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LG-FUZZY PARTITION OF UNITY

Marzieh Mostafavi

Department of Mathematics,
University of Qom, Qom, IRAN

E-mail : mmostafavi14279@gmail.com

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Abstract: In this paper, we define LG^c-fuzzy Euclidean topological space with countable basis, which L denotes a complete distributive lattice and we show that each LG^c-fuzzy open covering of this space can be refined to an LG^c-fuzzy open covering that is locally finite. We introduce C^∞ LG-fuzzy manifold $(X, \mathfrak{T}^c)$, with countable basis of LG-fuzzy open sets which X is an L-fuzzy subset of a crisp set M and $\mathfrak{T} : L_X^M \to L$, is an L-gradation of openness on X. We prove that for any LG-fuzzy topological manifold $(X, \mathfrak{T})$, there exists an LG-fuzzy exhaustion. We prove LG-Urysohn lemma and also existence of LG-partitions of unity on every LG-fuzzy topological manifold.

Keywords and Phrases: C^∞ LG^c-fuzzy topological manifold; LG-fuzzy exhaustion; LG-partitions of unity.

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1. Introduction and Definitions

In 1968 Chang [2] has introduced the concept of the fuzzy topological space and later many authors like Katsaras [15], Shostak [31], Chattopadhyay et. al. [3] and Gregori et. al. [10] have presented various kinds of definitions of fuzzy topological spaces. The approach in our manuscript [25] was different from what they have constructed here, since we have answered two questions: What will these structures look like if we assume that the fuzzy topological space X is itself an L-fuzzy subset of a crisp set in Goguen's sense [9], where L denotes a complete distributive lattice set with atleast 2 elements and also if we consider L-gradation of openness of L-fuzzy subsets of X instead of the collection of fuzzy subsets of X as a fuzzy topology on it? This approach has resulted into our definition of C^∞ LG-fuzzy manifolds in [9], which is different from C^1 fuzzy manifolds have been introduced by Ferraro and Foster [5] and others.

One of the main problems, in the theory of fuzzy topological spaces was to obtain an appropriate notion of a fuzzy metric space. Many authors have rapidly developed the fuzzy metric space theory. For example various kinds of fuzzy metric spaces and their properties are discussed in [6], [7-8], [11-14], [16-17], [28], [29], [33]. These different fuzzy metrics have applications in Artificial Intelligence, Computer Science, Economics and Geology. For example the process of digital signals and images, and particularly colour image processing, are two of the most modern applications of fuzzy metrics. (See [1], [12], [26], [30]).

In 1944 Dieudonne [4] has studied some of the properties of paracompact spaces and later Stone [32] and Michael, [24] have investigated paracompact spaces. About three decades later Lowen [19], [20] has discussed compact Housdorff fuzzy topological spaces and Kudri and Warner [18] have established L-fuzzy local compactness. Paracompactness is extraordinarily useful weaker than compactness and it is widely used in many fields of mathematics.

Luo [23] has initiated the concept of paracompactness in fuzzy topological spaces in 1988 and later Lupianez [21-22] has discussed three paracompactness-type properties of fuzzy topological spaces. Recently Wali [34] investigated the compactness of Hausdorff fuzzy metric spaces.

The most important tool to pass "from local to global" in many branches of Geometry and Analysis, is the theory of "partitions of unity". For more familiarity see [27]. The goal of this paper is to develop the partitions of unity to C^∞ LG-fuzzy manifolds. Firstly it would be extremely interesting to show that each LG-locally compact Hausdorff space $(X, \mathfrak{T})$ that is second countable, then it admits a countable base of LG-fuzzy open subsets $\{V_n\}$ with LG-compact closures. Next we show that the following properties of X are equivalent: its connected LG-components are countable unions of LG-compact sets, its connected LG-components are second countable, and it is LG-paracompact. Moreover we prove the Urysohn lemma for LG-normal LG-fuzzy topological spaces and then we use this to introduce LG-partitions of unity. Also we give several interesting examples.

2. Preliminary Theorems

We recall some of the fundamental concepts and definition, which are necessary for this paper. For more information about C^∞ LG-fuzzy manifolds, LGP-related functions of them and LG-fuzzy submanifolds, we refer the readers to [28], although we are sure now that the condition of countability on the LG-structure of an LG-fuzzy manifold in the Definition 3.3 is necessary.

Definition 2.1. *Let X be an L-fuzzy subset of the nonempty crisp set M. Then any L-fuzzy subset of M which is less than or equal to X is called an L-fuzzy subset of X and the set of such subsets is denoted by L_X^M. If $\mathfrak{T} : L_X^M \to L$, be a mapping satisfying:*

i) $\mathfrak{T}(X) = \mathfrak{T}(\tilde{0}) = 1.$

ii) $\mathfrak{T}(A \cap B) \geq \mathfrak{T}(A) \wedge \mathfrak{T}(B).$

iii) $\mathfrak{T}(\bigcup_{j\in J} A_j) \geq \bigwedge_{j\in J} \mathfrak{T}(A_j)$

Then $\mathfrak{T}$ is called a L-gradation of openness on X and $(X,\ \mathfrak{T})$ is called an LG-fuzzy topological space (L-gfts).

