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A FIXED POINT THEOREM ON ENRICHED (ψ, φ_λ)-WEAKLY CONTRACTIVE MAPS IN CONVEX METRIC SPACES

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Abstract: In this paper, we define enriched (ψ, φ_λ)-weakly contractive map in convex metric spaces where ψ is continuous on $[0, +\infty)$ and φ_λ is not continuous on $[0, +\infty)$ and prove the existence and uniqueness of fixed points of these maps in complete convex metric spaces. We provide an example in support of our result.

Keywords and Phrases: Contraction, enriched contraction, (λ, c)-enriched contraction, enriched (ψ, φ_λ)-weakly contractive map.

2020 Mathematics Subject Classification: 47H10, 54H25.

1. Introduction

Weakly contractive maps in Hilbert spaces were introduced by Alber and Guerre-Delabriere [2] as a generalization of contraction maps and they established the existence of fixed points in Hilbert spaces. Rhoades [11] extended it to the setting of metric spaces.

Definition 1. (Rhoades [11]) *Let (X, d) be a metric space. A map $T : X \to X$ is said to be weakly contractive if*

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)) \tag{1}$$

for all $x, y \in X$, where $\varphi : [0, +\infty) \to [0, +\infty)$ is continuous, monotone nondecreasing with $\varphi(t) = 0$ if and only if $t = 0$.

Here we note that every contraction map is weakly contractive, but its converse is not true (Example 2.1.4, [9]). Rhoades [11] proved that every weakly contractive map has a unique fixed point in complete metric spaces.

In 2008, Dutta and Choudhury [10] proved the following theorem.

Theorem 1. (Dutta and Choudhury [10]) *Let (X, d) be a complete metric space and let $T : X \to X$ be a map satisfying the inequality*

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \varphi(d(x, y)) \tag{2}$$

for all $x, y \in X$ where $\psi, \varphi : [0, +\infty) \to [0, +\infty)$ are both continuous and monotone nondecreasing functions with $\psi(t) = 0 = \varphi(t)$ if and only if $t = 0$. Then T has a unique fixed point in X.

If ψ is the identity map in (2), then clearly T is a weakly contractive map that was introduced by Rhoades [11]. For more works on the existence of fixed points of weakly contractive maps, we refer [3, 4, 8, 11] and for applications of fixed point theory in Science and Engineering, we refer [1].

On the other hand, Takahashi [12] introduced the concept of convexity in metric spaces namely convex metric spaces and studied the existence of fixed points of nonexpansive maps in convex metric spaces.

In 2020, Berinde and Păcurar [5] introduced enriched contraction maps in a normed linear space and proved the existence of fixed points in Banach spaces. Further, Berinde and Păcurar [6] extended enriched contraction maps in convex metric spaces and proved the existence of fixed points.

Motivated by the works of Takahashi [12], Berinde and Păcurar [5, 6] and Dutta and Choudhury [10], in this paper, we define enriched (ψ, φ_λ)-weakly contractive maps in convex metric spaces and prove the existence and uniqueness of fixed points. An example is provided in support of our result.

2. Comparison of various contraction maps

In the following, we compare contraction maps, nonexpansive maps, enriched contraction maps, (λ, c)-enriched contraction maps.

Definition 2. *Let (X, d) be a metric space. Let $T : X \to X$ be a selfmap of X. If there exists a real number $c \in [0, 1)$ such that*

$$d(Tx, Ty) \leq c\, d(x, y) \tag{3}$$

for all $x, y \in X$, then we say that T is a contraction on X.

In this case, we say that T is a contraction with contraction constant c. If $c = 1$ then we call T a nonexpansive map on X.

Definition 3. (Berinde and Păcurar [6]) *Let $(X, \|.\|)$ be a normed linear space. Let $T : X \to X$ be a selfmap. If there exist $b \in [0, +\infty)$ and $\theta \in [0, b+1)$ such that*

$$\|b(x-y) + Tx - Ty\| \leq \theta\|x-y\| \tag{4}$$

for all $x, y \in X$, then we say that T is a (b, θ)-enriched contraction.

Here we observe that if $b = 0$ then $\theta \in [0, b+1)$ so that every contraction is a $(0, \theta)$-enriched contraction.

