J. of Ramanujan Society of Mathematics and Mathematical Sciences
Vol. 10, No. 1 (2022), pp. 139-152

DOI: 10.56827/JRSMMS.2022.1001.12 **ISSN (Online): 2582-5461**

ISSN (Print): 2319-1023

β-PREREGULAR SPACE IN FUZZY SETTING

Anjana Bhattacharyya

Department of Mathematics,
Victoria Institution (College),
78 B, A.P.C. Road, Kolkata - 700009, INDIA

E-mail : anjanabhattacharyya@hotmail.com

(**Received:** *Aug. 28, 2022* **Accepted:** *Nov. 21, 2022* **Published:** *Dec. 30, 2022)*

Abstract: In this paper a new type of fuzzy regular space is introduced and studied by introducing fuzzy β-preopen set, the class of which is strictly larger than that of fuzzy open set as well as fuzzy preopen set. Here we also introduce two new types of functions, viz., fuzzy β-precontinuous and fuzzy β-preirresolute functions. Lastly, the application of fuzzy β-precontinuous function on fuzzy β-preregular space is shown here.

Keywords and Phrases: Fuzzy β-preopen set, fuzzy β-preregular space, fuzzy β-precontinuous function, fuzzy β-preirresolute function, fuzzy β-preopen q-nbd.

2020 Mathematics Subject Classification: 54A40, 03E72.

1. Introduction

In [4], fuzzy open set is introduced. Afterwards, different types of fuzzy open-like sets are introduced by many mathematicians. Here we first introduce fuzzy β-preopen set, the class of which is strictly larger than that of fuzzy open sets and fuzzy preopen sets [8]. Using this concept as a basic tool, we introduce fuzzy β-precontinous function, the class of which is strictly larger than that of fuzzy continuous function [4]. Afterwards, we introduce fuzzy β-preregular space in which fuzzy closed set and fuzzy β-preclosed sets coincide.

2. Preliminary

Throughout this paper, (X, τ) or simply by X we shall mean a fuzzy topological space. A fuzzy set A is a function from a non-empty set X into the closed interval

$I = [0, 1]$, i.e., $A \in I^X$ [11]. The support [11] of a fuzzy set A, denoted by $suppA$ or A_0 and is defined by $suppA = \{x \in X : A(x) \neq 0\}$. The fuzzy set with the singleton support $\{x\} \subseteq X$ and the value t $(0 < t \leq 1)$ will be denoted by x_t. 0_X and 1_X are the constant fuzzy sets taking values 0 and 1 respectively in X. The complement [11] of a fuzzy set A in an fts X is denoted by $1_X \setminus A$ and is defined by $(1_X \setminus A)(x) = 1 - A(x)$, for each $x \in X$. For any two fuzzy sets A, B in X, $A \leq B$ means $A(x) \leq B(x)$, for all $x \in X$ [11] while AqB means A is quasi-coincident (q-coincident, for short) [9] with B, i.e., there exists $x \in X$ such that $A(x) + B(x) > 1$. The negation of these two statements will be denoted by $A \not\leq B$ and $A \not{q} B$ respectively. For a fuzzy set A, clA and $intA$ will stand for fuzzy closure [4] and fuzzy interior [4] respectively. A fuzzy set A in X is called a fuzzy neighbourhood (nbd, for short) [9] of a fuzzy point x_t if there exists a fuzzy open set G in X such that $x_t \in G \leq A$. If, in addition, A is fuzzy open, then A is called fuzzy open nbd of x_t. A fuzzy set A is said to be a fuzzy quasi neighbourhood (q-nbd for short) of a fuzzy point x_t in an fts X if there is a fuzzy open set U in X such that $x_t q U \leq A$. If, in addition, A is fuzzy open, then A is called a fuzzy open q-nbd of x_t [9].

A fuzzy set A in an fts (X, τ) is called fuzzy β-open [6] (resp., fuzzy preopen [8]) if $A \leq cl(int(clA))$ (resp., $A \leq int(clA)$). The complement of a fuzzy β-open set is called fuzzy β-closed [6]. The union (intersection) of all fuzzy β-open (resp., fuzzy β-closed) sets contained in (resp., containing) a fuzzy set A is called fuzzy β-interior [6] (resp., fuzzy β-closure [6]) of A, denoted by $\beta intA$ (resp., βclA). A fuzzy set A in an fts X is called fuzzy β-neighbourhood (fuzzy β-nbd, for short) of a fuzzy point x_α in X if there exists a fuzzy β-open set U in X such that $x_\alpha q U \leq A$ [6].

3. Fuzzy β-Preopen Set : Some Properties

In this section, we introduce a new class of fuzzy sets which is strictly larger than that of fuzzy open set as well as fuzzy preopen set. Afterwards, we introduce a new type of closure-like operator which is an idempotent operator.

Definition 3.1. *A fuzzy set A in an fts (X, τ) is called fuzzy β-preopen if $A \leq \beta int(clA)$. The complement of this set is called fuzzy β-preclosed set.*

The collection of fuzzy β-preopen (resp., fuzzy β-preclosed) sets in (X, τ) is denoted by $F\beta PO(X)$ (resp., $F\beta PC(X)$).

The union (resp., intersection) of all fuzzy β-preopen (resp., fuzzy β-preclosed) sets contained in (containing) a fuzzy set A is called fuzzy β-preinterior (resp., fuzzy β-preclosure) of A, denoted by $\beta pintA$ (resp., $\beta pclA$).

Result 3.2. *Union of two fuzzy β-preopen sets in an fts X is also so.*

Proof. Let $A, B \in F\beta PO(X)$. Then $A \leq \beta int(clA), B \leq \beta int(clB)$. Now $\beta int(cl(A \bigvee B)) = \beta int(clA \bigvee clB) \geq \beta int(clA) \bigvee \beta int(clB) \geq A \bigvee B$.

