J. of Ramanujan Society of Mathematics and Mathematical Sciences
Vol. 10, No. 1 (2022), pp. 173-184

DOI: 10.56827/JRSMMS.2022.1001.15 **ISSN (Online): 2582-5461**

ISSN (Print): 2319-1023

EXPANDING THE LAURENT SERIES WITH ITS APPLICATIONS

Ganesh Prasad Adhikari

Central Department of Mathematics Education,
Tribhuvan University, NEPAL

E-mail : gpadhikarin@gmail.com

(**Received:** *Sep. 22, 2022* **Accepted:** *Nov. 20, 2022* **Published:** *Dec. 30, 2022)*

Abstract: In Nepal, there are many mathematics subjects taught at university level. Among them, complex analysis is the most powerful. In complex analysis, the Laurent series expansion is a well-known subject because it may be used to find the residues of complex functions around their singularities. It turns out that computing the Laurent series of a function around its singularities is an effective way to calculate the integral of the function along any closed contour around the singularities as well as the residue of the function. Learning the Laurent series concepts can be difficult, and many students struggle to develop adequate understanding, reasoning, and problem-solving skills. Therefore, this article presents multiple practical examples where the Laurent series of a function is found and then utilized to compute the integral of the function over any closed contour around the singularities of the function, based on the theory of the Laurent series.

Keywords and Phrases: Laurent series, integral, contour, applications, singularities.

2020 Mathematics Subject Classification: 65E05.

1. Introduction

The Laurent series expansion method is a vital tool in complex analysis. A Laurent series can only be used to work around a complex function's singularities. To accomplish this, we must first identify the function's singularities. Based on these singularities, we can then build a number of concentric rings, each with the same center z_0, and, in the case where the function is analytical, we can then obtain a different Laurent series of $z-z_0$ inside each ring. The Laurent series' construction is significant since it yields the function's residue via the coefficient associated with the $\frac{1}{(z-z_0)}$ term. Such a residue based on the Residue Theorem may be used to effectively calculate the integral of the function along any closed contour [1, 7]. The Laurent series has several other uses in physics and engineering in addition to producing an effective approach for integration. We seldom look at the coefficient of the $\frac{1}{(z-z_0)}$ component that appears in the outer rings of a Laurent series expansion, despite the fact that the residue of the function has been employed extensively in calculations of both complex and real integration. This study helps to discuss the relevance of this coefficient in the outer rings by offering various practical instances of the Laurent series outside of the center annulus and utilizing them to calculate the integral of the function along any closed curve outside of the center annulus [1].

Theorem 1.1. *If $f(z)$ is an analytic throughout an anular region of two concentric circles C_1 and C_1 with radius R_1, and R_2 such that $R_2 < |z-z_0| < R_1$ and C denotes any positively oriented simple closed contour around z_0 and lying in that domain, then at each z in the domain $f(z)$ has series representation $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} b_n(z-z_0)^{-n}$, where $a_n = \frac{1}{2\pi i}\int_{C_1} \frac{f(s)ds}{(s-z)^{n+1}}$, $n = 0, 1, 2, \cdots$ and $b_n = \frac{1}{2\pi i}\int_{C_2} \frac{f(s)ds}{(s-z)^{-n+1}}$, $n = 1, 2, 3, \cdots$ [1, 5].*

Proof. Let us suppose that $f(z)$ is an analytic in the closed annular region bounded by two concentric circles C_1 and C_1 with centre z_0 radius R_1, and R_2 respectively, then D is the annular region made by them. Let $|z-z_0| = R$ satisfying $R_2 < |z-z_0| < R_1$. Since, the function $f(z)$ is analytic throughout the annular region D. Hence by extension of Cauchy integral formula we have

$$f(z) = \frac{1}{2\pi i}\int_{C_1} \frac{f(s)ds}{(s-z)} - \frac{1}{2\pi i}\int_{C_2} \frac{f(s)ds}{(s-z)} \tag{1.1}$$

Now, the first integral of equation (1.1).