Definition 2.2. *Let $B(a,\ r,\ b)$ be an L-fuzzy subset of $1_{\mathbb{R}^n}$, that is equal to zero outside or on the sphere $B_r(a)$ for $a \in \mathbb{R}^n$, $r \in \mathbb{R}^+$ and equal to the function b with values in L, inside $B_r(a)$. Let $\mathfrak{T}_{Ln}$ be any L-gradation of openness on $1_{\mathbb{R}^n}$, such that $supp\mathfrak{T} = \tau_{_{Ln}}$, where $\tau_{_{Ln}}$ is the L-fuzzy topology induced by*

$$\beta_{Ln} = \{B(a,r,b),\ a \in \mathbb{R}^n,\ r \in \mathbb{R}^+,\ b : B_r(a) \to L \ is\ a\ function\}.$$

Then we call $(1_{\mathbb{R}^n},\ \mathfrak{T}_{Ln})$ the LG-fuzzy Euclidean topological space.

Example 2.3. As two useful examples of L-gradations of openness on $1_{\mathbb{R}^n}$, we define

$$\mathfrak{T}_{Ln}\ :\ I_X^M \to L \qquad \mathfrak{T}_{Ln}(B) = \begin{cases} 1 & B \in \tau_{_{Ln}}, \\ 0 & elsewhere. \end{cases}$$

and

$$\mathfrak{T}_{Linf}\ :\ L_X^M \to L, \qquad \mathfrak{T}_{Linf}(B) = \begin{cases} 1 & B = \tilde{0} \\ inf\{B(x) : x \in M\} & \tilde{0} \neq B \in \tau_{_{Ln}} \\ 0 & elsewhere, \end{cases}$$

Definition 2.4. *Let $(X, \mathfrak{T})$ be an LG-fuzzy topological space, $p \in X$ and A be an L-fuzzy subset of X, Set $supp\mathfrak{T} = \{A \in L_X^M :\ \mathfrak{T}(A) > 0\}$. then A is called an LG-fuzzy open subset of X if $A \in supp\mathfrak{T}$.*
An L-fuzzy subset V of X is called an LG-neighborhood of $p \in X$, if there exists an LG-fuzzy open subset U of X such that $p \in U \leq N$.

Definition 2.5. *Let $X \in L^{M_1}$, $Y \in L^{M_2}$ such that $(X, \mathfrak{T})$, $(Y, \mathfrak{R})$ are LG-fuzzy* *topological spaces. Let* $f : M_1 \to M_2$ *be a function and* $f[X]$ *be an L-fuzzy subset of* M_2*, defined by* $f[X](y) = \bigvee\{X(x) \mid x \in f^{-1}(y)\}$. *If we have* $f[X] \leq Y$*, then* f *is called an LG-related function from* X *to* Y *and the set of all these functions is denoted by* $LGRf(X, Y)$.

i) f *is called an one-to-one LG-related function if* $f|_{suppX} : suppX \to suppY$ *is one-to-one function.*

ii) f *is called an onto LG-related function if* $f[X] = Y$.

Further more if we have $\mathfrak{R}(H) \leq \mathfrak{T}(f^{-1}[H])$, *for all LG-fuzzy subset* H *of* Y*, then* f *is an L-gradation preserving LG-related function so it is called an LGP-related function from* X *to* Y *or briefly* $f \in LGPRf(X, Y)$.

Remark 2.6. *Let* $A \in supp\mathfrak{T}$ *and* $B \in supp\mathfrak{R}$. *Let* $f : M_1 \to M_2$ *be a function such that* $f[A] \leq B$, *then* f *can be considered as an L-related function of two Lgfts's,* $(A, \mathfrak{T}_A)$ *and* $(B, \mathfrak{R}_B)$. *So we write* $f \in LGRf(A, B)$.

Definition 2.7. *Let* $(X, \mathfrak{T})$, $(Y, \mathfrak{R})$ *be two Lgfts's and* $f \in LGRf(X, Y)$ *then*

i) f *is called LG-open if* $f[A] \in supp\mathfrak{R}, \ \forall A \in supp\mathfrak{T}$.

ii) f *is called LG-continuous if* $f^{-1}[G] \cap X \in supp\mathfrak{T}, \ \forall G \in supp\mathfrak{R}$.

iii) f *is called LG-homeomorphism if is one -to -one, onto, LG-continuous and LG-open.*

Definition 2.8. *An LG-fuzzy topological space* $(X, \mathfrak{T})$ *is called an LG-fuzzy topological space of dimention n, if for any* $x \in X$*, there exists an LG-fuzzy open subset* A *of* X *such that* $x \in A$ *and* $B \in \mathfrak{T}_{Ln}$ *along with an LGP-homeomorphism* $\psi \in LGPRf(A, B)$.
An LG-fuzzy manifold of dimension n is a second countable Hausdorff LG-fuzzy topological space of dimention n. An C^∞ *LG-fuzzy manifold is an LG-fuzzy topological manifold with an* C^∞ *LG-structure on it.*

3. C^∞ LG-fuzzy manifolds with countable basis of LG-fuzzy open sets

From now on we assume that there exists a countable subset J dence in the Lattice set L, hence $L = \bar{J}$.