Remark 3.3. *Intersection of two fuzzy β-preopen sets need not be so, follows from the next example.*

Example 3.4. Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A\}$ where $A(a) = 0.5, A(b) = 0.6$. Then (X, τ) is an fts. Here we consider two fuzzy sets U, V defined by $U(a) = 0.6, U(b) = 0.3, V(a) = V(b) = 0.5$. Then clearly $U, V \in F\beta PO(X)$. Let $W = U \bigwedge V$. Then $W(a) = 0.5, W(b) = 0.3$. Now $\beta int(clW) = \beta int(1_X \setminus A) = 0_X \not\geq W \Rightarrow W \notin F\beta PO(X)$.

Note 3.5. *So we can conclude that the set of all fuzzy β-preopen sets in an fts do not form a fuzzy topology.*

Remark 3.6. *Fuzzy open and fuzzy preopen sets are fuzzy β-preopen, but not conversely follows from the next example.*

Example 3.7. Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A\}$ where $A(a) = 0.5, A(b) = 0.45$. Then (X, τ) is an fts. Consider the fuzzy set B defined by $B(a) = B(b) = 0.5$. Then $\beta int(clB) = 1_X \setminus A \geq B \Rightarrow B \in F\beta PO(X)$. But $B \notin \tau$, $int(clB) = A \not\geq B \Rightarrow B$ is not fuzzy preopen set in (X, τ).
Now we introduce a new type of fuzzy neighbourhood of a fuzzy point, the class of which is strictly greater than that of fuzzy neighbourhood.

Definition 3.8. *A fuzzy set A in an fts (X, τ) is called fuzzy β-pre neighbourhood (fuzzy β-pre nbd, for short) of a fuzzy point x_α if there exists a fuzzy β-preopen set U in X such that $x_\alpha \leq U \leq A$. If, in addition, A is fuzzy β-preopen, then A is called fuzzy β-preopen nbd of x_α.*

Definition 3.9. *A fuzzy set A in an fts (X, τ) is called fuzzy β-pre quasi neighbourhood (fuzzy β-pre q-nbd, for short) of a fuzzy point x_α if there exists a fuzzy β-preopen set U in X such that $x_\alpha q U \leq A$. If, in addition, A is fuzzy β-preopen, then A is called fuzzy β-preopen q-nbd of x_α.*

Remark 3.10. *Since a fuzzy open set is fuzzy β-preopen, we can conclude that*
(i) fuzzy nbd (resp., fuzzy open nbd) of a fuzzy point is a fuzzy β-pre nbd (resp., fuzzy β-preopen nbd) of x_α,
(ii) fuzzy q-nbd (resp., fuzzy open q-nbd) of a fuzzy point x_α is a fuzzy β-pre q-nbd (resp., fuzzy β-preopen q-nbd) of x_α.
But the reverse implications are not necessarily true follow from the following examples.

Example 3.11. Consider Example 3.4 and a fuzzy point $a_{0.6}$. Then $B \in F\beta PO(X)$ with $a_{0.6} \in B$, but there does not exist any fuzzy open set U in X containing $a_{0.6}$ such that $a_{0.6} \in U \leq B$.

Example 3.12. Consider Example 3.4 and a fuzzy point $a_{0.41}$. Then $B \in F\beta PO(X)$ with $a_{0.41}qB$, but there does not exist any fuzzy open set U in X with $a_{0.41}qU \leq B$.

Theorem 3.13. *For any fuzzy set A in an fts (X, τ), $x_\alpha \in \beta pclA$ iff every fuzzy β-preopen q-nbd U of x_α, UqA.*
Proof. Let $x_\alpha \in \beta pclA$ for any fuzzy set A in an fts (X, τ). Let $U \in F\beta PO(X)$ with $x_\alpha qU$. Then $U(x) + \alpha > 1 \Rightarrow x_\alpha \notin 1_X \setminus U \in F\beta PC(X)$. Then by definition, $A \not\leq 1_X \setminus U \Rightarrow$ there exists $y \in X$ such that $A(y) > 1 - U(y) \Rightarrow A(y) + U(y) > 1 \Rightarrow UqA$.
Conversely, let the given condition hold. Let $U \in F\beta PC(X)$ with $A \leq U$ ... (1). We have to show that $x_\alpha \in U$, i.e., $U(x) \geq \alpha$. If possible, let $U(x) < \alpha$. Then $1 - U(x) > 1 - \alpha \Rightarrow x_\alpha q(1_X \setminus U)$ where $1_X \setminus U \in F\beta PO(X)$. By hypothesis, $(1_X \setminus U)qA \Rightarrow$ there exists $y \in X$ such that $1 - U(y) + A(y) > 1 \Rightarrow A(y) > U(y)$, contradicts (1).

Theorem 3.14. *$\beta pcl(\beta pclA) = \beta pclA$ for any fuzzy set A in an fts (X, τ).*
Proof. Let $A \in I^X$. Then $A \leq \beta pclA \Rightarrow \beta pclA \leq \beta pcl(\beta pclA)$ ... (1).
Conversely, let $x_\alpha \in \beta pcl(\beta pclA)$. If possible, let $x_\alpha \notin \beta pclA$. Then there exists $U \in F\beta PO(X)$,

$$x_\alpha qU, U \not{q} A ...(2)$$

But as $x_\alpha \in \beta pcl(\beta pclA)$, $Uq(\beta pclA) \Rightarrow$ there exists $y \in X$ such that $U(y) + (\beta pclA)(y) > 1 \Rightarrow U(y) + t > 1$ where $t = (\beta pclA)(y)$. Then $y_t \in \beta pclA$ and $y_t qU$ where $U \in F\beta PO(X)$. Then by definition, UqA, contradicts (2). So

$$\beta pcl(\beta pclA) \leq \beta pclA ...(3)$$

Combining (1) and (3), we get the result.

4. Fuzzy β-Precontinuous Function: Some Characterizations

In this section we introduce and characterize a new type of function, i.e., fuzzy β-precontinuous function, the class of which is strictly larger than that of fuzzy continuous function [4] and fuzzy almost continuous function [7].