Example 1. Let $X = [0, 1]$ and $T : X \to X$ be given by $Tx = 1 - x$, $x \in [0, 1]$. Then

$$\begin{aligned}|b(x-y) + Tx - Ty| &= |b(x-y) + 1 - x - 1 + y| \\ &= |b(x-y) - (x-y)| \\ &= |b-1||x-y| \\ &\leq \theta|x-y| \text{ for any } 0 \leq \theta < b+1, b \geq 0,\end{aligned}$$

so that T is a (b, θ)-enriched contraction. But it is not a contraction. In fact, it is nonexpansive.

Now the following question is possible.
Is every non-expansive map a (b, θ)-enriched contraction map?
The following example shows that its answer is not affirmative.

Example 2. Define $T : X \to X$ by $Tx = x$. Then, for any x, y in X,

$$\begin{aligned}|b(x-y) + Tx - Ty| &= |b(x-y) + x - y| \\ &= |(b+1)(x-y)| \\ &= |b+1||x-y| \\ &\not\leq \theta|x-y| \text{ for any } \theta \in [0, b+1).\end{aligned}$$

Therefore nonexpansive maps are not (b, θ)-enriched contraction maps.

Definition 4. (Takahashi [12]) *Let (X, d) be a metric space. Let $W : X \times X \times [0, 1] \to X$ be a map. If for all $x, y \in X$ and for any $\lambda \in [0, 1]$,*

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1-\lambda)d(u, y) \tag{5}$$

for any $u \in X$, then we say that W is a convex structure on X.

A metric space (X, d) endowed with a convex structure W is called a convex metric space and we denote by (X, d, W). We observe that any normed linear space is a convex metric space, with convex structure

$$W(x, y, \lambda) = \lambda x + (1-\lambda)y \tag{6}$$

for all $x, y \in X$ and $\lambda \in [0, 1]$. But its converse is not valid; there are several examples of convex metric spaces which cannot be embedded in any Banach space (see [12], Example 1 and Example 2).

The following lemmas present some fundamental properties of a convex metric space.

Lemma 1. (Takahashi [12]) *Let (X, d, W) be a convex metric space. For each $x, y \in X$ and $\lambda, \lambda_1, \lambda_2 \in [0, 1]$, we have the following:*
(i) $d(x, W(x, y, \lambda)) = (1-\lambda)d(x, y)$ (ii) $d(W(x, y, \lambda), y) = \lambda d(x, y)$
(iii) $W(x, x, \lambda) = x$ (iv) $W(x, y, 0) = y$ (v) $W(x, y, 1) = x$ and
(vi) $|\lambda_1 - \lambda_2| d(x, y) \leq d(W(x, y, \lambda_1), W(x, y, \lambda_2))$.

Let (X, d, W) be a convex metric space and $T : X \to X$ be a selfmap. We denote the set of all fixed points of T by $Fix(T)$, i.e., $Fix(T) = \{x \in X : Tx = x\}$.

Lemma 2. (Berinde and Păcurar [6]) *Let (X, d, W) be a convex metric space and $T : X \to X$ be a selfmap. Let $\lambda \in [0, 1)$. We define the map $T_\lambda : X \to X$ by*

$$T_\lambda x = W(x, Tx; \lambda) \tag{7}$$

$x \in X$. Then, for any $\lambda \in [0, 1)$,

$$Fix(T) = Fix(T_\lambda). \tag{8}$$

Definition 5. (Berinde and Păcurar [5]) *Let (X, d, W) be a convex metric space. Let $T : X \to X$ be a selfmap. If there exist $0 \leq c < 1$ and $\lambda \in [0, 1)$ such that*

$$d(W(x, Tx, \lambda), W(y, Ty, \lambda)) \leq c\, d(x, y) \tag{9}$$

for all $x, y \in X$, then we say that T is a (λ, c)-enriched contraction.

We define $T_\lambda : X \to X$ by $T_\lambda(x) = W(x, Tx, \lambda), x \in X$. Then (9) reduces to $d(T_\lambda x, T_\lambda y) \leq c\, d(x, y)$ for all $x, y \in X$.

Note. *In a normed linear space, every (b, θ)-enriched contraction is a (λ, c)-enriched contraction with $\lambda = \frac{1}{b+1}$ and $c = \lambda\theta$.*

Example 3. Let $X = [-2, 0]$ and $T : X \to X$ be defined by $Tx = -x - 2$. Then

$$\begin{aligned} d(W(x, Tx, \lambda), W(y, Ty, \lambda)) &= d((1-\lambda)x + \lambda Tx, (1-\lambda)y + \lambda Ty) \\ &= |(1-\lambda)x + \lambda Tx - (1-\lambda)y - \lambda Ty| \\ &= |(1-\lambda)x + \lambda(-x-2) - (1-\lambda)y - \lambda(-y-2)| \\ &= |(x-y) - 2\lambda(x-y)| \\ &= |1-2\lambda||x-y| \\ &\leq c|x-y| \text{ where } \lambda = \tfrac{1}{3} \text{ and } c = \tfrac{1}{3}. \end{aligned}$$

Therefore T is a (λ, c)-enriched contraction with $\lambda = \frac{1}{3}, c = \frac{1}{3}$ but not a contraction.