Definition 3.1. *We denote by* β^c_{Ln} *the set of all constant L-fuzzy subsets* $B(a, r, b)$ *as in Definition 2.2. Since for each real number, there exists an increasing sequences of rational numbers limited to it, hence the L-fuzzy topology* τ^c_{Ln}*, induced by* β^c_{Ln} *has a countable basis.*

$$\{ \ B(a, r, b), \ a \in \mathbb{Q}^n, \ r \in \mathbb{Q}^+, \ b : B_r(a) \to \ J \ \ is \ a \ constant \ function \ \}$$

We call $(1_{\mathbb{R}^n},\ \mathfrak{T}^c_{Ln})$*, the* LG^c*-fuzzy Euclidean topological space.*

Proposition 3.2. *Each* LG^c*-fuzzy open covering* $\{A_i\}$ *of the* LG^c*-fuzzy Euclidean topological space can be refined to an* LG^c*-fuzzy open covering that is locally finite.*
Proof. For each $x \in \mathbb{R}^n$, we can consider an LG^c-fuzzy open subset $B(x, r_x, b_x)$ contained in some $A_{i(x)}$ with $r_x \leq 1$ in this manner: Since $A_{i(x)} \in \tau^c_{In}$, then $A_{i(x)} = \bigcup_{j \in J} B(a_j, r_j, b_j)$. Hence there exists at least one $j_1 \in J$ such that $x \in B(a_{j_1}, r_{j_1}, b_{j_1})$. Setting $r_x = \min\{1, (r_{j_1} - \|x - a_{j_1}\|)\}$ and $b_x = b_{j_1}$, we have $r_x \leq 1$ and $B(x, r_x, b_x) \leq B(a_{j_1}, r_{j_1}, b_{j_1})$. If we haxe $x \in \bigcap_{k=1}^{s} B(a_{j_k}, r_{j_k}, b_{j_k})$, then $A_{i(x)}(x) = \sup\{\ b_{j_k} \mid 1 \leq k \leq s\}$. Thus $B(x, r_x, b_x) \leq A_{i(x)}$.
For each integer $N > 0$ finitely many of LG^c-fuzzy open subsets $B(x, r_x, b_0)$ cover the LG-fuzzy compact set $\overline{B}(0, N, b_0) - B(0, N-1, b_0)$, say $B(x_1, r_{x_1}, b_0),\ \ldots,$ $B(x_m, r_{x_m}, b_0)$. Hence we may write $\{V_{j,N}\}$ to denote these finitely many LG^c-fuzzy open subsets. As we rechange j and N, the $V_{j,N}$'s assuredly cover the whole $(1_{\mathbb{R}^n},\ \mathfrak{T}^c_{In})$ (even the origin), and this covering refines $\{A_i\}$ in the sense that every $V_{j,N}$ lies in some A_i and the collection $V_{j,N}$ is locally finite in the sense that any point $x \in \mathbb{R}^n$ has an LG^c-neighborhood meeting only finitely many $V_{j,N}$'s. Indeed, since $V_{j,N}$ is an LG^c-fuzzy open subset of radius at most 1 and it intersects $\overline{B}(0, N, b_0) - B(0, N-1, b_0)$, by elementary investigation with the triangle inequality we see that a bounded region of $\mathbb{R}^n$ encounter only finitely many $V_{j,N}$'s. Thus, we have refined $\{A_i\}$ to an LG^c-fuzzy open covering that is locally finite.

Example 3.3. Let $I = [0, 1]$. Then $J = \mathbb{Q} \cap [0, 1]$ is dence in I. Consider the IG^c-fuzzy Euclidean topological space $(1_{\mathbb{R}},\ \mathfrak{T}^c_{I1})$ and define for each $n \in \mathbb{Z}$, the IG^c-fuzzy subset A_n by

$$A_n(x) = \begin{cases} \dfrac{1}{n} & \text{if } x \in \big(n - \dfrac{1}{n},\ n + 1 + \dfrac{1}{n}\big) \\ \\ 0 & \textit{elsewhere} \end{cases}$$

Since for each $x \in \mathbb{R}$, we have $n \leq x < n + 1$ for some $n \in \mathbb{Z}$. Then $x \in (n - \dfrac{1}{n},\ n + 1 + \dfrac{1}{n})$. Hence $x \in A_n$. Also setting $B_{n,k} = B\big(n + \dfrac{k}{n},\ \dfrac{1}{n},\ \dfrac{1}{n}\big)$, we have $B_{n,k} \in \tau^c_{I1}$ and we see $A_n = \bigcup_{k=0}^{n} B_{n,k}$. Therefore $\{A_n\}$ is a locally finite IG^c-fuzzy open covering of $1_{\mathbb{R}}$.

Definition 3.4. *Let* $(X, \mathfrak{T})$ *be an LG-fuzzy topological space and* $\mathcal{A}$ *be a given set of real valued LG-continuous related functions of* X*. Then*

i) $\mathcal{A}$ *is called LG-normal if for any two LG-fuzzy closed disjoint subsets* $A, B \subseteq X$*, there exists* $f : X \to [0, 1]$ *which belongs to* $\mathcal{A}$ *and such that* $f|_A = 0,\ f|_B = 1$.

ii) A is called LG-locally compact if each $p \in A$ admits a compact LG-neighborhood V such that $V \leq A$. It means that for each $p \in A$, there exists an open set U and an LG-compact set K with $p \in U \leq K$.