Definition 4.1. *A function $f : X \to Y$ is said to be fuzzy β-precontinuous if for each fuzzy point x_α in X and every fuzzy nbd V of $f(x_\alpha)$ in Y, $cl(f^{-1}(V))$ is a fuzzy β-nbd of x_α in X.*

Theorem 4.2. *For a function $f : X \to Y$, the following statements are equivalent:*
(a) f is fuzzy β-precontinuous,
(b) $f^{-1}(B) \leq \beta int(cl(f^{-1}(B)))$, for all fuzzy open set B of Y,
(c) $f(\beta clA) \leq cl(f(A))$, for all fuzzy open set A in X.
Proof. (a) $\Rightarrow$ (b). Let B be any fuzzy open set in Y and $x_\alpha \in f^{-1}(B)$. Then $f(x_\alpha) \in B \Rightarrow B$ is a fuzzy nbd of $f(x_\alpha)$ in Y. By (a), $cl(f^{-1}(B))$ is a fuzzy β-nbd of x_α in X. So $x_\alpha \in \beta int(cl(f^{-1}(B)))$. Since x_α is taken arbitrarily, $f^{-1}(B) \leq \beta int(cl(f^{-1}(B)))$.
(b) $\Rightarrow$ (a). Let x_α be a fuzzy point in X and B be a fuzzy nbd of $f(x_\alpha)$ in Y. Then $x_\alpha \in f^{-1}(B) \leq \beta int(cl(f^{-1}(B)))$ (by (b)) $\leq cl(f^{-1}(B))$. So $cl(f^{-1}(B))$ is a fuzzy β-nbd of x_α in X.
(b) $\Rightarrow$ (c). Let A be a fuzzy open set in X. Then $1_Y \setminus cl(f(A))$ is a fuzzy open set in Y. By (b), $f^{-1}(1_Y \setminus cl(f(A))) \leq \beta int(cl(f^{-1}(1_Y \setminus cl(f(A))))) = \beta int(cl(1_X \setminus f^{-1}(cl(f(A))))) \leq \beta int(cl(1_X \setminus f^{-1}(f(A)))) \leq \beta int(cl(1_X \setminus A)) = \beta int(1_X \setminus A) = 1_X \setminus \beta clA$. Then $\beta clA \leq 1_X \setminus f^{-1}(1_Y \setminus cl(f(A))) = f^{-1}(cl(f(A)))$. So $f(\beta clA) \leq cl(f(A))$.
(c) $\Rightarrow$ (b). Let B be any fuzzy open set in Y. Then $int(f^{-1}(1_Y \setminus B))$ is a fuzzy open set in X. By (c), $f(\beta cl(int(f^{-1}(1_Y \setminus B)))) \leq cl(f(int(f^{-1}(1_Y \setminus B)))) \leq cl(f(f^{-1}(1_Y \setminus B))) \leq cl(1_Y \setminus B) = 1_Y \setminus B \Rightarrow B \leq 1_Y \setminus f(\beta cl(int(f^{-1}(1_Y \setminus B))))$. Then $f^{-1}(B) \leq f^{-1}(1_Y \setminus f(\beta cl(int(f^{-1}(1_Y \setminus B))))) = 1_X \setminus f^{-1}(f(\beta cl(int(f^{-1}(1_Y \setminus B))))) \leq 1_X \setminus \beta cl(int(f^{-1}(1_Y \setminus B))) = 1_X \setminus \beta cl(int(1_X \setminus f^{-1}(B))) = \beta int(cl(f^{-1}(B)))$.

Note 4.3. *It is clear from above theorem that the inverse image under fuzzy β-precontinuous function of any fuzzy open set is fuzzy β-preopen.*

Theorem 4.4. *For a function $f : X \to Y$, the following statements are equivalent:*
(a) f is fuzzy β-precontinuous,
(b) $f^{-1}(B) \leq \beta int(cl(f^{-1}(B)))$, for all fuzzy open set B of Y,
(c) for each fuzzy point x_α in X and each fuzzy open nbd V of $f(x_\alpha)$ in Y, there exists $U \in F\beta PO(X)$ containing x_α such that $f(U) \leq V$,
(d) $f^{-1}(F) \in F\beta PC(X)$, for all fuzzy closed sets F in Y,
(e) for each fuzzy point x_α in X, the inverse image under f of every fuzzy nbd of $f(x_\alpha)$ is a fuzzy β-pre nbd of x_α in X,
(f) $f(\beta pclA) \leq cl(f(A))$, for all fuzzy set A in X,
(g) $\beta pcl(f^{-1}(B)) \leq f^{-1}(clB)$, for all fuzzy set B in Y,
(h) $f^{-1}(intB) \leq \beta pint(f^{-1}(B))$, for all fuzzy set B in Y,
(i) for every basic open fuzzy set V in Y, $f^{-1}(V) \in F\beta PO(X)$.
Proof. (a) $\Leftrightarrow$ (b). Follows from Theorem 4.2 (a) $\Leftrightarrow$ (b).
(b) $\Rightarrow$ (c). Let x_α be a fuzzy point in X and V be a fuzzy open nbd of $f(x_\alpha)$ in

Y. By (b), $f^{-1}(V) \leq \beta int(cl(f^{-1}(V)))$ ... (1). Now $f(x_\alpha) \in V \Rightarrow x_\alpha \in f^{-1}(V)$ $(= U$, say). Then $x_\alpha \in U$ and by (1), $U(= f^{-1}(V)) \in F\beta PO(X)$ and $f(U) = f(f^{-1}(V)) \leq V$.