Example 4. Let $X = \mathbb{R}$ and $T : \mathbb{R} \to \mathbb{R}$ be defined by $Tx = 2x - 1$. Then

$$\begin{aligned} d(W(x,Tx,\lambda), W(y,Ty,\lambda)) &= d((1-\lambda)x + \lambda Tx, (1-\lambda)y + \lambda Ty) \\ &= |(1-\lambda)x + \lambda Tx - (1-\lambda)y - \lambda Ty| \\ &= |(1-\lambda)x + \lambda(2x-1) - (1-\lambda)y - \lambda(2y-1)| \\ &= |(x-y) + \lambda(x-y)| \\ &= |1+\lambda||x-y| \\ &\nleq c|x-y| \end{aligned}$$

for any $0 \leq c < 1$ and for any $\lambda \in [0,1)$.
Therefore T is not a (λ, c)-enriched contraction.

Theorem 2. (Berinde and Păcurar [6]) *Let (X, d, W) be a complete convex metric space and $T : X \to X$ be a (λ, c)-enriched contraction. Let $x_0 \in X$. Then the sequence $\{x_n\}_{n=1}^{+\infty}$ defined by the iterative process*

$$x_{n+1} = W(x_n, Tx_n, \lambda), n = 0, 1, 2, ... \tag{10}$$

converges to p (say) in X and p is the unique fixed point of T.

3. Enriched $(\psi, \varphi_\lambda)-$weakly contractive maps and fixed points

We denote $\Psi = \{\psi / \psi : [0, +\infty) \to [0, +\infty)$ satisfying
(i) ψ is continuous,
(ii) ψ is nondecreasing and
(iii) $\psi(t) = 0$ if and only if $t = 0\}$.

We now introduce enriched $(\psi, \varphi_\lambda)-$weakly contractive maps.

Definition 6. *Let (X, d, W) be a convex metric space. Let $T : X \to X$ be a selfmap. If there exist $\psi \in \Psi$ and $\lambda \in (0,1)$ and corresponding to this λ, there exists $\varphi_\lambda : [0, +\infty) \to [0, +\infty)$ such that*

$$\psi(d(W(x,Tx,\lambda), W(y,Ty,\lambda))) \leq \psi(d(x,y)) - \varphi_\lambda(d(x,y)) \tag{11}$$

for all $x, y \in X$, where φ_λ is nondecreasing and $\varphi_\lambda(t) = 0$ if and only if $t = 0$, then we say that T is an enriched (ψ, φ_λ)-weakly contractive map on the convex metric space X.

If ψ is the identity map of X then we say that T is an enriched φ_λ-weakly contractive map on X.

Remark 1. *(i) We observe that every (λ, c)-enriched contraction is an enriched φ_λ-weakly contractive map on X. For, $d(W(x,Tx,\lambda), W(y,Ty,\lambda)) \leq cd(x,y) = d(x,y) - (1-c)d(x,y) = d(x,y) - \varphi_\lambda(d(x,y))$ where $\varphi_\lambda(t) = (1-c)t, t \geq 0$ and $\lambda \in (0,1)$.*

(ii) Every contraction is a (b,θ)-enriched contraction, every (b,θ)-enriched contraction is a (λ,c)-enriched contraction and hence it is an enriched (ψ,φ_λ)-weakly contractive map.

Example 5. Let $X=\mathbb{R}$ and $T:X\to X$ be defined by $Tx=-2x-1, x\in\mathbb{R}$. We define ψ on $[0,+\infty)$ by $\psi(t)=t, t\geq 0$. For $\lambda\in(0,1)$, we define φ_λ on $[0,+\infty)$ by

$$\varphi_\lambda(t)=3\lambda t, t\geq 0.$$

Then clearly φ_λ satisfies nondecreasing property and $\varphi_\lambda(t)=0$ if and only if $t=0$ for each $\lambda\in(0,1)$. Now, consider