$$\frac{1}{s-z} = \frac{1}{(s-z_0)-(z-z_0)} = \frac{1}{(s-z_0)}\left[\frac{1}{1-\left(\frac{z-z_0}{s-z_0}\right)}\right]$$

$$= \frac{1}{s-z_0}\left[1+\left(\frac{z-z_0}{s-z_0}\right)+\left(\frac{z-z_0}{s-z_0}\right)^2+\cdots+\left(\frac{z-z_0}{s-z_0}\right)^n \frac{1}{\left\{1-\left(\frac{z-z_0}{s-z_0}\right)\right\}}\right]$$

$$\frac{1}{s-z}=\left[\frac{1}{(s-z_0)}+\frac{(z-z_0)}{(s-z_0)^2}+\frac{(z-z_0)^2}{(s-z_0)^3}+\frac{(z-z_0)^3}{(s-z_0)^4}+\cdots+\frac{(z-z_0)^n}{(s-z_0)^n}\frac{1}{(s-z)}\right] \tag{1.2}$$

Now, multiply both side of equation (1.2) by $\frac{f(s)}{2\pi i}$ and integrating along C_1, then we have

$$\frac{1}{2\pi i}\int\limits_{C_1}\frac{f(s)ds}{(s-z)}=\frac{1}{2\pi i}\int\limits_{C_1}\frac{f(s)ds}{(s-z_0)}+\frac{1}{2\pi i}\int\limits_{C_1}\frac{(z-z_0)\,f(s)ds}{(s-z_0)^2}+\cdots+R_n$$

$$where\ R_n=\frac{1}{2\pi i}\int\limits_{C_1}\frac{(z-z_0)^n\,f(s)ds}{(s-z_0)^n(s-z)} \tag{1.3}$$

Now, we have $a_n=\frac{1}{2\pi i}\int\limits_{C_1}\frac{f(s)ds}{(s-z_0)^{n+1}}$ for $n=0,1,2,3,\cdots$, then we have

$$\left.\begin{aligned} a_0 &= \tfrac{1}{2\pi i}\int\limits_{C_1}\tfrac{f(s)ds}{(s-z_0)} \\ a_1 &= \tfrac{1}{2\pi i}\int\limits_{C_1}\tfrac{f(s)ds}{(s-z_0)^2} \\ a_2 &= \tfrac{1}{2\pi i}\int\limits_{C_1}\tfrac{f(s)ds}{(s-z_0)^3} \end{aligned}\right\} \tag{1.4}$$

Now from equations (1.3) and (1.4), we have
$\frac{1}{2\pi i}\int\limits_{C_1}\frac{f(s)ds}{(s-z)}=a_0+a_1(z-z_0)+a_2(z-z_0)^2+a_3(z-z_0)^3+\cdots+a_n(z-z_0)^n+\cdots+R_n.$
Here, C_1, we have $|z-z_0|=R$,$|f(s)|\leq M$, $|s-z_0|=R_1$, $|s-z|=|s-z_0+z_0-z|\geq |s-z_0|-|z-z_0|=R_1-R.$
Therefore,

$$R_n=\frac{1}{2\pi i}\int\limits_{C_1}\frac{(z-z_0)^n}{(s-z_0)^n}\frac{f(s)ds}{(s-z)}$$

$$|R_n|=\frac{1}{2\pi}\int\limits_{C_1}\frac{|z-z_0|^n}{|s-z_0|^n}\frac{|f(s)||ds|}{|s-z|}\leq\frac{1}{2\pi}\frac{R^n}{R_1^n}\frac{M}{R_1-R}2\pi R_1$$

$$|R_n|\leq\left(\frac{R}{R_1}\right)^n\frac{MR_1}{R_1-R}\to 0\ \ as\ \ n\to\infty.$$

Thus,

$$\frac{1}{2\pi i}\int\limits_{C_1}\frac{f(s)ds}{(s-z)}=\sum_{n=0}^{\infty}a_n\,(z-z_0)^n \tag{1.5}$$

Again, for second integral of equation (1.5), then we have

$$\frac{-1}{s-z} = \frac{1}{z-s} = \frac{1}{(z-z_0)-(s-z_0)} = \frac{1}{(z-z_0)}\left[\frac{1}{1-\left(\frac{s-z_0}{z-z_0}\right)}\right]$$
$$= \frac{1}{z-z_0}\left[1+\left(\frac{s-z_0}{z-z_0}\right)+\left(\frac{s-z_0}{z-z_0}\right)^2+\cdots+\left(\frac{s-z_0}{z-z_0}\right)^n\frac{1}{\left\{1-\left(\frac{s-z_0}{z-z_0}\right)\right\}}\right]$$
$$= \left[\frac{1}{(z-z_0)}+\frac{(s-z_0)}{(z-z_0)^2}+\frac{(s-z_0)^2}{(z-z_0)^3}+\frac{(s-z_0)^3}{(z-z_0)^4}+\cdots+\frac{(s-z_0)^n}{(z-z_0)^n}\frac{1}{(z-s)}\right] \tag{1.6}$$