(c) $\Rightarrow$ (b). Let V be a fuzzy open set in Y and let $x_\alpha \in f^{-1}(V)$. Then $f(x_\alpha) \in V \Rightarrow V$ is a fuzzy open nbd of $f(x_\alpha)$ in Y. By (c), there exists $U \in F\beta PO(X)$ containing x_α such that $f(U) \leq V$. Then $x_\alpha \in U \leq f^{-1}(V)$. Now $U \leq \beta int(clU)$. Then $U \leq \beta int(clU) \leq \beta int(cl(f^{-1}(V))) \Rightarrow x_\alpha \in U \leq \beta int(cl(f^{-1}(V)))$. Since x_α is taken arbitrarily, $f^{-1}(V) \leq \beta int(cl(f^{-1}(V)))$.

(b) $\Leftrightarrow$ (d). Obvious.

(b) $\Rightarrow$ (e). Let W be a fuzzy nbd of $f(x_\alpha)$ in Y. Then there exists a fuzzy open set V in Y such that $f(x_\alpha) \in V \leq W \Rightarrow V$ is a fuzzy open nbd of $f(x_\alpha)$ in Y. Then by (b), $f^{-1}(V) \in F\beta PO(X)$ and $x_\alpha \in f^{-1}(V) \leq f^{-1}(W) \Rightarrow f^{-1}(W)$ is a fuzzy β-pre nbd of x_α in X.

(e) $\Rightarrow$ (b). Let V be a fuzzy open set in Y and $x_\alpha \in f^{-1}(V)$. Then $f(x_\alpha) \in V$. Then V is a fuzzy open nbd of $f(x_\alpha)$ in Y. By (e), there exists $U \in F\beta PO(X)$ containing x_α such that $U \leq f^{-1}(V) \Rightarrow x_\alpha \in U \leq \beta int(clU) \leq \beta int(cl(f^{-1}(V)))$. Since x_α is taken arbitrarily, $f^{-1}(V) \leq \beta int(cl(f^{-1}(V)))$.

(d) $\Rightarrow$ (f). Let $A \in I^X$. Then $cl(f(A))$ is a fuzzy closed set in Y. By (d), $f^{-1}(cl(f(A))) \in F\beta PC(X)$ containing A. Therefore, $\beta pclA \leq \beta pcl(f^{-1}(cl(f(A)))) = f^{-1}(cl(f(A))) \Rightarrow f(\beta pclA) \leq cl(f(A))$.

(f) $\Rightarrow$ (d). Let B be a fuzzy closed set in Y. Then $f^{-1}(B) \in I^X$. By (f), $f(\beta pcl(f^{-1}(B))) \leq cl(f(f^{-1}(B))) \leq clB = B \Rightarrow \beta pcl(f^{-1}(B)) \leq f^{-1}(B) \Rightarrow f^{-1}(B) \in F\beta PC(X)$.

(f) $\Rightarrow$ (g). Let $B \in I^Y$. Then $f^{-1}(B) \in I^X$. By (f), $f(\beta pcl(f^{-1}(B))) \leq cl(f(f^{-1}(B))) \leq clB \Rightarrow \beta pcl(f^{-1}(B)) \leq f^{-1}(clB)$.

(g) $\Rightarrow$ (f). Let $A \in I^X$. Let $B = f(A)$. Then $B \in I^Y$. By (g), $\beta pcl(f^{-1}(f(A))) \leq f^{-1}(cl(f(A))) \Rightarrow \beta pclA \leq f^{-1}(cl(f(A))) \Rightarrow f(\beta pclA) \leq cl(f(A))$.

(b) $\Rightarrow$ (h). Let $B \in I^Y$. Then $intB$ is a fuzzy open set in Y. By (b), $f^{-1}(intB) \leq \beta int(cl(f^{-1}(intB))) \Rightarrow f^{-1}(intB) \in F\beta PO(X) \Rightarrow f^{-1}(intB) = \beta pint(f^{-1}(intB)) \leq \beta pint(f^{-1}(B))$.

(h) $\Rightarrow$ (b). Let A be any fuzzy open set in Y. Then $f^{-1}(A) = f^{-1}(intA) \leq \beta pint(f^{-1}(A))$ (by (h)) $\Rightarrow f^{-1}(A) \in F\beta PO(X)$.

(b) $\Rightarrow$ (i). Obvious.

(i) $\Rightarrow$ (b). Let W be any fuzzy open set in Y. Then there exists a collection $\{W_\alpha : \alpha \in \Lambda\}$ of fuzzy basic open sets in Y such that $W = \bigvee_{\alpha \in \Lambda} W_\alpha$. Now $f^{-1}(W) = f^{-1}(\bigvee_{\alpha \in \Lambda} W_\alpha) = \bigvee_{\alpha \in \Lambda} f^{-1}(W_\alpha) \in F\beta PO(X)$ (by (i) and by Result 3.2).

Hence (b) follows.

Theorem 4.5. *A function $f : X \to Y$ is fuzzy β-precontinuous if and only if for each fuzzy point x_α in X and each fuzzy open q-nbd V of $f(x_\alpha)$ in Y, there exists a fuzzy β-pre q-nbd W in X such that $f(W) \leq V$.*

Proof. Let f be fuzzy β-precontinuous function and x_α be a fuzzy point in X and V be a fuzzy open q-nbd of $f(x_\alpha)$ in Y. Then $f(x_\alpha)qV$. Let $f(x) = y$. Then $V(y) + \alpha > 1 \Rightarrow V(y) > 1 - \alpha \Rightarrow V(y) > \beta > 1 - \alpha$, for some real number β. Then V is a fuzzy open nbd of y_β. By Theorem 4.4 (a)$\Rightarrow$(c), there exists $W \in F\beta PO(X)$ containing x_β, i.e., $W(x) \geq \beta$ such that $f(W) \leq V$. Then $W(x) \geq \beta > 1 - \alpha \Rightarrow x_\alpha qW$ and $f(W) \leq V$.

