\ldots,m\,and\,\Re(\kappa)>0$.

- In 2012, Kalla, Haidey and Virchenko [2], introduced multiparameter M-L type function in the following form

$$HE_{\mu_1,\mu_2,\ldots,\mu_r}^{\lambda_1,\lambda_2,\ldots,\lambda_r}(z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{\prod_{i=1}^{r}\Gamma(1+\mu_i+\lambda_i n)}\left(\frac{z}{\Lambda}\right)^{\Lambda n+M} \tag{7}$$

where $\mu_i\in\mathbb{C},\lambda_i>0,i=1,2,\ldots,r;\sum_{i=1}^{r}\mu_i=M\,and\sum_{i=1}^{r}\lambda_i=\Lambda$.

2. Mittag-Leffler type E-function

In 2014, Bhatter and Faisal [1], defined a unified M-L type E-function as follows

$$_{\tau}E_k^h(z)={}_{\tau}E_k^h\left[z\left|\begin{matrix}(\rho,a):(\gamma_i,q_i,s_i)_{1,h}\\(\alpha,\beta);(\delta_j,p_j,r_j)_{1,k}\end{matrix}\right.\right]={}_{\tau}E_k^h\left[z\left|\begin{matrix}(\rho,a);(\gamma_1,q_1,s_1),...,(\gamma_h,q_h,s_h)\\(\alpha,\beta);(\delta_1,p_1,r_1),...,(\delta_k,p_k,r_k)\end{matrix}\right.\right]$$

$$=\sum_{n=0}^{\infty}\frac{\left[(\gamma_1)_{q_1 n}\right]^{s_1}\left[(\gamma_2)_{q_2 n}\right]^{s_2}\cdots\left[(\gamma_h)_{q_h n}\right]^{s_h}(-1)^{\rho n}z^{an+\tau}}{\left[(\delta_1)_{p_1 n}\right]^{r_1}\left[(\delta_2)_{p_2 n}\right]^{r_2}\cdots\left[(\delta_k)_{p_k n}\right]^{r_k}\Gamma(\alpha n+\beta)} \tag{8}$$

where

$z, \alpha, \beta, \gamma_i, \delta_j \in \mathbb{C}; \Re(\alpha) \geq 0, \Re(\beta) > 0, \Re(\gamma_i) > 0, \Re(\delta_j) > 0, \Re(q_i) \geq 0,$

$\Re(p_j) \geq 0; s_i, r_j, a, \tau \in \mathbb{R}; \rho \in \{0, 1\}, \left(\sum_{i=1}^{h} q_i s_i < \sum_{j=1}^{k} p_j r_j + \Re(\alpha) \right)$ *or*

$$\left(\sum_{i=1}^{h} q_i s_i = \sum_{j=1}^{k} p_j r_j + \Re(\alpha) \ when \ \prod_{i=1}^{h} (q_i)^{q_i s_i} \left[\alpha^{\alpha} \prod_{j=1}^{k} (p_j)^{p_j r_j} \right]^{-1} |z^a| < 1 \right)$$

$$for \ \ i = 1, 2, ..., h; j = 1, 2, ..., k. \tag{9}$$

3. The image of M-L type E-function under the Riemann-Liouville (R-L) operator I^{θ}_{c+}

Theorem 3.1. *If convergence conditions (9) are satisfied also $\theta \in \mathbb{C}$ and $\Re(\theta) > 0$ then the R-L transform I^{θ}_{c+} of the E-function is*

$$\left(I^{\theta}_{c+} \left[{}_{\tau}E^{h}_{k} (t - c) \right] \right) (x) = \frac{1}{(\tau + 1)_{\theta}} {}_{\theta+\tau}E^{h+1}_{k+1} \left[(x - c) \mid \begin{matrix} (\rho, a) ; (\gamma_i, q_i, s_i)_{1,h} , (\tau + 1, a, 1) \\ (\alpha, \beta) ; (\delta_j, p_j, r_j)_{1,k} , (\tau + \theta + 1, a, 1) \end{matrix} \right] \tag{10}$$