Multiplying both sides of equation (1.6) by $\frac{f(s)}{2\pi i}$ and integrating along C_1, then we have

$$-\frac{1}{2\pi i}\int_{C_2}\frac{f(s)ds}{(s-z)} = \frac{1}{2\pi i}\int_{C_2}\frac{f(s)ds}{(z-z_0)} + \frac{1}{2\pi i}\int_{C_2}\frac{(s-z_0)\,f(s)ds}{(z-z_0)^2} + \cdots + Q_n$$
$$where\ Q_n = \frac{1}{2\pi i}\int_{C_2}\frac{(s-z_0)^n\,f(s)ds}{(z-z_0)^n(z-s)} \tag{1.7}$$

Now, we have $b_n = \frac{1}{2\pi i}\int\limits_{C_2}\frac{f(s)ds}{(s-z_0)^{-n+1}}$ for $n = 1, 2, 3, \cdots$, then we have

$$\left.\begin{aligned} b_1 &= \frac{1}{2\pi i}\int\limits_{C_2}\frac{f(s)ds}{(s-z_0)^0} = \frac{1}{2\pi i}\int\limits_{C_2} f(s)ds \\ b_2 &= \frac{1}{2\pi i}\int\limits_{C_2}\frac{f(s)ds}{(s-z_0)^{-1}} \\ b_3 &= \frac{1}{2\pi i}\int\limits_{C_2}\frac{f(s)ds}{(s-z_0)^{-2}} \\ . &= ... \\ . &= ... \\ . &= ... \end{aligned}\right\} \tag{1.8}$$

Now, from equation (1.7), we have

$$-\frac{1}{2\pi i}\int_{C_2}\frac{f(s)ds}{(s-z)} = \frac{1}{(z-z_0)}\frac{1}{2\pi i}\int_{C_2} f(s)ds + \frac{1}{(z-z_0)^2}\frac{1}{2\pi i}\int_{C_2}\frac{f(s)ds}{(s-z)^{-1}} + \cdots + Q_n \tag{1.9}$$

Thus, from equations (1.8) and (1.9), then we have

$$-\frac{1}{2\pi i}\int\limits_{C_2}\frac{f(s)ds}{(s-z)}=\frac{b_1}{(z-z_0)}+\frac{b_2}{(z-z_0)^2}+\frac{b_3}{(z-z_0)^3}+\cdots+Q_n \tag{1.10}$$

For, C_2 :- $|z-z_0|=R$, $|s-z_0|=R_2$ and $|f(s)|\leq M$. Also, $|z-s|=|(z-z_0)-(s-z_0)|\geq|z-z_0|-|s-z_0|=R-R_2$

Therefore, $|Q_n|\leq\left(\frac{R_2}{R}\right)^n\frac{MR_2}{R-R_2}\to 0$ as $n\to\infty$

$\therefore$

$$-\frac{1}{2\pi i}\int\limits_{C_2}\frac{f(s)ds}{(s-z)}=\sum_{n=1}^{\infty}\frac{b_n}{(z-z_0)\,n} \tag{1.11}$$

Therefore, from the equations (1.1), (1.5) and (1.11), we have

$$f(z)=\sum_{n=1}^{\infty}a_n(z-z_0)^n+\sum_{n=1}^{\infty}\frac{b_n}{(z-z_0)^n} \tag{1.12}$$

Hence, $f(z)=\sum\limits_{n=1}^{\infty}a_n(z-z_0)^n+\sum\limits_{n=1}^{\infty}\frac{b_n}{(z-z_0)^n}$ is called the Laurent series of $f(z)$. The first part $\sum\limits_{n=1}^{\infty}a_n(z-z_0)^n$ is called the **Analytic part** and the second part $\sum\limits_{n=1}^{\infty}\frac{b_n}{(z-z_0)^n}$ is called the **principal part** of the function $f(z)$.