Proposition 3.5. *The existence of an LG-normal set $\mathcal{A}$ of real valued LG-continuous related functions of X, implies that X must be LG-normal.*

Proof. We show that any two LG-fuzzy closed disjoint subsets $A, B \subseteq X$ can be separated topologically. Since there exists LG-continuous related function $f : X \to [0,1]$ which belongs to $\mathcal{A}$ and such that $f|_A = 0,\ f|_B = 1$. Hence for $H_1 = B(0, \frac{1}{3}, 1),\ H_2 = B(1, \frac{1}{3}, 1)$ we have $H_1 \cap H_2 = \phi$ and $0 \in H_1,\ 1 \in H_2$. Thus $f^{-1}[H_1]$ is an LG-fuzzy open subset of X containing B and $f^{-1}[H_2]$ is an LG-fuzzy open subset of X containing A and $f^{-1}[H_1] \cap f^{-1}[H_2] = \phi$.

Lemma 3.6. *In an LG-normal space X, for any LG-fuzzy closed subset A and LG-fuzzy open subset U, that $A \subseteq U \subseteq X$, there exists an LG-fuzzy open subset V in X such that $A \subseteq V \subseteq LG\overline{V} \subseteq U$.*

Proof. Since $A \subseteq U$, A and $X - U$ are disjoint. They are both LG-fuzzy closed, hence we know that we can find disjoint LG-fuzzy open subsets W and V such that $A \subseteq V,\ X - U \subseteq W$. The condition $V \cap W = \phi$ is equivalent to $V \subseteq X - W$. Since $X - W$ is an LG-fuzzy closed subset containing V, this implies $V \subseteq X - W$. On the other hand, $X - U \subseteq W$ can be re-written as $X - W \subseteq U$. Hence $V \subseteq X - W \subseteq U$.

Lemma 3.7. *If $(X, \mathfrak{T})$ is an LG-locally compact Hausdorff space that is second countable, then it admits a countable base of LG-fuzzy open subsets $\{V_n\}$ with LG-compact closure.*

Proof. Since X is an LG-locally compact, each $p \in X$ admits an LG-compact LG-neighborhood N_p. Hence by Proposition 3.5, N_p is LG-fuzzy closed and so N_p contains the closure of N_p° around p. Hence, in such cases every point $p \in X$ lies in an LG-fuzzy open subset U_p whose closure is LG-compact. Let $\{V_n\}$ be a countable base of LG-fuzzy open subsets of X. Then some $V_{n(p)}$ contains p and is contained in U_p. The LG-closure of $V_{n(p)}$ is an LG-closed subset of the LG-compact set $\overline{U}_p$, and so $V_{n(p)}$ is also LG-compact. Thus, the $\{V_n\}$'s with LG-compact closure are a countable base of LG-fuzzy open subsets.

Theorem 3.8. *Any second countable Hausdorff LG-fuzzy space $(X, \mathfrak{T})$ that is LG-locally compact is LG-paracompact.*

Proof. Let $\{V_n\}$ be a countable base of LG-fuzzy open subsets in X. Let $\{U_i\}$ be an LG-fuzzy open cover of X for which we search a locally finite refinement. Each $p \in X$ lies in some U_i and so there exists a $V_n(p)$ containing p with $V_n(p) \subseteq U_i$. The $V_n(p)$'s therefore organize a refinement of U_i that is countable. Since the exclusivity of one LG-fuzzy open covering refining another is transitive, we therefore lose no generality by finding locally finite refinements of countable LG-fuzzy covers. Assume that all $\overline{V}_n$ are LG-compact. Hence, we can curb our attention to countable covers by LG-fuzzy opens U_n for which $\overline{U}_n$ is LG-compact. Since closure commutes with finite unions, by replacing U_n with $\bigcup_{j<n} U_j$, we retain the LG-compactness condition (as a finite union of LG-compact subsets is LG-compact) and so we can suppose that U_n is an increasing collection of LG-opens with LG-compact closure (with $n \geq 0$). Since $\overline{U}_n$ is LG-compact yet is covered by the open U_i's, for sufficiently large N we have $\overline{U}_n \subseteq U_N$. If we recursively replace U_{n+1} with such a U_N for each n, then we can arrange that $\overline{U}_n \subseteq U_{n+1}$ for each n. Let $K_0 = \overline{U}_0$ and for $n \geq 1$ let $K_n = \overline{U}_n - U_{n-1} = \overline{U}_n \cap (X - U_{n-1})$, so K_n is LG-compact for every n (as it is LG-fuzzy closed subset in the LG-compact $\overline{U}_n$ but for any fixed N we see that U_N is disjoint from K_n for all $n > N$. Now we have a situation similar to the concentric shells in our earlier proof of paracompactness of $\mathbb{R}^n$, and so we can carry over the argument from LG^c-fuzzy Euclidean spaces as follows. We search a locally finite refinement of $\{U_n\}$. For $n \geq 2$ the LG-fuzzy open set $W_n = U_{n+1} - LG\overline{U}_n$ contains K_n, so for each $p \in K_n$ there exists some $V_m \subseteq W_n$ around p. There are finitely many such V_m's that cover the LG-compact K_n, and the collection of V_m's that arise in this way as we vary $n \geq 2$ is a locally finite collection of LG-fuzzy open subsets in X whose union contains $X - U_0$. Throwing in finitely many V_m's contained in U_1 that cover the LG-compact U_0 thereby gives an open cover of X that refines $\{U_i\}$ and is locally finite.