Proof. We obtain the R-L transform I^{θ}_{c+} of the E-function as follows

$$\left(I^{\theta}_{c+} \left[{}_{\tau}E^{h}_{k} (t - c) \right] \right) (x) = \frac{1}{\Gamma(\theta)} \int_{c}^{x} (x - t)^{\theta - 1} \sum_{n=0}^{\infty} \Phi(n) \, (t - c)^{an+\tau} \, dt, \quad \Re(\theta) > 0 \tag{11}$$

where

$$\Phi(n) = \frac{\left[(\gamma_1)_{q_1 n} \right]^{s_1} \left[(\gamma_2)_{q_2 n} \right]^{s_2} ... \left[(\gamma_h)_{q_h n} \right]^{s_h} (-1)^{\rho n}}{\left[(\delta_1)_{p_1 n} \right]^{r_1} \left[(\delta_2)_{p_2 n} \right]^{r_2} ... \left[(\delta_k)_{p_k n} \right]^{r_k} \Gamma(\alpha n + \beta)} \tag{12}$$

Then

$$\left(I^{\theta}_{c+} \left[{}_{\tau}E^{h}_{k} (t - c) \right] \right) (x) = \frac{1}{(\tau + 1)_{\theta}} \sum_{n=0}^{\infty} \Phi(n) \frac{(\tau + 1)_{an}}{(\tau + \theta + 1)_{an}} (x - c)^{an+\theta+\tau} \tag{13}$$

$$= \frac{1}{(\tau + 1)_{\theta}} {}_{\theta+\tau}E^{h+1}_{k+1} \left[(x - c) \mid \begin{matrix} (\rho, a) ; (\gamma_i, q_i, s_i)_{1,h} , (\tau + 1, a, 1) \\ (\alpha, \beta) ; (\delta_j, p_j, r_j)_{1,k} , (\tau + \theta + 1, a, 1) \end{matrix} \right] \tag{14}$$

3.1 Special Cases

1. R-L transform I^{θ}_{c+} of the M-L type function (5)

$$\left[I^{\theta}_{c+}\left\{E_{(1/\rho_i),(\mu_i)}(t)\right\}\right](x)=\frac{1}{(\theta)!\prod_{j=1}^{m-1}\Gamma\left(\mu_j\right)}\times$$

$$\times_{\theta}E^{1}_{m}\left[(x-c)\mid\begin{array}{c}(0,1);(1,1,1)\\ (1/\rho_m,\mu_m);(\mu_1,1/\rho_1,1),\ldots,\left(\mu_{m-1},1/\rho_{m-1},1\right),(\theta+1,1,1)\end{array}\right]. \tag{15}$$

2. R-L transform I^{θ}_{0+} of the M-L type function (6)

$$\left[I^{\theta}_{0+}\left\{E_{\gamma,\kappa}\left[(\alpha_1,\beta_1),\ldots,(\alpha_m,\beta_m);t\right]\right\}\right](x)$$

$$=\frac{1}{(\theta)!\prod_{j=1}^{m}\Gamma\left(\beta_j\right)}{}_{\theta}E^{2}_{m+1}\left[x\mid\begin{array}{c}(0,1);(\gamma,\kappa,1),(1,1,1)\\ (1,1);(\beta_1,\alpha_1,1),\ldots,(\beta_m,\alpha_m,1),(\theta+1,1,1)\end{array}\right]. \tag{16}$$

3. R-L transform I^{θ}_{0+} of the M-L type function (7)

$$\left[I^{\theta}_{0+}\left\{HE^{\lambda_1,\ldots,\lambda_\nu}_{\mu_1,\ldots,\mu_\nu}(t)\right\}\right](x)=\frac{x^{\theta}}{(M+1)_{\theta}\prod_{j=1}^{\nu-1}\Gamma\left(1+\mu_j\right)}\times$$

$$\times_{M}E^{1}_{\nu}\left[\frac{x}{\Lambda}\mid\begin{array}{c}(1,\Lambda);(M+1,\Lambda,1)\\ (\lambda_\nu,1+\mu_\nu);(1+\mu_1,\lambda_1,1),\ldots,\left(1+\mu_{\nu-1},\lambda_{\nu-1},1\right),(M+\theta+1,\Lambda,1)\end{array}\right]. \tag{17}$$