Example 1.1. Expand $f(z)=\frac{z}{(z+1)(z+2)}$ in Laurent's series with centre at $z=-2$ [1].

Solution. Since, $f(z)=\frac{z}{(z+1)(z+2)}$. Let us $z+2=u$, then $z=u-2$

Thus,

$$\begin{aligned}
f(z)&=\frac{u-2}{(u-1)\,u}\\
&=\frac{(2-u)}{u}\frac{1}{(1-u)}\\
&=\frac{(2-u)}{u}\left[1+u+u^2+u^3+\cdots\right]\\
&=\frac{2}{u}\left(1+u+u^2+u^3+\cdots\right)-\left(1+u+u^2+u^3+\cdots\right)\\
&=\frac{2}{u}+1+u+u^2+u^3+\cdots\\
&=\frac{2}{(z+2)}+1+(z+2)+(z+2)^2+(z+2)^3+\cdots
\end{aligned}$$

Hence, $f(z) = 1 + (z+2) + (z+2)^2 + (z+2)^3 + \cdots + \frac{2}{(z+2)}$, which is required Laurent's series at $z = -2$.

Example 1.2. Find the Laurent's series of $f(z) = \frac{7z-2}{z(z+1)(z-2)}$ in the region $1 < z+1 < 3$ [1, 2].

Solution: Since, $f(z) = \frac{7z-2}{z(z+1)(z-2)}$. Let $u = z+1$, then $z = u-1$

Therefore,

$$\begin{aligned}
f(u-1) &= \frac{7(u-1)-2}{u(u-1)(u-3)} = \frac{(7u-9)}{u(u-1)(u-3)} \\
&= \frac{-3}{u} + \frac{1}{(u-1)} + \frac{2}{(u-3)} \\
&= \frac{-3}{u} + \frac{1}{u\left(1-\frac{1}{u}\right)} + \frac{2}{-3\left(1-\frac{u}{3}\right)} \\
&= \frac{-3}{u} + \frac{1}{u}\left[\left(1-\frac{1}{u}\right)^{-1}\right] - \frac{2}{3}\left[\left(1-\frac{u}{3}\right)^{-1}\right] \\
&= -\left[\frac{-2}{u} + \frac{1}{u^2} + \frac{1}{u^3} + \cdots\right] - \frac{2}{3}\left[1 + \frac{u}{3} + \frac{u^2}{3^2} + \cdots\right] \\
f(z) &= \left[\frac{-2}{(z+1)} + \frac{1}{(z+1)^2} + \frac{1}{(z+1)^3} + \cdots\right] - \frac{2}{3}\left[1 + \frac{(z+1)}{3} + \frac{(z+1)^2}{3^2} + \cdots\right]
\end{aligned}$$

which is the required expansion in a Laurent's series and is valid for $1 < z+1 < 3$. Let's go on to discuss classification of singularity, which is essential to creating a Laurent series. A point z_0 is said to be singular point of an analytic function $f(z)$, if $f(z)$ is not analytic at point z_0, but it is analytic at some every point of the neighbourhood of z_0. For a complex function, singularities are not always simple to find. Singularities come in a variety of forms and categories, which we will now define.

Definition 1.1. (Isolated Singularity) *An isolated singularity of the function $f(z)$ is one that has no other singularity within a small circle surrounding the point $z = z_0$ and is expressed as* [2]:

$$f(z) = \cdots + \frac{b_n}{(z-z_0)^n} + \cdots + \frac{b_1}{(z-z_0)} + \cdots + a_0 + a_1(z-z_0) + \cdots$$

The Laurent series is convergent if and only if we can discover a laurent expansion centered at a single singularity in an annulus that omits that singularity. In a region excluding points where f is not differentiable, the Laurent expansion provides for a series representation in both negative and positive powers of $(z-z_0)$.

Example 1.3. Let $f(z) = \frac{1}{z}$, then $z = 0$ is an isolated singular point of $f(z)$. $f(z)$ is clearly an analytic in the domain $0 < |z| < \infty$.
Isolated singularities can be categorized in several ways, as follows:

Definition 1.2. (Removable Singularity) *If all b_n coefficients in the principal part of $f(z)$ are zero at that point $z = z_0$, In other words, if there are no terms in the principal parts, z_0 is referred to as the removable singular point of $f(z)$* [1].

Example 1.4. Let $f(z) = \frac{sinz}{z}$, then $f(z)$ has removable singularity at point $z = 0$.