$$\begin{aligned}\psi(d(W(x,Tx,\lambda),W(y,Ty,\lambda))) &= d(W(x,Tx,\lambda),W(y,Ty,\lambda))\\ &= d((1-\lambda)x+\lambda Tx,(1-\lambda)y+\lambda Ty)\\ &= |(1-\lambda)x+\lambda Tx-((1-\lambda)y+\lambda Ty)|\\ &= |(1-\lambda)x+\lambda(-2x-1)-(1-\lambda)y-\lambda(-2y-1)|\\ &= |(x-y)-3\lambda(x-y)|\\ &= \begin{cases}(x-y)-3\lambda(x-y) \text{ if } (x>y \text{ and } \lambda\leq\frac{1}{3}) \text{ or } (x<y \text{ and } \lambda\geq\frac{1}{3})\\ (y-x)-3\lambda(y-x) \text{ if } (x<y \text{ and } \lambda<\frac{1}{3}) \text{ or } (x>y \text{ and } \lambda\geq\frac{1}{3})\end{cases}\\ &= \psi(d(x,y))-\varphi_\lambda(d(x,y)).\end{aligned}$$

Hence T is an enriched (ψ,φ_λ)-weakly contractive map.

We use the following lemma to prove our main result.

Lemma 3. (Babu and Sailaja [4], Berzig, Karapinar, Radenović, Kadelburg, Jandrlić and Jandrlić [7]) *Suppose (X,d) is a metric space. Let $\{x_n\}$ be a sequence in X such that $d(x_n,x_{n+1})\to 0$ as $n\to+\infty$. If $\{x_n\}$ is not a Cauchy sequence then there exist $\epsilon>0$ and sequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $m_k>n_k>k$ such that $d(x_{m_k},x_{n_k})\geq\epsilon$, $d(x_{m_k-1},x_{n_k})<\epsilon$ and*

(i) $\lim\limits_{k\to+\infty} d(x_{m_k},x_{n_k})=\epsilon$ *(ii)* $\lim\limits_{k\to+\infty} d(x_{m_k-1},x_{n_k})=\epsilon$

(iii) $\lim\limits_{k\to+\infty} d(x_{m_k-1},x_{n_k+1})=\epsilon$ *(iv)* $\lim\limits_{k\to+\infty} d(x_{m_k-1},x_{n_k-1})=\epsilon.$

Theorem 3. *Let (X,d,W) be a complete convex metric space. Suppose that $T:X\to X$ is an enriched $(\psi,\varphi_\lambda)-$weakly contractive map. Then T has a unique fixed point in X.*

Proof. Let λ be as in the Definition 6. We define $T_\lambda:X\to X$ by
$T_\lambda x=W(x,Tx;\lambda), x\in X.$
Then (11) becomes

$$\psi(d(T_\lambda x,T_\lambda y))\leq\psi(d(x,y))-\varphi_\lambda(d(x,y)) \tag{12}$$

for all $x,y\in X$. Let $x_0\in X$. We define $x_{n+1}=T_\lambda x_n, n=0,1,2,\ldots$.
By taking $x=x_{n-1}$ and $y=x_n$ in (12), we get

$$\begin{aligned}\psi(d(T_\lambda x_{n-1}, T_\lambda x_n)) &\leq \psi(d(x_{n-1}, x_n)) - \varphi_\lambda(d(x_{n-1}, x_n))\\ &\leq \psi(d(x_{n-1}, x_n)) \text{ that implies}\end{aligned}$$

$d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)$ for $n = 1, 2, 3, \ldots$.

For, if there exists $n \in \mathbb{Z}^+$ such that $d(x_n, x_{n+1}) > d(x_{n-1}, x_n)$ that implies

$$\begin{aligned}\psi(d(x_{n-1}, x_n)) &\leq \psi(d(x_n, x_{n+1}))\\ &\leq \psi(d(x_{n-1}, x_n)) - \varphi_\lambda(d(x_{n-1}, x_n))\\ &< \psi(d(x_{n-1}, x_n)),\end{aligned}$$

a contradiction.

Hence, we have $\{d(x_n, x_{n+1})\}$ is a decreasing sequence of non-negative reals.

Suppose that $\lim\limits_{n\to+\infty} d(x_n, x_{n+1}) = \beta, \beta \geq 0$.

We now show that $\beta = 0$.