Corollary 3.9. *Let $(X, \mathfrak{T})$ be an LG-fuzzy topological space of dimention n. The following properties of X are equivalent: its connected LG-components are countable unions of LG-compact sets, its connected LG-components are second countable, and it is LG-paracompact.*

Proof. If $\{U, V\}$ is a separation of X and X is LG-paracompact then it is clear that both U and V are LG-paracompact. Hence, since the connected LG-components of X are LG-fuzzy open, X is LG-paracompact if and only if its connected LG-components are LG-paracompact. We may therefore restrict our attention to connected X. For such X, we claim that it is equivalent to require that X be a countable union of LG- compact sets, that X be second countable, and that X be LG-paracompact. By the preceding theorem, if X is second countable then it is LG-paracompact. Since X is connected, Hausdorff, and locally LG- compact, if it is LG-paracompact then it is a countable union of LG-compacts. Hence, to complete the cycle of implications it remains to check that if X is a countable union of LG-compacts then it is second countable. Let $\{K_n\}$ be a countable collection of LG-compacts that cover X, so if $\{U_i\}$ is a covering of X by LG-fuzzy open sets $LGPRf$-homeomorphic to an LG-fuzzy open set in an LG^c-fuzzy Euclidean space we may find finitely many U_i's that cover each K_n. As there are only countably many K_n's, in this way we find countably many U_i's that cover X. Since each U_i is certainly second countable (being open in a Euclidean space), a countable base of LG-fuzzy opens for X is given by the union of countable bases of LG-fuzzy open subsets for each of the U_i's. Hence, X is second countable.

Lemma 3.10. *For any LG-fuzzy topological n-manifold* $(X, \mathfrak{T})$*, there exists an LG-fuzzy exhaustion of* X*, that is a countable collection of LG-fuzzy open subsets* $\{Z_j\}$ *such that*

(1) For each j*, the LG-closure* $LG\overline{Z}_j$ *is LG-compact,*

(2) For each j*,* $LG\overline{Z}_j \subseteq Z_{j+1}$*,*

(3) $X = \bigcup_j Z_j$*.*

Proof. According to the definition 2.8, X is second countable, hence there is a countable basis of the LG-topology of X. We choose those LG-fuzzy open subsets of this countable basis that have LG-compact LG-closures, and denote them by $Y_1, Y_2, \dots$ Since X is locally LG^c-fuzzy Euclidean, it is easy to see that $\mathcal{Y} = \{Y_j\}$ is an LG-fuzzy open cover of X. Set $Z_1 = Y_1$. Since $LG\overline{Z_1}$ is compact, there exist finitely many LG-fuzzy open sets $Y_{i_1}, \dots, Y_{i_k}$ so that $LG\overline{Z_1} \subseteq Y_{i_1} \cup \dots \cup Y_{i_k}$. Let $Z_2 = Y_2 \cup Y_{i_1} \cup \dots \cup Y_{i_k}$ Clearly Z_2 is LG-compact. By repeating this procedure, we obtain a sequence of LG-fuzzy open sets $\{Z_j\}$ which obviously satisfies (1) and (2). It satisfies (3) since $\bigcup_{j=1}^{k} Y_j \subseteq Z_k$ and $\mathcal{Y}$ is is an LG-fuzzy open cover of X.

Lemma 3.11. *Let* $(X, \mathfrak{T})$ *be any LG-fuzzy topological manifold. For any LG-fuzzy open cover* $\mathcal{U} = \{U_\alpha\}$ *of* X*, one can find two countable family of LG-fuzzy open covers* $\mathcal{V} = \{V_j\}$ *and* $\mathcal{W} = \{W_j\}$ *of* X *so that*

(1) For each j*, the LG-closure* $LG\overline{V}_j$ *is LG-compact and* $\overline{V}_j \subseteq W_j$*,*

(2) $\mathcal{W}$ *is a refinement of* $\mathcal{U}$*,*

(3) $\mathcal{W}$ *is a locally finite cover of* X*.*

Proof. For each $p \in X$, there is an j and an $\alpha(p)$ such that $p \in LG\overline{Z}_{j+1} - Z_j$ and $p \in U_{\alpha(p)}$. Since X is locally LG^c-fuzzy Euclidean, we can choose open neighborhoods V_p, W_p of p so that $LG\overline{V}_p$ is LG-compact and

$$LG\overline{V}_p \subseteq W_p \subseteq U_{\alpha(p)} \cap \left(X_{j+2} - LG\overline{Z_{j-1}}\ \right)$$

Now for each j, since the "stripe" $p \in LG\overline{Z}_{j+1} - X_j$ is compact, one can choose finitely many points $p^j_1, \dots, p^j_{k_j}$ so that $V_{p^j_1}, \dots, V_{p^j_{k_j}}$ an open cover of $LG\overline{Z}_{j+1} - Z_j$. Denote all these $V_{p^j_k}$'s by $V_1, V_2, \dots$ and the corresponding $W_{p^j_k}$'s by $W_1, W_2, \dots$. Then $\mathcal{V} = \{V_j\}$ and $\mathcal{W} = \{W_j\}$ are open covers of X that satisfies all above conditions.