Conversely, let the given condition hold and let V be a fuzzy open set in Y. Put $W = f^{-1}(V)$. If $W = 0_X$, then we are done. Suppose $W \neq 0_X$. Then for any $x \in W_0$, let $y = f(x)$. Then $W(x) = [f^{-1}(V)](x) = V(f(x)) = V(y)$. Let us choose $m \in \mathcal{N}$ where $\mathcal{N}$ is the set of all natural numbers such that $1/m \leq W(x)$. Put $\alpha_n = 1 + 1/n - W(x)$, for all $n \in \mathcal{N}$. Then for $n \in \mathcal{N}$ and $n \geq m$, $1/n \leq 1/m \Rightarrow 1 + 1/n \leq 1 + 1/m \Rightarrow \alpha_n = 1 + 1/n - W(x) \leq 1 + 1/m - W(x) \leq 1$. Again $\alpha_n > 0$, for all $n \in \mathcal{N} \Rightarrow 0 < \alpha_n \leq 1$ so that $V(y) + \alpha_n > 1 \Rightarrow y_{\alpha_n} qV \Rightarrow V$ is a fuzzy open q-nbd of y_{α_n}. By the given condition, there exists $U_n^x \in F\beta PO(X)$ such that $x_{\alpha_n} qU_n^x$ and $f(U_n^x) \leq V$, for all $n \geq m$. Let $U^x = \bigvee\{U_n^x : n \in \mathcal{N}, n \geq m\}$. Then $U^x \in F\beta PO(X)$ (by Result 3.2) and $f(U^x) \leq V$. Again $n \geq m \Rightarrow U_n^x(x) + \alpha_n > 1 \Rightarrow U_n^x(x) + 1 + 1/n - W(x) > 1 \Rightarrow U_n^x(x) > W(x) - 1/n \Rightarrow U_n^x(x) \geq W(x)$, for each $x \in W_0$. Then $W \leq U_n^x$, for all $n \geq m$ and for all $x \in W_0 \Rightarrow W \leq U^x$, for all $x \in W_0 \Rightarrow W \leq \bigvee_{x \in W_0} U^x = U$ (say) ... (1) and $f(U^x) \leq V$, for all $x \in W_0 \Rightarrow f(U) \leq V \Rightarrow U \leq f^{-1}(f(U)) \leq f^{-1}(V) = W$ ... (2). By (1) and (2), $U = W = f^{-1}(V) \Rightarrow f^{-1}(V) \in F\beta PO(X)$. Hence by Theorem 4.2, f is fuzzy β-precontinuous function.

Note 4.6. *The inverse image of a fuzzy preopen set under fuzzy β-precontinuous function need not be so follows from the following example.*

Example 4.7. Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, A\}$, $\tau_2 = \{0_X, 1_X, B\}$ where $A(a) = 0.5, A(b) = 0.6, B(a) = B(b) = 0.5$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Clearly i is fuzzy β-precontinous function. Consider the fuzzy set C defined by $C(a) = 0.5, C(b) = 0.4$. Then $int_{\tau_2}(cl_{\tau_2} C) = B \geq C \Rightarrow C \in FPO(X, \tau_2)$. Now $i^{-1}(C) = C \not\leq \beta int_{\tau_1}(cl_{\tau_1} C) = 0_X \Rightarrow C \notin F\beta PO(X, \tau_1)$.

Remark 4.8. *Composition of two fuzzy β-precontinuous functions need not be so,* *follows from the following example.*

Example 4.9. Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, A\}$, $\tau_2 = \{0_X, 1_X\}$, $\tau_3 = \{0_X, 1_X, B\}$ where $A(a) = 0.5, A(b) = 0.6, B(a) = 0.5, B(b) = 0.4$. Then (X, τ_1), (X, τ_2) and (X, τ_3) are fts's. Consider two identity functions $i_1 : (X, \tau_1) \to (X, \tau_2)$ and $i_2 : (X, \tau_2) \to (X, \tau_3)$. Clearly i_1 and i_2 are fuzzy β-precontinuous functions. Now $B \in \tau_3$. $(i_2 \circ i_1)^{-1}(B) = B \not\leq \beta int_{\tau_1}(cl_{\tau_1} B) = 0_X \Rightarrow i_2 \circ i_1$ is not fuzzy β-precontinuous function.

Lemma 4.10. [2] *Let Z, X, Y be fts's and $f_1 : Z \to X$ and $f_2 : Z \to Y$ be functions. Let $f : Z \to X \times Y$ be defined by $f(z) = (f_1(z), f_2(z))$ for $z \in Z$, where $X \times Y$ is provided with the product fuzzy topology. Then if B, U_1, U_2 are fuzzy sets in Z, X, Y respectively such that $f(B) \leq U_1 \times U_2$, then $f_1(B) \leq U_1$ and $f_2(B) \leq U_2$.*

Theorem 4.11. *Let Z, X, Y be fts's. For any functions $f_1 : Z \to X, f_2 : Z \to Y$, if $f : Z \to X \times Y$, defined by $f(x) = (f_1(x), f_2(x))$, for all $x \in Z$, is fuzzy β-precontinuous function, so are f_1 and f_2.*
Proof. Let U_1 be any fuzzy open q-nbd of $f_1(x_\alpha)$ in X for any fuzzy point x_α in Z. Then $U_1 \times 1_Y$ is a fuzzy open q-nbd of $f(x_\alpha)$, i.e., $(f(x))_\alpha$ in $X \times Y$. Since f is fuzzy β-precontinuous, there exists $V \in F\beta PO(Z)$ with $x_\alpha q V$ such that $f(V) \leq U_1 \times 1_Y$. By Lemma 4.10, $f_1(V) \leq U_1$, $f_2(V) \leq 1_Y$. Consequently, f_1 is fuzzy β-precontinuous.
Similarly, f_2 is fuzzy β-precontinuous.