4. The image of M-L type *E*-function under the Erdélyi-Kober (E-K) operator $\Xi^{\eta,\theta}_{0+}$

Theorem 4.1 *If convergence conditions (9) are satisfied also* $\eta,\theta\in\mathbb{C},\Re(\eta)>0$ *and* $\Re(\theta)>0$ *then the E-K transform* $\Xi^{\eta,\theta}_{0+}$ *of the* E*-function is*

$$\left(\Xi^{\eta,\theta}_{0+}\left[{}_{\tau}E^{h}_{k}(t)\right]\right)(x)=\frac{1}{(\tau+\theta+1)_{\eta}}{}_{\tau}E^{h+1}_{k+1}\left[x\mid\begin{array}{c}(\rho,a);(\gamma_i,q_i,s_i)_{1,h},(\tau+\theta+1,a,1)\\ (\alpha,\beta);\left(\delta_j,p_j,r_j\right)_{1,k},(\tau+\eta+\theta+1,a,1)\end{array}\right]. \tag{18}$$

Proof : We obtain the E-K transform $\Xi^{\eta,\theta}_{0+}$ of the E-function as follows

$$\left(\Xi^{\eta,\theta}_{0+}\left[{}_{\tau}E^{h}_{k}(t)\right]\right)(x)=\frac{x^{-\eta-\theta}}{\Gamma(\eta)}\int_0^x(x-t)^{\eta-1}t^{\theta}\sum_{n=0}^{\infty}\Phi(n)\,t^{an+\tau}dt \tag{19}$$

where

$$\Phi(n) = \frac{\left[(\gamma_1)_{q_1 n}\right]^{s_1}\left[(\gamma_2)_{q_2 n}\right]^{s_2}\ldots\left[(\gamma_h)_{q_h n}\right]^{s_h}(-1)^{\rho n}}{\left[(\delta_1)_{p_1 n}\right]^{r_1}\left[(\delta_2)_{p_2 n}\right]^{r_2}\ldots\left[(\delta_k)_{p_k n}\right]^{r_k}\Gamma(\alpha n+\beta)} \tag{20}$$

Then

$$\left(\Xi_{0+}^{\eta,\theta}\left[{}_{\tau}E_k^h(t)\right]\right)(x) = \frac{1}{(\tau+\theta+1)_\eta}\sum_{n=0}^{\infty}\Phi(n)\frac{(\tau+\theta+1)_{an}}{(\tau+\theta+\eta+1)_{an}}x^{an+\tau} \tag{21}$$

$$= \frac{1}{(\tau+\theta+1)_\eta}{}_{\tau}E_{k+1}^{h+1}\left[x \mid \begin{matrix}(\rho,a);(\gamma_i,q_i,s_i)_{1,h},(\tau+\theta+1,a,1)\\(\alpha,\beta);(\delta_j,p_j,r_j)_{1,k},(\tau+\eta+\theta+1,a,1)\end{matrix}\right]. \tag{22}$$

4.1 Special Cases

1. E-K transform $\Xi_{0+}^{\eta,\theta}$ of the M-L type function (5)

$$\left[\Xi_{0+}^{\eta,\theta}\left\{E_{(1/\rho_i),(\mu_i)}(t)\right\}\right](x)$$

$$= \frac{1}{(\theta+1)_\eta\prod_{j=1}^{m-1}\Gamma(\mu_j)}{}_0E_m^1\left[x \mid \begin{matrix}(0,1);(\theta+1,1,1)\\(1/\rho_m,\mu_m);(\mu_1,1/\rho_1,1),\ldots,(\mu_{m-1},1/\rho_{m-1},1),(\eta+\theta+1,1,1)\end{matrix}\right] \tag{23}$$

2. E-K transform $\Xi_{0+}^{\eta,\theta}$ of the M-L type function (6)

$$\left[\Xi_{0+}^{\eta,\theta}\left\{E_{\gamma,\kappa}\left[(\alpha_1,\beta_1),\ldots,(\alpha_m,\beta_m);t\right]\right\}\right](x)$$

$$= \frac{1}{(\theta+1)_\eta\prod_{j=1}^{m}\Gamma(\beta_j)}{}_0E_{m+1}^2\left[x \mid \begin{matrix}(0,1):(\gamma,\kappa,1),(\theta+1,1,1)\\(1,1);(\beta_1,\alpha_1,1),\ldots,(\beta_m,\alpha_m,1),(\eta+\theta+1,1,1)\end{matrix}\right]. \tag{24}$$