Example 3.12. Let $M = \mathbb{R}^2,\ X : \mathbb{R}^2 \to I,\ X(x) = \begin{cases} 1 & \|x\| = 1, \\ 0 & \|x\| \neq 1. \end{cases}$

Then $suppX = S^1$, the unit circle. Set

$$\mathfrak{T} : I^M_X \to I, \quad \mathfrak{T}(A) = \begin{cases} inf\{A(x) \mid x \in X\} & A \in \tau_{I1},\ A \leq X, \\ 0 & elsewhere. \end{cases}$$

Let $J = \{1, 2\}$. We define four IG-open subsets covering X by:

$$\forall x = (x_1,\ x_2), \quad A^{\pm}_j(x) = \begin{cases} \pm x_j & \pm\, x_j > 0,\ \|x\| = 1, \\ 0 & otherwise. \end{cases}$$

Then we show that all $A^{\pm}_j$ are diffeomorphic to IG-open subset $B : \mathbb{R} \to I$, defined by:

$$B(y) = \begin{cases} \sqrt{1 - y^2} & \|y\| < 1, \\ 0 & otherwise. \end{cases}$$

Since $suppB = B(0, 1)$, so $B \in \tau_{I2}$. We define four bijections $\psi^{\pm}_j$ from $suppA^{\pm}_j = \{(x_1,\ x_2) \mid \pm\, x_j > 0,\ \ \|x\| = 1\}$ to $suppB = (0, 1)$, for all $j \in J$ by:

$$\psi^{\pm}_1(x_1, x_2) = (x_2)\ , \quad (\psi^{\pm}_1)^{-1}(y) = (\pm\sqrt{1 - y^2}\ ,\ y)$$

$$\psi^{\pm}_2(x_1, x_2) = (x_1)\ , \quad (\psi^{\pm}_2)^{-1}(y) = (y, \pm\sqrt{1 - y^2})$$

Then we can prove in a similar manner to the Example 3.5 of [27], that $\psi^{\pm}_j \in IGPRf(A^{\pm}_j, B)$ is an IGP-homeomorphism for all $j \in J$ and therefore $(X, \mathfrak{T})$ is an IG-fuzzy manifold of dimension 1. Now we define an uncountable LG-fuzzy open covering $\mathcal{U} = \{U_y \mid y \in (0, 1)\}$ of $(0, 1)$ defined by

$$U_y = B\left(y,\ \frac{1}{2\lceil\frac{1}{y}\rceil(\lceil\frac{1}{y}\rceil - 1)},\ \lceil\frac{1}{y}\rceil\right)$$

Where $\lceil x \rceil$ is the smallest integer grater than or equal to x. So $\mathcal{U}^{\pm}_j = \{(\psi^{\pm}_j)^{-1}(U_y) \mid y \in (0, 1)\}$ is an uncountable LG-fuzzy open covering of $A^{\pm}_j$. Hence The union of these four family is an LG-fuzzy open covering of X. We define two countable family of LG-fuzzy open covers $\mathcal{W} = \{V_n\}$ and $\mathcal{W} = \{W_n\}$ of $(0, 1)$ by

$$W_n = B\Big(\frac{n+\frac{1}{2}}{n(n+1)},\ (\frac{1}{n(n+1)} + \frac{1}{2^{n-1}}),\ \frac{1}{n+2}\Big)$$

$$\implies \quad suppW_n = \Big((\frac{1}{n+1} - \frac{1}{2^n}),\ (\frac{1}{n} + \frac{1}{2^n})\Big)$$

$$V_n = B\Big(\frac{n+\frac{1}{2}}{n(n+1)},\ \frac{1}{2n(n+1)},\ \frac{1}{n+2}\Big)$$

$$\implies \quad suppV_n = (\frac{1}{n+1},\ \frac{1}{n})$$

We see that $suppV_n \subseteq suppW_n$ and furtheremore $LG\overline{V_n} \subset W_n$ for all $n \in \mathbb{N}$ and for all $y \in V_n$

$$\frac{1}{n+1} < y < \frac{1}{n} \quad \implies \quad n < \frac{1}{r} < n+1 \quad \implies \quad \frac{1}{\lceil \frac{1}{y} \rceil} = \frac{1}{n+1}$$

Hence $\mathcal{W}$ is a refinement of $\mathcal{U}$.

4. LG-fuzzy partition of unity

Theorem 4.1. *If X is an LG-normal LG-fuzzy topological space then for any two LG-fuzzy closed disjoint subsets $A, B \subseteq X$, there exists an LG-continuous function $f : X \to [0, 1]$ such that $f|_A = 0,\ f|_B = 1$, so $C(X)$ is LG-normal.*

Proof. Our proof contains three steps:

Step 1:

We show that there is a family of LG-fuzzy open subsets $\{U_q : q \in \mathbb{Q}\}$ such that

(i) $U_q = \phi$ for $q < 0$, U_0 contains A, $U_1 = X - B$, $U_q = X$ for $q > 1$.

(ii) $U_q \subseteq U_{q'}$ for all $q < q'$.