Lemma 4.12. [1] *Let X, Y be fts's and let $g : X \to X \times Y$ be the graph of a function $f : X \to Y$. Then if A, B are fuzzy sets in X and Y respectively, $g^{-1}(A \times B) = A \bigwedge f^{-1}(B)$.*

Theorem 4.13. *Let $f : X \to Y$ be a function from an fts X to an fts Y and $g : X \to X \times Y$ be the graph function of f. If g is fuzzy β-precontinuous function, then f is so.*
Proof. Let g be fuzzy β-precontinuous function and B be a fuzzy set in Y. Then by Lemma 4.12, $f^{-1}(B) = 1_X \bigwedge f^{-1}(B) = g^{-1}(1_X \times B)$. Now if B is fuzzy open in Y, then $1_X \times B$ is fuzzy open in $X \times Y$. Again, $g^{-1}(1_X \times B) = f^{-1}(B) \in F\beta PO(X)$ as g is fuzzy β-precontinuous function. Hence f is fuzzy β-precontinuous.
Let us now recall the following definitions from [4, 7] for ready references.

Definition 4.14. [4] *A function $f : X \to Y$ is called fuzzy continuous function if the inverse image of every fuzzy open set in Y is fuzzy open set in X.*

Definition 4.15. [7] *A function $f : X \to Y$ is called fuzzy almost continuous if $f^{-1}(B) \leq int(cl(f^{-1}(B)))$ for all fuzzy open set B in Y.*

Note 4.16. *It is clear from above definitions that fuzzy continuous and fuzzy almost continuous functions are fuzzy β-precontinuous. But the converses are not necessarily true follow from the next example.*

Example 4.17. Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, A\}$, $\tau_2 = \{0_X, 1_X, B\}$ where $A(a) = 0.5, A(b) = 0.4, B(a) = B(b) = 0.5$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Now $B \in \tau_2, i^{-1}(B) = B \notin \tau_1$ (also $B \not\leq int_{\tau_1}(cl_{\tau_1} B)$). Clearly i is not fuzzy continuous as well as fuzzy almost continuous function. Now $\beta int_{\tau_1}(cl_{\tau_1} B) = 1_X \setminus A \geq B \Rightarrow B \in F\beta PO(X, \tau_1) \Rightarrow i$ is fuzzy β-precontinuous function.

5. Fuzzy β-Preirresolute Function: Some Properties

In this section we introduce a new type of function, viz., fuzzy β-preirresolute function, the class of which is coarser than that of fuzzy β-precontinuous function.

Definition 5.1. *A function $f : X \to Y$ is called fuzzy β-preirresolute if the inverse image of every fuzzy β-preopen set in Y is fuzzy β-preopen in X.*

Theorem 5.2. *For a function $f : X \to Y$, the following statements are equivalent:*
(a) f is fuzzy β-preirresolute,
(b) for each fuzzy point x_α in X and each fuzzy β-preopen nbd V of $f(x_\alpha)$ in Y, there exists a fuzzy β-preopen nbd U of x_α in X and $f(U) \leq V$,
(c) $f^{-1}(F) \in F\beta PC(X)$, for all $F \in F\beta PC(Y)$,
(d) for each fuzzy point x_α in X, the inverse image under f of every fuzzy β-preopen nbd of $f(x_\alpha)$ is a fuzzy β-preopen nbd of x_α in X,
(e) $f(\beta pclA) \leq \beta pcl(f(A))$, for all $A \in I^X$,
(f) $\beta pcl(f^{-1}(B)) \leq f^{-1}(\beta pclB)$, for all $B \in I^Y$,
(g) $f^{-1}(\beta pintB) \leq \beta pint(f^{-1}(B))$, for all $B \in I^Y$.
Proof. The proof is similar to that of Theorem 4.4 and hence is omitted.

Theorem 5.3. *A function $f : X \to Y$ is fuzzy β-preirresolute if and only if for each fuzzy point x_α in X and corresponding to any fuzzy β-preopen q-nbd V of $f(x_\alpha)$ in Y, there exists a fuzzy β-preopen q-nbd W of x_α in X such that $f(W) \leq V$.*
Proof. The proof is similar to that of Theorem 4.5 and hence is omitted.

Note 5.4. *Composition of two fuzzy β-preirresolute functions is also so.*

Theorem 5.5. *If $f : X \to Y$ is fuzzy β-preirresolute and $g : Y \to Z$ is fuzzy β-precontinuous (resp., fuzzy continuous), then $g \circ f : X \to Z$ is fuzzy β-precontinuous.*
Proof. Obvious.

Remark 5.6. *Every fuzzy β-preirresolute function is fuzzy β-precontinuous, but* *the converse is not true, in general, follows from the following example.*

Example 5.7. Fuzzy β-precontinuous function $\not\Rightarrow$ fuzzy β-preirresolute function
Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A\}$, $\tau_1 = \{0_X, 1_X\}$ where $A(a) = 0.5, A(b) = 0.6$. Then (X, τ) and (X, τ_1) are fts's. Consider the identity function $i : (X, \tau) \to (X, \tau_1)$. Clearly i is fuzzy β-precontinuous function. Now every fuzzy set in (X, τ_1) is fuzzy β-preopen set in (X, τ_1). Consider the fuzzy set B defined by $B(a) = B(b) = 0.4$. Then $B \in F\beta PO(X, \tau_1)$. Now $i^{-1}(B) = B \not\leq \beta int_\tau(cl_\tau B) = 0_X \Rightarrow B \notin F\beta PO(X, \tau) \Rightarrow i$ is not fuzzy β-preirresolute function.
Let us now recall the following definitions from [3, 10] for ready references.