Definition 1.3. (Essential Singularity) *If the principal part of $f(z)$ at $z = z_0$ contains an infinite number of terms, then the point z_0 is called the essential singular point of $f(z)$* [1, 4].

Example 1.5. Let $f(z) = e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2!}\left(\frac{1}{z}\right)^2 + \frac{1}{3!}\left(\frac{1}{z}\right)^3 + \cdots$, then $f(z)$ has essential singularity at point $z = 0$.

Definition 1.4. (Pole Singularity) *If the principal part of $f(z)$ at $z = z_0$ contains finite number of terms say $'m'$ such that $b_m \neq 0$, and $b_{m+1} = b_{m+1} = \cdots = 0$, then the series (1.12) becomes as $f(z) = \sum\limits_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_m}{(z-z_0)^m}$* [1,3]. *In this case, the singular point z_0 is called the pole of order m. If $m = 1$, then z_0 is the pole of simple order.*

Example 1.6. Let
$f(z) = \frac{sin(z-z_0)}{(z-z_0)^4} = \frac{1}{(z-z_0)^4}\left[(z - z_0) - \frac{(z-z_0)^3}{3!} + \frac{(z-z_0)^5}{5!} - \frac{(z-z_0)^7}{7!} + \cdots\right]$, then the point $z = z_0$ is a pole of order 3 of $f(z)$.
We will concentrate on the Laurent series' main application: determining the residue of a function. While some complex functions have useful formulas for calculating the residue, it is primarily dependent on the type of singularity you are dealing with.

Definition 1.5. (Residue of a function f(z)) *Let z_0 be an isolated singularity of a function $f(z)$, then there is a positive number R_2 such that $f(z)$ is analytic at all points z for which $0 < |z - z_0| < R_1$. Then $f(z)$ has Laurent series of expansion [1, 5]*

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} b_n(z - z_0)^{-n}$$
$$= \sum_{n=0}^{\infty} a_n(z - z_0)^n + \frac{b_1}{(z - z_0)} + \frac{b_2}{(z - z_0)^2} + \frac{b_1}{(z - z_0)^2} + \cdots$$

$$where\ \ a_n = \frac{1}{2\pi i}\int\limits_C \frac{f(z)dz}{(z-z_0)^{n+1}},\ \ n = 0, 1, 2, \cdots$$

$$and\ \ b_n = \frac{1}{2\pi i}\int\limits_C \frac{f(z)dz}{(z-z_0)^{-n+1}},\ \ n = 1, 2, 3, \cdots$$

where C *is any positively oriented closed contour arround* z_0 *lying in the domain* $0 < |z - z_0| < R_2$. *When* $n = 1$, *then* $b_1 = \frac{1}{2\pi i}\int\limits_C f(z)dz \Rightarrow \int\limits_C f(z)dz = 2\pi i \times b_1$, *where* b_1 *is coefficient of* $\frac{1}{(z-z_0)}$. *This complex number* b_1 *called residue of* $f(z)$ *at* $z = z_0$. *It can be expressed as* $b_1 = \underset{z=z_0}{Res} f(z)$.*It can assist us in computing the integral of* $f(z)$ *along any closed contour situated inside those annuli.*

Theorem 1.2. *Let* $f(z)$ *be analytic inside and on simple closed contour* C *except at finite number of singularities* $z_1, z_2, \cdots, z_n$ *inside* C *at which the residual are* $\beta_1, \beta_2, \beta_3, \cdots, \beta_n$ *respectively of* $f(z)$, *then* $\int\limits_C f(z)dz = 2\pi i \times \sum\limits_{k=1}^{n} \beta_k$ [1, 5, 3].