The condition (i) force the definition of U_q for $q < 0$ and for $q \geq 1$. We set $U_1 = X - B$. Since $A \cap B = \phi$ so $A \subseteq X - B = U_1$, hence we can apply Lemma 3.6 and choose U_0 to be any LG-fuzzy open set such that $A \subseteq U_0 \subseteq LG\overline{U}_0 \subseteq U_1$. We will repeatedly use Lemma 3.6 to construct U_q for $q \in \mathbb{Q} \cap (0, 1)$. Writing $\mathbb{Q} \cap [0, 1] = \{q_0,\ q_1,\ q_2,\ \ldots\}$, with $q_0 = 0,\ q_1 = 1$, we will define U_{q_n} by induction on n such that (ii) holds for all $q = q_i,\ q' = q_j$ with $0 \leq i, j \leq n$. Assume that U_q is constructed for $q \in \{q_0,\ q_1,\ \ldots,\ q_n\}$ and we construct it for $q = q_{n+1}$. Looking at all intervals of type (q_i, q_j) with $0 \leq i, j \leq n$, there is a smallest one containing q_{n+1}. Call it (q_a, q_b). Since $q_a < q_b$, by the induction hypothesis we have $LG\overline{U}_a \subseteq U_b$ hence, by Lemma 3.6, we find an open U such that

$$LG\overline{U}_a \subseteq U \subseteq LG\overline{U} \subseteq U_b$$

Define $U_{q_{n+1}} = U$. We have to check that (ii) holds for $q, q' \in \{q_0,\ \ldots,\ q_{n+1}\}$. Fix q, q' If $q \neq q_{n+1}$ and $q' \neq q_{n+1}$, $LG\overline{U}_q \subseteq U_{q'}$ holds by the induction hypothesis. Hence we may assume that $q = q_{n+1}$ or $q' = q_{n+1}$. In the case $q = q_{n+1}$, our behavior is similar. Write $q' = q_j$ with $j \in \{0, 1, \ldots, n\}$. The assumption is that $q_{n+1} < q_j$ and we want to show that

$$LG\overline{U}_{q_{n+1}} \subseteq U_{q_j}$$

But, since $q_{n+1} < q_j$ and (q_a, q_b) is the smallest interval of this type containing q_{n+1}, we must have $q_j \geq q_b$. Then

$$LG\overline{U}_{q_{n+1}} = LG\overline{U} \subseteq U_{q_b} \subseteq U_{q_j}.$$

step 2:
We define the function $f : X \to [0, 1]$, $f(x) = \inf\{q \in \mathbb{Q} : x \in U_q\}$ satisfies:

(1) $f(x) > q \Rightarrow x \notin LG\overline{U}_q$.

(2) $f(x) < q \Rightarrow x \in U_q$. (in particular, $f(x) = q$ for $x \in LG\partial U_q$).

For (1), we prove its negation, i.e. that $x \in LG\overline{U}_q$ implies $f(x) \leq q$. Hence assume that $x \in LG\overline{U}_q$. From (i) we deduce that $x \in U_{q'}$ for all $q' > q$. Hence $f(x) \leq q'$ for all $q' > q$. This implies $f(x) \leq q$. For (2), we assume that $f(x) < q$. By the definition of $f(x)$ (as an infimum), there exists $q' < q$ such that $x \in U_{q'}$. But $q' < q$ implies $U_{q'} \subseteq U_q$, hence $x \in U_q$.
Step 3:
We show that $f|_A = 0$, $f|_B = 1$, and f is LG-continuous.
The first two conditions are immediate from the definition of f and properties (i) of the first step. We now prove that f is LG-continuous. We have to prove that for any LG-fuzzy open subset $\chi_{(a,b)}$ of $1_{\mathbb{R}}$ and any $x \in f^{-1}(\chi_{(a,b)})$, there exists an LG-fuzzy open U containing x such that $f(U) \subseteq (a, b)$. Fix (a, b) and x such that $f(x) \in (a, b)$ and look for U satisfying this condition. Choosing $p, q \in \mathbb{Q}$ such that $a < p < f(x) < q < b$, then $U := U_q - LG\overline{U}_p$ will do the job. Indeed: 1. using step 2, $f(x) > p$ implies $x \notin LG\overline{U}_p$, while $f(x) < q$ implies $x \in U_q$. Hence $x \in U$.

2. for $y \in U$ arbitrary, we have $f(y) \in (a, b)$ because: $y \in U_q \subseteq LG\overline{U}_q$ which, by the previous step, implies $f(y) \leq q < b$. $y \notin LG\overline{U}_p$, hence $y \notin U_p$ which, by the previous step, implies $f(y) \geq p > a$.

Now we define the support of each LG-related function $\sigma : X \to \mathbb{R}$ by

$$supp\ \sigma = LG\overline{\{p|\sigma(p) \neq 0\}}$$

The condition LG-closure in this definition allows us to perform globalization. Thus we prove the fundamental theorem of existence of partitions of unity of any LG-fuzzy open cover of any LG-fuzzy topological manifold:

Theorem 4.2. *Let $(X, \mathfrak{T})$ be any LG-fuzzy topological manifold and $\{U_\alpha\}$ be an LG-fuzzy open cover of X, Then there exists an LG-partitions of unity $\{\sigma_\alpha\}$ subordinate to $\{U_\alpha\}$. It means we need to find a family $\{\sigma_\alpha\}$ of smooth LG-related functions $\sigma_\alpha : X \to [0, 1]$ so that*

(1) supp $\sigma_\alpha \subseteq U_\alpha$ for all α,

(2) For each $p \in X$, there is only a finite number of α such that $\sigma_\alpha(x) \neq 0$ (point finiteness condition)