Definition 5.8. [3] *A function $f : X \to Y$ is called fuzzy β-irresolute if the inverse image of fuzzy β-open set in Y is fuzzy β-open in X.*

Definition 5.9. [10] *A function $f : X \to Y$ is called fuzzy open function if $f(U)$ is fuzzy open set in Y for every fuzzy open set U in X.*

Lemma 5.10. *If $f : X \to Y$ is fuzzy open function, then $f^{-1}(clB) \leq cl(f^{-1}(B))$, for all fuzzy set B in Y.*
Proof. Let $x_\alpha \notin cl(f^{-1}(B))$ for any fuzzy set B in Y. Then there exists a fuzzy open set U in X with $x_\alpha qU, U \not{q} f^{-1}(B)$. Then $f(x_\alpha)qf(U)$, $f(U) \not{q} B$ where $f(U)$ is a fuzzy open set in Y as f is a fuzzy open function. Then $f(x_\alpha) \notin clB \Rightarrow x_\alpha \notin f^{-1}(clB)$.

Theorem 5.11. *If $f : X \to Y$ is fuzzy open and fuzzy β-irresolute function, then f is fuzzy β-preirresolute function.*
Proof. Let V be a fuzzy β-preopen set in Y. Then $V \leq \beta int(clV)$. As f is fuzzy β-irresolute function, $f^{-1}(V) \leq f^{-1}(\beta int(clV)) = \beta int(f^{-1}(\beta int(clV))) \leq \beta int(cl(f^{-1}(\beta int(clV)))) \leq \beta int(cl(f^{-1}(clV))) \leq \beta int(cl(cl(f^{-1}(V))))$ (by Lemma 5.10) $= \beta int(cl(f^{-1}(V))) \Rightarrow f^{-1}(V) \in F\beta PO(X)$. Hence f is fuzzy β-preirresolute function.

6. Fuzzy β-Preregular Space

In this section we introduce fuzzy β-preregular space in which space fuzzy β-precontinuity and fuzzy β-preirresoluteness coincide.

Definition 6.1. *An fts (X, τ) is said to be fuzzy β-preregular space if for each fuzzy β-preclosed set F in X and each fuzzy point x_α in X with $x_\alpha \notin F$, there exist a fuzzy open set U in X and a fuzzy β-preopen set V in X such that $x_\alpha qU$, $F \leq V$ and $U \not{q} V$.*

Theorem 6.2. *For an fts (X, τ), the following statements are equivalent:*
(a) X is fuzzy β-preregular,

(b) for each fuzzy point x_α in X and each fuzzy β-preopen set U in X with $x_\alpha qU$, there exists a fuzzy open set V in X such that $x_\alpha qV \leq \beta pclV \leq U$,
(c) for each fuzzy β-preclosed set F in X, $\bigwedge\{clV : F \leq V, V \in F\beta PO(X)\} = F$,
(d) for each fuzzy set G in X and each fuzzy β-preopen set U in X such that GqU, there exists a fuzzy open set V in X such that GqV and $\beta pclV \leq U$.
Proof (a)⇒(b). Let x_α be a fuzzy point in X and U, a fuzzy β-preopen set in X with $x_\alpha qU$. Then $x_\alpha \notin 1_X \setminus U \in F\beta PC(X)$. By (a), there exist a fuzzy open set V and a fuzzy β-preopen set W in X such that $x_\alpha qV$, $1_X \setminus U \leq W$, $V \not{q} W$. Then $x_\alpha qV \leq 1_X \setminus W \leq U \Rightarrow x_\alpha qV \leq \beta pclV \leq \beta pcl(1_X \setminus W) = 1_X \setminus W \leq U$.
(b)⇒(a). Let F be a fuzzy β-preclosed set in X and x_α be a fuzzy point in X with $x_\alpha \notin F$. Then $x_\alpha q(1_X \setminus F) \in F\beta PO(X)$. By (b), there exists a fuzzy open set V in X such that $x_\alpha qV \leq \beta pclV \leq 1_X \setminus F$. Put $U = 1_X \setminus \beta pclV$. Then $U \in F\beta PO(X)$ and $x_\alpha qV$, $F \leq U$ and $U \not{q} V$.
(b)⇒(c). Let F be fuzzy β-preclosed set in X. Then $F \leq \bigwedge\{clV : F \leq V, V \in F\beta PO(X)\}$.
Conversely, let $x_\alpha \not\leq F \in F\beta PC(X)$. Then $F(x) < \alpha \Rightarrow x_\alpha q(1_X \setminus F)$ where $1_X \setminus F \in F\beta PO(X)$. By (b), there exists a fuzzy open set U in X such that $x_\alpha qU \leq \beta pclU \leq 1_X \setminus F$. Put $V = 1_X \setminus \beta pclU$. Then $F \leq V$ and $U \not{q} V \Rightarrow x_\alpha \notin clV \Rightarrow \bigwedge\{clV : F \leq V, V \in F\beta PO(X)\} \leq F$.
(c)⇒(b). Let V be any fuzzy β-preopen set in X and x_α be any fuzzy point in X with $x_\alpha qV$. Then $V(x) + \alpha > 1 \Rightarrow x_\alpha \not\leq (1_X \setminus V)$ where $1_X \setminus V \in F\beta PC(X)$. By (c), there exists $G \in F\beta PO(X)$ such that $1_X \setminus V \leq G$ and $x_\alpha \notin clG$. Then there exists a fuzzy open set U in X with $x_\alpha qU$, $U \not{q} G \Rightarrow U \leq 1_X \setminus G \leq V$ $\Rightarrow x_\alpha qU \leq \beta pclU \leq \beta pcl(1_X \setminus G) = 1_X \setminus G \leq V$.
(c)⇒(d). Let G be any fuzzy set in X and U be any fuzzy β-preopen set in X with GqU. Then there exists $x \in X$ such that $G(x) + U(x) > 1$. Let $G(x) = \alpha$. Then $x_\alpha qU \Rightarrow x_\alpha \not\leq 1_X \setminus U$ where $1_X \setminus U \in F\beta PC(X)$. By (c), there exists $W \in F\beta PO(X)$ such that $1_X \setminus U \leq W$ and $x_\alpha \notin clW \Rightarrow (clW)(x) < \alpha \Rightarrow x_\alpha q(1_X \setminus clW)$. Let $V = 1_X \setminus clW$. Then V is fuzzy open set in X and $V(x) + \alpha > 1 \Rightarrow V(x) + G(x) > 1 \Rightarrow VqG$ and $\beta pclV = \beta pcl(1_X \setminus clW) \leq \beta pcl(1_X \setminus W) = 1_X \setminus W \leq U$.
(d)⇒(b). Obvious.

