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FIXED POINT THEOREMS FOR θ-EXPANSIONS IN BRANCIARI METRIC SPACES

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Abstract: In this paper, we define θ-expansions on Branciari metric spaces by complementing the concept of θ-contractions introduced by Jleli and Samet (J. Inequal. Appl. 2014:38, 2014). Also, we present some new fixed point results for θ-expansion mappings on a Branciari metric space.

Keywords and Phrases: Expansive mapping, metric space, fixed point, θ-contraction.

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1. Introduction and Definitions

Researchers in mathematics and other areas of science and technology, both past and present, have drawn inspiration from the Banach Contraction Principle [2]. Even in the twenty-first century, researchers in the fields of computer science, physics, applied mathematics, etc. are working to apply the Banach Contraction Principle to improve the quality of life for people. The Banach contraction principle, which states that every contraction mapping defined on a complete metric space X to itself permits a unique fixed point, is one of the key findings of nonlinear analysis. This rule is a very useful and well-liked instrument for ensuring the existence and originality of answers to specific issues that arise in and outside of mathematics. The Banach contraction principle has been extended and generalized in many directions (see [11], [16], [18], [19], [20], [22], [29] and references therein).

It is a natural activity in the field of mathematics, and particularly nonlinear functional analysis, to generalise existing concepts in order to go beyond the boundaries of present understanding. In an attempt to generalize the idea of metric, Branciari [5] in 2000 created a new concept known as generalised metric by substituting a quadrilateral inequality for the triangle inequality axiom in the definition of standard metric. Existence of fixed point in Branciari metric space has been considered recently by many authors (see, [1], [3], [4], [6], [8], [10-13], [15], [17], [21] and the references therein.)

In 2014, Jleli and Samet [9] presented a new generalization of the Banach contraction principle in the setting of Branciari metric spaces by introducing a new type of contractive maps. Thereafter, Jleli et al. [7] established a new fixed point theorem in the setting of Branciari metric spaces which was an extension of the fixed point theorem established by Jleli and Samet [9].

The aim of this paper is to define θ-expansions in Branciari metric spaces by complementing the concept of θ-contractions introduced by Jleli and Samet [9]. Also, we present a new fixed point result for θ-expansion mappings on a Branciari metric space.
A very intriguing idea known as "ν-generalized metric space" was first suggested by Branciari in [5] in the year 2000.

Definition 1.1. ([5]) *Let X be a set, let d be a function from $X \times X$ into $[0, \infty)$ and let $\nu \in \mathbb{N}$. Then (X, d) is said to be a ν-generalized metric space if the following hold:*
$(N1)$ $d(x, y) = 0$ if and only if $x = y$ for any $x, y \in X$.
$(N2)$ $d(x, y) = d(y, x)$ for any $x, y \in X$.
$(N3)_\nu$ $d(x, y) = d(x, u_1) + d(u_1, u_2) + + d(u_\nu, y)$ for any $x, u_1, u_2, ..., u_\nu, y \in X$ such that $x, u_1, u_2, ..., u_\nu, y$ are all different.

In the case where $\nu = 2$, X is simply called a generalized metric space.

Definition 1.2. ([5]) *Let X and d be as in Definition 1.1. Then (X, d) is said to be a generalized metric space if $(N1)$, $(N2)$ and the following hold: $(N3)_2$ $d(x, y) = d(x, u) + d(u, v) + d(v, y)$ for any $x, u, v, y \in X$ such that x, u, v, y are all different. Metric space and the idea of "generalised metric space" are extremely similar. However, because X may not always have the topology that is compatible with d, it is exceedingly challenging to approach this concept. If (X, d) is a 1-generalized metric space, it is apparent that (X, d) is a metric space. Therefore,* *every 1-generalized metric space has a topology that is compatible with d.*

However, it is proved in [27] that every 3-generalized metric space has the compatible topology. On the other hand, see Example 7 in [23] and Example 4.2 in [27] for examples of ν-generalized metric spaces that do not have the compatible topology for $\nu \in \{2, 4, 5, ...\}$. For further elaboration on this idea, see ([14], [24-26], [28] and references therein.

Note. [12] *Since there are several distinct notions named as generalized metric, we prefer to use Branciari metric space (BMS) instead of generalized metric space.*

Definition 1.3. [5] *Let (X, d) be a BMS, $\{x_n\}$ be a sequence in X and $x \in X$. We say that $\{x_n\}$ is convergent to x if and only if $d(x_n, x) \to 0$ as $n \to \infty$. We denote this by $x_n \to x$.*

Definition 1.4. [5] *Let (X, d) be a BMS and $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is Cauchy if and only if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.*

Definition 1.5. [5] *Let (X, d) be a BMS. We say that (X, d) is complete if and only if every Cauchy sequence in X converges to some element in X.*

2. Preliminary Theorems

Lemma 2.1. [8] *Let (X, d) be a BMS, $\{x_n\}$ be a Cauchy sequence in (X, d), and $x, y \in X$. Suppose that there exists a positive integer N such that*
(i) $x_n \neq x_m$, for all $n, m > N$;
(ii) x_n and x are distinct points in X, for all $n > N$;
(iii) x_n and y are distinct points in X, for all $n > N$;
(iv) $\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x_n, y)$.
Then we have $x = y$.