3. E-K transform $\Xi_{0+}^{\eta,\theta}$ of the M-L type function (7)

$$\left[\Xi_{0+}^{\eta,\theta}\left\{HE_{\mu_1,\ldots,\mu_\nu}^{\lambda_1,\ldots,\lambda_\nu}(t)\right\}\right](x) = \frac{1}{(M+\theta+1)_\eta\prod_{j=1}^{m}\Gamma(1+\mu_j)}\times$$

$$\times{}_ME_\nu^1\left[\frac{x}{\Lambda} \mid \begin{matrix}(1,\Lambda);(M+\theta+1,\Lambda,1)\\(\lambda_\nu,1+\mu_\nu);(1+\mu_1,\lambda_1,1),\ldots,(1+\mu_{\nu-1},\lambda_{\nu-1},1),(M+\eta+\theta+1,\Lambda,1)\end{matrix}\right]. \tag{25}$$

5. The image of M-L type *E*-function under a generalized integral operator

Theorem 5.1. *If convergence conditions (9) are satisfied also* $\eta, \theta, \sigma \in \mathbb{C}, \Re(\eta) > 0, \Re(\theta) > 0,$ $\Re(\sigma) > 0,$ *and* $t, x, v \in \mathbb{R}$ *then*

$$\int_t^x (x-s)^{\eta-1}(s-t)^{\theta-1}\, {}_\tau E_k^h\{v(s-t)^\sigma\}\, ds$$

$$= (x-t)^{\eta+\theta-1} B(\theta+\sigma\tau, \eta)\, {}_\tau E_{k+1}^{h+1}\left[v(x-t)^\sigma \mid \begin{matrix} (\rho, a); (\gamma_i, q_i, s_i)_{1,h}, (\theta+\sigma\tau, \sigma a, 1) \\ (\alpha, \beta); (\delta_j, p_j, r_j)_{1,k}, (\eta+\theta+\sigma\tau, \sigma a, 1) \end{matrix} \right]. \quad (26)$$

Corollary 5.2 *If convergence conditions (9) are satisfied also* $\eta, \theta, \sigma \in \mathbb{C}, \Re(\eta) > 0, \Re(\theta) > 0,$ $\Re(\sigma) > 0,$ *and* $x, v \in \mathbb{R}$ *then*

$$\int_0^x (x-s)^{\eta-1} s^{\theta-1}\, {}_\tau E_k^h\{vs^\sigma\}\, ds$$

$$= x^{\eta+\theta-1} B(\theta+\sigma\tau, \eta)\, {}_\tau E_{k+1}^{h+1}\left[vx^\sigma \mid \begin{matrix} (\rho, a); (\gamma_i, q_i, s_i)_{1,h}, (\theta+\sigma\tau, \sigma a, 1) \\ (\alpha, \beta); (\delta_j, p_j, r_j)_{1,k}, (\eta+\theta+\sigma\tau, \sigma a, 1) \end{matrix} \right]. \quad (27)$$

Corollary 5.3. *If convergence conditions (9) are satisfied also* $\theta, \sigma \in \mathbb{C}, \Re(\theta) > 0, \Re(\sigma) > 0,$ *and* $x, v \in \mathbb{R}$ *then*

$$\int_0^x s^{\theta-1}\, {}_\tau E_k^h(vs^\sigma)\, ds$$

$$= \left(\frac{x^\theta}{\sigma\tau+\theta}\right) {}_\tau E_{k+1}^{h+1}\left[vx^\sigma \mid \begin{matrix} (\rho, a); (\gamma_i, q_i, s_i)_{1,h}, (\theta+\sigma\tau, \sigma a, 1) \\ (\alpha, \beta); (\delta_j, p_j, r_j)_{1,k}, (\theta+\sigma\tau+1, \sigma a, 1) \end{matrix} \right]. \quad (28)$$

Proof : We prove the theorem as follows

$$\int_t^x (x-s)^{\eta-1}(s-t)^{\theta-1}\, {}_\tau E_k^h\{v(s-t)^\sigma\}\, ds$$

$$= \int_t^x (x-s)^{\eta-1}(s-t)^{\theta-1} \sum_{n=0}^{\infty} \Phi(n)\, v^{an+\tau}\{(s-t)^\sigma\}^{an+\tau}\, ds \quad (29)$$

where

$$\Phi(n) = \frac{\left[(\gamma_1)_{q_1 n}\right]^{s_1}\left[(\gamma_2)_{q_2 n}\right]^{s_2} \ldots \left[(\gamma_h)_{q_h n}\right]^{s_h} (-1)^{\rho n}}{\left[(\delta_1)_{p_1 n}\right]^{r_1}\left[(\delta_2)_{p_2 n}\right]^{r_2} \ldots \left[(\delta_k)_{p_k n}\right]^{r_k} \Gamma(\alpha n+\beta)} \quad (30)$$