Note 6.3. *It is clear from Theorem 6.2 that in a fuzzy β-preregular space, every fuzzy β-preclosed set is fuzzy closed and hence every fuzzy β-preopen set is fuzzy open. As a result, in a fuzzy β-preregular space, the collection of all fuzzy closed (resp., fuzzy open) sets and fuzzy β-preclosed (resp., fuzzy β-preopen) sets coincide.*

Theorem 6.4. *If $f : X \to Y$ is fuzzy β-precontinuous function where Y is fuzzy β-preregular space, then f is fuzzy β-preirresolute function.*

Proof. Let x_α be a fuzzy point in X and V be any fuzzy β-preopen q-nbd of $f(x_\alpha)$ in Y where Y is fuzzy β-preregular space. By Theorem 6.2 (a)⇒(b), there exists a fuzzy open set W in Y such that $f(x_\alpha)qW \leq \beta pclW \leq V$. Since f is fuzzy β-precontinuous function, by Theorem 4.5, there exists $U \in F\beta PO(X)$ with $x_\alpha qU$ and $f(U) \leq W \leq V$. By Theorem 5.3, f is fuzzy β-preirresolute function.
Let us now recall following definitions from [4, 5] for ready references.

Definition 6.5. [4] *A collection $\mathcal{U}$ of fuzzy sets in an fts X is said to be a fuzzy cover of X if $\bigcup \mathcal{U} = 1_X$. If, in addition, every member of $\mathcal{U}$ is fuzzy open, then $\mathcal{U}$ is called a fuzzy open cover of X.*

Definition 6.6 [4] *A fuzzy cover $\mathcal{U}$ of an fts X is said to have a finite subcover $\mathcal{U}_0$ if $\mathcal{U}_0$ is a finite subcollection of $\mathcal{U}$ such that $\bigcup \mathcal{U}_0 = 1_X$.*

Definition 6.7. [5] *An fts (X, τ) is said to be fuzzy almost compact if every fuzzy open cover $\mathcal{U}$ of X has a finite proximate subcover, i.e., there exists a finite subcollection $\mathcal{U}_0$ of $\mathcal{U}$ such that $\{clU : U \in \mathcal{U}_0\}$ is again a fuzzy cover of X.*

Theorem 6.8. *If $f : X \to Y$ is a fuzzy β-precontinuous, surjective function where X is fuzzy β-preregular and almost compact space, then Y is fuzzy almost compact space.*
Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ be a fuzzy open cover of Y. Then as f is fuzzy β-precontinuous function, $\mathcal{V} = \{f^{-1}(U_\alpha) : \alpha \in \Lambda\}$ is a fuzzy cover of X by fuzzy β-preopen and hence fuzzy open sets of X as X is fuzzy β-preregular space (by Note 6.3). Since X is fuzzy almost compact, there are finitely many members $U_1, U_2, ..., U_n$ of $\mathcal{U}$ such that $\bigcup_{i=1}^{n} cl(f^{-1}(U_i)) = 1_X$. Since X is fuzzy β-preregular, by Note 6.3, $clA = \beta pclA$ for all $A \in I^X$ and so $1_X = \bigcup_{i=1}^{n} \beta pcl(f^{-1}(U_i)) \Rightarrow 1_Y = f(\bigcup_{i=1}^{n} \beta pcl(f^{-1}(U_i))) = \bigcup_{i=1}^{n} f(\beta pcl(f^{-1}(U_i))) \leq \bigcup_{i=1}^{n} cl(f(f^{-1}(U_i)))$ (by Theorem 4.4 (a)⇒(f)) $\leq \bigcup_{i=1}^{n} cl(U_i) \Rightarrow \bigcup_{i=1}^{n} cl(U_i) = 1_Y \Rightarrow Y$ is fuzzy almost compact space.

References

[1] Azad, K. K., On fuzzy semi-continuity, fuzzy almost continuity and fuzzy weakly continuity, J. Math. Anal. Appl., 82 (1981), 14-32.

[2] Bhattacharyya, Anjana, On fuzzy δ-almost continuous and δ^*-almost continuous functions, J. Tripura Math. Soc., 2 (2000), 45-57.

[3] Bhattacharyya, Anjana, Fuzzy β-irresolute mapping, International Research Journal of Mathematics, Engineering & IT, Vol. 1, Issue 7 (2014), 30-37.

[4] Chang, C. L., Fuzzy topological spaces, J. Math. Anal. Appl., 24 (1968), 182-190.

[5] DiConcillio, A. and Gerla, G., Almost compactness in fuzzy topological spaces, Fuzzy Sets and Systems, 13 (1984), 187-192.

[6] Fath Alla, M. A., On fuzzy topological spaces, Ph.D. Thesis, Assiut Univ., Sohag, Egypt, 1984.

[7] Mukherjee, S. P. and Sinha, S. P., On some weaker forms of fuzzy continuous and fuzzy open mappings on fuzzy topological spaces, Fuzzy Sets and Systems, Vol. 32 (1989), 103-114.

[8] Nanda S., Strongly compact fuzzy topological spaces, Fuzzy Sets and Systems, 42 (1991), 259-262.

[9] Pu, Pao Ming and Liu, Ying Ming, Fuzzy topology I. Neighbourhood structure of a fuzzy point and Moore-Smith Convergence, J. Math Anal. Appl., 76 (1980), 571-599.

[10] Wong, C. K., Fuzzy points and local properties of fuzzy topology, J. Math. Anal. Appl., Vol. 46 (1974), 316-328.

[11] Zadeh, L. A., Fuzzy Sets, Inform. Control, 8 (1965), 338-353.