(2) $\sum_\alpha \sigma_\alpha(x) = 1$

Proof. By Lemma 3.11, we can find two countable family of LG-fuzzy open covers $\mathcal{V}$ and $\mathcal{W}$ of X, such that for each j, the LG-closure $LG\overline{V}_j$ is LG-compact and $LG\overline{V}_j \subseteq W_j$. Now use Urysohn's Lemma to find for each j an LG-continuous function such that $0 \leq \varphi_j \leq 1$, $\varphi_j \equiv 1$ on $LG\overline{V}_j$ and $\varphi(x) = 0$ for $x \notin W_j$. Since $\mathcal{W}$ is a locally finite cover, the function $\varphi = \sum_j \varphi_j$ is a well-defined smooth function on X. Since each φ_j is non-negative, and $\mathcal{V}$ is cover of X, φ is strictly positive on X. It follows that the functions $\rho_j = \dfrac{\varphi_j}{\varphi}$ are smooth and satisfy $0 \leq \rho_j \leq 1$ and $\sum_j \rho_j = 1$ Now we re-index the family ρ_j to get the demanded partial of unity. For each j, we fix an index $\alpha(j)$ such that $W_j \subseteq U_{\alpha(j)}$, and define

$$\sigma_\alpha = \sum_{\alpha(j)=\alpha} \rho_j$$

Since the right hand side is a finite sum near any point $p \in X$, so it defines a smooth function.

$$supp\ \sigma_\alpha = LG\overline{\bigcup_{\alpha(j)=\alpha} supp\ \rho_j} = \bigcup_{\alpha(j)=\alpha} LG\overline{(\rho_j^{-1}(0, 1])} = \bigcup_{\alpha(j)=\alpha} supp\ \rho_j \subseteq U_\alpha$$

Obviously the family σ_α is a partial of unity subordinate to $\{U_\alpha\}$.

Definition 4.3. *Let ρ be a metric on the nonempty set M and X be an L-fuzzy subset of M. Let $S(p,r)$ be the sphere with center p and radius r. Then the L-fuzzy topology τ_{L_ρ} induced by $\beta_{L\rho} = \{S(p,r,s),\, p \in X,\, r \in \mathbb{R}^+,\, s : S(p,r) \to L$ is a constant function less than or equal to $X\}$ is called L-fuzzy topology induced by the metric ρ.*
Also we call any L-gradation of openness on X, with support equal to τ_{L_ρ}, the L-gradation of openness induced by the metric ρ and denote by $\mathfrak{T}_{L\rho}$. Also $(X, \mathfrak{T}_{L\rho})$ is called an LG-fuzzy topological metric space.

Example 4.4. Let $M = \mathbb{R}$ and $\rho(x,y) = |x-y|$ be the ordinary metring on it. Let X be an I-fuzzy subset of M defined by $X(x) = \dfrac{1}{|\lfloor x \rfloor| + 2}$ where $|\lfloor x \rfloor|$ denotes the absolute value of the greatest integer less than or equal to x. For each $x \in S(k,1)$, we have two cases:

$$\text{if } x \in (k-1,k) \implies X(x) = \frac{1}{k+1} = S\big(k-1,1,\frac{1}{k+1}\big)(x) \bigvee S\big(k,1,\frac{1}{k+2}\big)(x)$$

$$\text{if } x \in [k,\ k+1) \implies X(x) = \frac{1}{k+2} = S(k,1,\frac{1}{k+2})(x) \bigvee S\big(k+1,1,\frac{1}{k+3}\big)(x)$$

Hence $X = \bigcup_{k\in\mathbb{Z}} S\big(k,1,\dfrac{1}{k+2}\big)$. Therefore $(X, \tau_{L\rho})$ has an countable LG-fuzzy open covering.

Theorem 4.5. *Let $(X, \mathfrak{T}_{L\rho})$ be an LG-fuzzy topological metric space. Then X is an LG-paracompact.*

Corollary 4.6. *Let $(X, \mathfrak{T})$ be an LG-fuzzy manifold. Let the collection $\{(A_\alpha, \phi_\alpha)\}_{\alpha\in I}$ be an LG-fuzzy open covering of X such that for each $\alpha \in I$ the pair $\{(A_\alpha, \phi_\alpha)\}$ is an LG-chart on X. Then there exists*

(i) an locally finite LG-fuzzy open refinement $\{V_\beta\}_{\beta\in J}$ such that for all $\beta \in J$, V_β is an LG-fuzzy open neighbourhood for a chart $\{(V_\beta, \phi_\beta)\}_{\beta\in J}$, and

(ii) a partition of unity $\{f_\beta\}_{\beta\in J}$ such that $supp(f_\beta) \subset V_\beta$.

5. Conclusion

In this article, using the notions we have introduced in [28], we construct C^∞ LG-fuzzy manifolds with countable basis of LG-fuzzy open sets which X is itself an L-fuzzy subset of a crisp set and $\mathfrak{T} : L_X^M \to L$, is an L-gradation of openness on X. Also we show the existence of an LG-fuzzy exhaustion. Finally we prove LG-Urysohn lemma and LG-partitions of unity.

Since the existence of suitable LG-partitions of unity, plays a very important role in LG-fuzzification of Riemannian Geometry and Finsler Geometry, for a development of knowledge frontiers, an interesting question is that under what conditions we can construct LG-fuzzy Minkowski or Finsler manifolds?

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