In 2014, Jleli and Samet [9] utilized the following set of functions to present a new generalization of the Banach contraction principle in the setting of Branciari metric spaces.

Definition 2.2. *Let Θ denote the set of functions $\theta : (0, \infty) \to (1, \infty)$ satisfying the following conditions:*
(Θ_1) *θ is non-decreasing;*
(Θ_2) *for each sequence $\{t_n\} \subset (0, \infty)$, $\lim_{n\to\infty} \theta(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0^+$;*
(Θ_3) *there exists $r \in (0, 1)$ and $l \in (0, \infty]$ such that $\lim_{t\to 0^+} \frac{\theta(t) - 1}{t^r} = l$.*
Following is the main result of [9]:

Theorem 2.3. *Let (X, d) be a complete g.m.s. and $T : X \to X$ be a given map.*

Suppose that there exist $\theta \in \Theta$ and $k \in (0,1)$ such that

$$x, y \in X, d(Tx, Ty) \neq 0 \Rightarrow \theta(d(Tx, Ty)) \leq [\theta(d(x, y))]^k.$$

Then T has a unique fixed point.

3. Main Theorems

Now, we can state and prove our main result in this section.

Theorem 3.1. *Let (X, d) be a complete BMS and $R : X \to X$ be a given map. Suppose that there exist $\theta \in \Theta$ and $k > 1$ such that*

$$x, y \in X, d(Rx, Ry) \neq 0 \Rightarrow \theta(d(Rx, Ry)) \geq [\theta(d(x, y))]^k \tag{1}$$

Then R has a unique fixed point.

Proof. Let us define the sequence $\{x_n\}$ in X by $x_n = Rx_{n+1}$
Consider,

$$\begin{aligned}
\theta(d(R^n x, R^{n+1} x)) &\leq [\theta(d(R(R^n x), R(R^{n+1} x)))]^{\frac{1}{k}} \\
&= [\theta(d(Rx_n, Rx_{n+1}))]^{\frac{1}{k}} \\
&= [\theta(d(x_{n-1}, x_n))]^{\frac{1}{k}} \\
&\leq [\theta(d(Rx_{n-1}, Rx_n))]^{\frac{1}{k^2}} = [\theta(d(x_{n-2}, x_{n-1}))]^{\frac{1}{k^2}} \\
&= [\theta(d(R^{n-2} x, R^{n-1} x))]^{\frac{1}{k^2}} \\
&\leq ...[\theta(d(x, Rx))]^{\frac{1}{k^n}}
\end{aligned}$$

Therefore, we have for all $n \in \mathbb{N}$

$$1 \leq \theta(d(R^n x, R^{n+1} x)) \leq \theta(d(x, Rx))]^{\frac{1}{k^n}} \tag{2}$$

Assuming $n \to \infty$ in (2), we get

$$\theta(d(R^n x, R^{n+1} x)) \to 1,$$

thereby implying from Definition 2.2 that

$$\lim_{n\to\infty} d(R^n x, R^{n+1} x) = 0 \tag{3}$$

In view of condition (Θ_3) of Definition 2.2, there exists $a \in (0, 1)$ and $b \in (0, \infty]$ such that

$$\lim_{n\to\infty} \frac{\theta(d(R^n x, R^{n+1} x)) - 1}{[d(R^n x, R^{n+1} x)]^a} = b.$$

Assume that b is a finite number and $P = b/2 > 0$. The definition of the limit implies that there exists $n_0 \in \mathbb{N}$ such that

$$\left| \frac{\theta(d(R^n x, R^{n+1}x)) - 1}{[d(R^n x, R^{n+1}x)]^a} - l \right| \leq P, \forall n \geq n_0,$$

which further implies that for all $n \geq n_0$,

$$\frac{\theta(d(R^n x, R^{n+1}x)) - 1}{[d(R^n x, R^{n+1}x)]^a} \geq b - P = P.$$

Thus, for all $n \geq n_0$,

$$n[d(R^n x, R^{n+1}x)]^a \leq Qn[\theta(d(R^n x, R^{n+1}x)) - 1],$$

where $Q = 1/P$.
Suppose now that $b = \infty$ and $P > 0$ be an arbitrary positive number. By the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$\frac{\theta(d(R^n x, R^{n+1}x)) - 1}{[d(R^n x, R^{n+1}x)]^a} \geq P.$$

This suggests that

$$n[d(R^n x, R^{n+1}x)]^a \leq Qn[\theta(d(R^n x, R^{n+1}x)) - 1],$$

for all $n \geq n_0$, where $Q = 1/P$.
Consequently, there is always $Q > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$n[d(R^n x, R^{n+1}x)]^a \leq Q.n[\theta(d(R^n x, R^{n+1}x)) - 1].$$