Proof. Let $C_1, C_2, C_3, \cdots, C_n$ be the circle with centre at $z_1, z_2, z_3, \cdots, z_n$ respectively and radii so small such that they lie entirely within positively oriented simple closed contour C and having no common point, then by extension of Cauchy Goursat's theorem we have

$$\int\limits_C f(z)dz - \sum_{k=1}^{n}\int\limits_{C_k} f(z)dz = 0$$

$$\Rightarrow \int\limits_C f(z)dz = \sum_{k=1}^{n}\int\limits_{C_k} f(z)dz \tag{1.13}$$

But, by definition of residue, we have

$$\int\limits_{C_k} f(z)dz = 2\pi i \times \underset{z=z_0}{Res} f(z)\ \ for\ \ k = 1, 2, \cdots, n. \tag{1.14}$$

Therefore, from equations (1.13) and (1.14), we have

$$\int\limits_C f(z)dz = \sum_{k=1}^{n}\int\limits_{C_k} f(z)dz = 2\pi i \sum_{k=1}^{n} \underset{z=z_k}{Res} f(z).$$

If we suppose $\underset{z=z_k}{Res} f(z) = \beta_k$, for $k = 1, 2, 3, \cdots, n$. Thus, we have $\int\limits_C f(z)dz = 2\pi i \sum\limits_{k=1}^{n} \beta_k for\ k = 1, 2, 3, \cdots, n.$

Example 1.7. Let us find the Laurent series of $f(z) = \frac{sinz}{z^2}$ at $z = 0$ and find $\int\limits_C \frac{sinz}{z^2} dz$ [1, 4].

Solution. Since, $f(z) = \frac{sinz}{z^2} = \frac{1}{z^2}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \cdots\right] = \frac{1}{z} - \frac{z}{3!} + \frac{z^3}{5!} - \frac{z^5}{7!} + \cdots$, which is required Laurent series at $z = 0$.
Now, the residue of $f(z)$ at $z = 0$ is the coefficient of $\frac{1}{(z-0)}$.
Therefore, $b_1 = \underset{z=0}{Res} f(z) = 1$. As a result of the Cauchy Residue theorem, we get $\int\limits_C \frac{sinz}{z^2} dz = 2\pi i \times \underset{z=0}{Res} f(z) = 2\pi i \times 1$. Hence, $\int\limits_C \frac{sinz}{z^2} dz = 2\pi i$.

Example 1.8. Find the Laurent's series of $f(z) = \frac{z}{(z-1)(2-z)}$ in the regions (a) $|z-1| > 1$ and (b) $0 < |z-2| < 1$. Also, find $\int\limits_{c_1} \frac{z}{(z-1)(2-z)} dz$,where $C_1 : |z-1| > 1$ and $\int\limits_{c_2} \frac{z}{(z-1)(2-z)}$,where $C_2 : 0 < |z-1| < 1$ [3].

Solution. Since, $f(z) = \frac{z}{(z-1)(2-z)}$ for (a) $C_1 : |z-1| > 1$. Let us suppose $u = z - 1$, then $z = u + 1$. Therefore, $f(u+1) = \frac{u+1}{(u+1-1)(2-u-1)} = \frac{u+1}{u(1-u)} = \frac{-(u+1)}{u^2\left(1-\frac{1}{u}\right)} = \frac{-(u+1)}{u^2}\left(1 - \frac{1}{u}\right)^{-1}$.

$$\Rightarrow f(u+1) = \frac{-(u+1)}{u^2}\left[1 + \frac{1}{u} + \left(\frac{1}{u}\right)^2 + \left(\frac{1}{u}\right)^3 + \cdots\right]$$
$$= \frac{-1}{u}\left[1 + \frac{1}{u} + \left(\frac{1}{u}\right)^2 + \left(\frac{1}{u}\right)^3 + \cdots\right] - \frac{1}{u^2}\left[1 + \frac{1}{u} + \left(\frac{1}{u}\right)^2 + \left(\frac{1}{u}\right)^3 + \cdots\right]$$
$$= \left[-\frac{1}{u} - \frac{1}{u^2} - \frac{2}{u^3} - \cdots\right]$$
$$\Rightarrow f(z) = \left[\frac{-1}{(z-1)} - \frac{1}{(z-1)^2} - \frac{2}{(z-1)^3} - \frac{2}{(z-1)^4} - \cdots\right],$$

which is the required expansion in a Laurent's series and valid for $C_1 : |z-1| > 1$.
Now, the residue of $f(z)$ at $z = 1$ is the coefficient of $\frac{1}{(z-1)}$.
Therefore, $b_1 = \underset{z=1}{Res} f(z) = -1$.As a result of the Cauchy Residue theorem, we get $\int\limits_{C_1} \frac{z}{(z-1)(2-z)} dz = 2\pi i \times \underset{z=1}{Res} f(z) = 2\pi i \times -1$. Hence, $\int\limits_{C_1} \frac{z}{(z-1)(2-z)} dz = -2\pi i$.
For (b) $C_2 : 0 < |z-2| < 1$. Let $u = z - 2$, then $z = u + 2$. Since, $f(z) = \frac{z}{(z-1)(2-z)}$.