Then

$$\int_t^x (x-s)^{\eta-1}(s-t)^{\theta-1}\,{}_{\tau}E_k^h\left\{v(s-t)^{\sigma}\right\}ds$$

$$=\frac{\Gamma(\eta)(x-t)^{\eta+\theta-1}}{(\tau+1)_{\theta}}\sum_{n=0}^{\infty}\Phi(n)\frac{(\theta+\sigma\tau)_{\sigma an}}{(\theta+\sigma\tau+\eta)_{\sigma an}}\left\{v(x-t)^{\sigma}\right\}^{an+\tau} \tag{31}$$

$$=(x-t)^{\eta+\theta-1}B(\theta+\sigma\tau,\eta)\,{}_{\tau}E_{k+1}^{h+1}\left[v(x-t)^{\sigma}\mid\begin{matrix}(\rho,a);(\gamma_i,q_i,s_i)_{1,h},(\theta+\sigma\tau,\sigma a,1)\\(\alpha,\beta);(\delta_j,p_j,r_j)_{1,k},(\eta+\theta+\sigma\tau,\sigma a,1)\end{matrix}\right]. \tag{32}$$

5.1 Special Cases

1. General integral transform of the M-L type function (5)

$$\int_t^x (x-s)^{\eta-1}(s-t)^{\theta-1}E_{(1/\rho_i),(\mu_i)}\left[v(s-t)^{\sigma}\right]ds=\frac{(x-t)^{\eta+\theta-1}B(\theta,\eta)}{\prod_{j=1}^{m-1}\Gamma(\mu_j)}\times$$

$$\times{}_0E_m^1\left[v(x-t)^{\sigma}\mid\begin{matrix}(0,1);(\theta,\sigma,1)\\(1/\rho_m,\mu_m);(\mu_1,1/\rho_1,1),\ldots,(\mu_{m-1},1/\rho_{m-1},1),(\eta+\theta,\sigma,1)\end{matrix}\right]. \tag{33}$$

2. General integral transform of the M-L type function (6)

$$\int_t^x (x-s)^{\eta-1}(s-t)^{\theta-1}E_{\gamma,\kappa}\left[(\alpha_1,\beta_1),\ldots,(\alpha_m,\beta_m);(s-t)\right]ds=\frac{(x-t)^{\eta+\theta-1}B(\theta,\eta)}{\prod_{j=1}^{m}\Gamma(\beta_j)}\times$$

$$\times{}_0E_{m+1}^2\left[(x-t)\mid\begin{matrix}(0,1);(\gamma,\kappa,1),(\theta,1,1)\\(1,1);(\beta_1,\alpha_1,1),\ldots,(\beta_m,\alpha_m,1),(\eta+\theta,1,1)\end{matrix}\right]. \tag{34}$$

3. General integral transform of the M-L type function (7)

$$\int_t^x (x-s)^{\eta-1}(s-t)^{\theta-1}HE_{\mu_1,\ldots,\mu_\nu}^{\lambda_1,\ldots,\lambda_\nu}\left[v(s-t)^{\sigma}\right]ds=\frac{(x-t)^{\eta+\theta-1}B(\theta+\sigma M,\eta)}{\prod_{j=1}^{\nu-1}\Gamma(1+\mu_j)}\times$$

$$\times{}_ME_{\nu}^1\left[\frac{v(x-t)^{\sigma}}{\Lambda}\mid\begin{matrix}(1,\Lambda):(\theta+\sigma M,\sigma\Lambda,1)\\(\lambda_\nu,1+\mu_\nu);(1+\mu_1,\lambda_1,1),\ldots,(1+\mu_{\nu-1},\lambda_{\nu-1},1),(\eta+\theta+\sigma M,\sigma\Lambda,1)\end{matrix}\right] \tag{35}$$

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