Utilizing (2), we get for all $n \geq n_0$

$$n[d(R^n x, R^{n+1}x)]^a \leq Q.n([\theta(d(x, Rx))]^{\frac{1}{k^n}} - 1).$$

In the above inequality, if n approaches to ∞, we get

$$\lim_{n \to \infty} n[d(R^n x, R^{n+1}x)]^a = 0.$$

As a result, there exists $n_1 \in \mathbb{N}$ such that

$$d(R^n x, R^{n+1}x) \leq \frac{1}{n^{1/a}}, \tag{4}$$

for all $n \geq n_1$.
We will now demonstrate that R has a periodic point. If it is not the case, then $R^n x \neq R^m x$ for every $n, m \in \mathbb{N}$ such that $n \neq m$. Using (1), we obtain

$$\begin{aligned} \theta(d(R^n x, R^{n+2}x)) &\leq [\theta(d(x_{n-1}, x_{n+1}))]^{\frac{1}{k}} \leq [\theta(d(x_{n-2}, x_n))]^{\frac{1}{k^2}} \\ &\leq \ldots \leq [\theta(d(x, R^2 x))]^{\frac{1}{k^n}}. \end{aligned}$$

Using (2) and assuming that n approaches to ∞ in the aforementioned inequality, we get

$$\lim_{n\to\infty} d(R^n x, R^{n+2}x) = 0. \tag{5}$$

Similarly, there exists $n_2 \in \mathbb{N}$ such that for all $n \geq n_2$

$$d(R^n x, R^{n+2}x) \leq \frac{1}{n^{1/r}}. \tag{6}$$

Let $N = \max\{n_0, n_1\}$. Let us consider the following two cases:
Case 1. If $m > 2$ is odd, that is, $m = 2L + 1$, $L \geq 1$, then using (4), we get for all $n \geq N$

$$\begin{aligned} d(R^n x, R^{n+m}x) &\leq d(R^n x, R^{n+1}x) + d(R^{n+1}x, R^{n+2}x) + \ldots + d(R^{n+2L}x, R^{n+2L+1}x) \\ &\leq \frac{1}{n^{1/r}} + \frac{1}{(n+1)^{1/r}} + \ldots + \frac{1}{(n+2L)^{1/r}} \\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{1/r}} \end{aligned}$$

Case 2. If $m > 2$ is even, that is, $m = 2L$, $L \geq 2$, then using (4) and (6), we get for all $n \geq N$

$$\begin{aligned} d(R^n x, R^{n+m}x) &\leq d(R^n x, R^{n+2}x) + d(R^{n+2}x, R^{n+3}x) + \ldots + d(R^{n+2L-1}x, R^{n+2L}x) \\ &\leq \frac{1}{n^{1/r}} + \frac{1}{(n+2)^{1/r}} + \ldots + \frac{1}{(n+2L-1)^{1/r}} \\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{1/r}} \end{aligned}$$

Consequently, merging all of our cases, we get for all $n \geq N$, $m \in \mathbb{N}$

$$d(R^n x, R^{n+m}x) \leq \sum_{i=n}^{\infty} \frac{1}{i^{1/r}}$$

Due to the fact that the series $\sum_i \frac{1}{i^{1/r}}$ is convergent, we obtain that $\{R^n x\}$ is a Cauchy sequence. As X is complete, there is $u \in X$ such that $R^n x \to u$. In view of continuity of R, we have from 1

$$\ln[\theta(d(Rx, Ry))] \geq k \ln(\theta(d(x, y))) \geq \ln(\theta(d(x, y))),$$

which implies that $d(Rx, Ry) \geq d(x, y)$ for all $x, y \in X$. Thus, we have for all $n \in \mathbb{N}$,

$$d(R^n x, u) \geq d(R^{n+1}x, Ru).$$

On letting $n \to \infty$ in the above inequality, we get $R^{n+1}x \to Ru$. In view of Lemma 1, we get $u = Ru$. This is a contradiction with the fact that R does not have a periodic point. Thus, R has a periodic point, say u, of period p.
Let us assume that the set of fixed points of R is empty. Then we have $p > 1$ and $d(u, Ru) > 0$.
Utilizing (1), we get

$$\begin{aligned} d(u, Ru) &\leq [\theta(d(Ru, R^2u))]^{\frac{1}{k}} \\ &\leq ... \leq [\theta(d(R^n u, R^{n+1}u))]^{\frac{1}{k^n}} \\ &= [\theta(d(u, Ru))]^{\frac{1}{k^n}} < \theta(d(u, Ru)), \end{aligned}$$

which is a contradiction. This implies that the set of fixed points of R is non-empty, that is, R has at least one fixed point. Now, we assume that $u, v \in X$ are two fixed points of R such that $d(u, v) = d(Ru, Rv) > 0$. From (1), we have

$$\theta(d(u, v)) = \theta(d(Ru, Rv)) \geq \theta(d(u, v))^k > \theta(d(u, v)),$$

which is a contradiction. Thus, there is a unique fixed point.

Corollary 3.2. *Let (X, d) be a complete metric space and $R : X \to X$ be a given mapping. Suppose that there exists $\theta \in \Theta$ and $k > 1$ such that*

$$x, y \in X, d(Rx, Ry) \neq 0 \Rightarrow \theta(d(Rx, Ry)) \geq [\theta(d(x, y))]^k.$$

Then R has a unique fixed point.

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