$$\Rightarrow f(u+2) = \frac{u+2}{(u+1)(-u)} = -\frac{(u+2)}{u(1+u)} = -\frac{(u+2)}{u}(1+u)^{-1}$$

$$= -\left(1 + \frac{2}{u}\right)\left[1 - u + u^2 - u^3 + u^4 - \cdots\right]$$

$$= -1\left[1 - u + u^2 - u^3 + u^4 - \cdots\right] - \frac{2}{u}\left[1 - u + u^2 - u^3 + u^4 - \cdots\right]$$

$$= \frac{-2}{u} + 1 - u + u^2 - u^3 + \cdots$$

$$\Rightarrow f(z) = \frac{-2}{(z-2)} + 1 - (z-2) + (z+2)^2 - (z-2)^3 + \cdots ,$$

which is the required expansion in a Laurent's series and valid for $0 < |z-2| < 1$.
Now, the residue of $f(z)$ at $z = 2$ is the coefficient of $\frac{1}{(z-2)}$.
Therefore, $b_1 = \underset{z=2}{Res} f(z) = -2$. As a result of the Cauchy Residue theorem, we get $\int\limits_{C_2} \frac{z}{(z-1)(2-z)} dz = 2\pi i \times \underset{z=2}{Res} f(z) = 2\pi i \times -2$. Hence, $\int\limits_{C_2} \frac{z}{(z-1)(2-z)} dz = -4\pi i$.

Example 1.9. Evaluate $\int\limits_{C} \frac{1-2z}{z(z-1)(2-z)} dz$,where $C : |z| = 1.5$ [1].

Solution. Since, the function is $\int\limits_{C} \frac{1-2z}{z(z-1)(2-z)} dz$, then the poles of $f(z)$ are given by
$z(z-1)(2-z) = 0$ *i.e.* $z = 0, 1,$ *and* 2 are simple poles where 0 and 1 lies inside the circle $C : |z| = 1.5$. Therefor, the residue of $z = 0$ of $f(z)$ is

$$\beta_1 = \underset{z=0}{Res} f(z) = \lim_{z\to 0} z \frac{(1-2z)}{z(z-1)(z-2)}$$

$$= \lim_{z\to 0} \frac{(1-2z)}{(z-1)(z-2)} = \frac{1}{2}$$

$$Thus,\ \beta_1 = \underset{z=0}{Res} f(z) = \frac{1}{2}$$

Similarly, $\beta_2 = \underset{z=1}{Res} f(z) = 1$.
Now, by Cauchy Residue theorem, we have

$$\int\limits_{C} f(z) dz = \int\limits_{C} \frac{1-2z}{z(z-1)(2-z)} dz = 2\pi i \left(\frac{1}{2} + 1\right) = 3\pi i$$

2. Conclusion

Complex analysis and its applications are fundamentally based on the Laurent series. It is possible to quickly locate the residue of some functions using well-known formulae, however this is not always the case. For instance, the Laurent expansion is the sole means to ascertain the function residue at a singularity for a function with an essential singularity for which no such simple formula exists. Complex integration, which has applications outside of pure mathematics, will be substantially facilitated by the residues uncovered by the Laurent series [1].

References

[1] Adhikari G. P., Complex analysis, Dikshanta Prakashan, Kirtipur, Kathmandu, Nepal, 2021.

[2] Ahlfors, L., Complex analysis : An introduction to the theory of analytic functions of one complex variable, McGraw-Hill, 1979.

[3] Alpay, D., An advanced complex analysis problem, Springer International Publishing, New York, 2015.

[4] Bulthee, A., Laurent series and their pade approximetions, Birkhauser Verlag Baset Boston, Beljeum, 1987.

[5] Chatterji, P. N., Complex analysis, Rajhans Agencies, Dharma- Alok, Ram Nagar, Meerut U.P., India, 2013.

[6] Kasana, H. S., Complex analysis theory and applications, PHL Learning Private Limited, New Delhi, India, 2012.

[7] Marsden, J. E., and Hoffiman, J. E., Basic complex analysis, W. H. Freeman and Company, New Work, NY, 2001.