Since ψ, φ_λ are nondecreasing, we have

$$\psi(d(x_n, x_{n+1})) \leq \psi(d(x_n, x_{n-1})) - \varphi_\lambda(d(x_n, x_{n-1})). \tag{13}$$

Let $\alpha_n = d(x_n, x_{n+1})$. Thus, from (13), we have

$$\psi(\alpha_n) \leq \psi(\alpha_{n-1}) - \varphi_\lambda(\alpha_{n-1}). \tag{14}$$

Since $\beta \leq \alpha_n$ for all n, we have $\varphi_\lambda(\beta) \leq \varphi_\lambda(\alpha_n)$ for all n.

Hence from (14) we have $\psi(\alpha_n) \leq \psi(\alpha_{n-1}) - \varphi_\lambda(\beta)$.

Now on letting $n \to +\infty$ and using the continuity of ψ, it follows that $\psi(\beta) \leq \psi(\beta) - \varphi_\lambda(\beta)$. Hence $\varphi_\lambda(\beta) = 0$. So $\beta = 0$.

Therefore $\lim\limits_{n\to+\infty} d(x_n, x_{n+1}) = 0$.

We now prove that $\{x_n\}$ is a Cauchy sequence.

If $\{x_n\}$ is not Cauchy, then by Lemma 3, there exist $\epsilon > 0$ and sequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $n_k > m_k > k$ such that

$$d(x_{n_k}, x_{m_k}) \geq \epsilon \text{ and } d(x_{n_k-1}, x_{m_k}) < \epsilon, \tag{15}$$

and (i) to (iv) of Lemma 3 hold.

From (15), we have

$\psi(\epsilon) \leq \psi(d(x_{n_k}, x_{m_k})) \leq \psi(d(x_{n_k-1}, x_{m_k-1})) - \varphi_\lambda(d(x_{n_k-1}, x_{m_k-1}))$ and it implies that

$\varphi_\lambda(d(x_{n_k-1}, x_{m_k-1})) \leq \psi(d(x_{n_k-1}, x_{m_k-1})) - \psi(\epsilon)$.

$$\begin{aligned}\text{Therefore } 0 \leq \liminf \varphi_\lambda(d(x_{n_k-1}, x_{m_k-1})) &\leq \limsup \varphi_\lambda(d(x_{n_k-1}, x_{m_k-1}))\\ &\leq \limsup \psi(d(x_{n_k-1}, x_{m_k-1})) - \psi(\epsilon)\\ &= \psi(\epsilon) - \psi(\epsilon) \qquad \text{(By (iv) of Lemma 3)}\end{aligned}$$

$$= 0.$$

Therefore $\liminf \varphi_\lambda(d(x_{n_k-1}, x_{m_k-1})) = \limsup \varphi_\lambda(d(x_{n_k-1}, x_{m_k-1})) = 0$ so

$$\lim_{n\to+\infty} \varphi_\lambda(d(x_{n_k-1}, x_{m_k-1})) = 0. \tag{16}$$

Let $\eta = \frac{\epsilon}{2} > 0$. So there exists $L_1 \in \mathbb{Z}^+$ such that $|d(x_{n_k-1}, x_{m_k-1}) - \epsilon| < \eta$ for all $l \geq L_1$, by Lemma 3 (iv).
Therefore for $l \geq L_1$, we have $\epsilon - \eta < d(x_{n_k-1}, x_{m_k-1}) < \epsilon + \eta$.
That is, $\epsilon - \frac{\epsilon}{2} < d(x_{n_k-1}, x_{m_k-1}) < \epsilon + \frac{\epsilon}{2}$.
That is, $\frac{\epsilon}{2} < d(x_{n_k-1}, x_{m_k-1})$.
Therefore $\varphi_\lambda(\frac{\epsilon}{2}) \leq \varphi_\lambda(d(x_{n_k-1}, x_{m_k-1})) \longrightarrow 0$ as $l \longrightarrow +\infty$, by (16).
Hence it follows that $\varphi_\lambda(\frac{\epsilon}{2}) = 0$ so that $\epsilon = 0$,
a contradiction.

Hence $\{x_n\}$ is a Cauchy sequence.
Since X is complete, there exists $x^* \in X$ such that $d(x_n, x^*) \longrightarrow 0$ as $n \longrightarrow +\infty$.
Now, from (12) we have

$$\begin{aligned}\psi(d(T_\lambda x^*, x_{n+1})) &= \psi(T_\lambda x^*, T_\lambda x_n)\\ &\leq \psi(d(x^*, x_n)) - \varphi_\lambda(d(x^*, x_n))\\ &\leq \psi(d(x^*, x_n)) \longrightarrow 0 \text{ as } n \to +\infty.\end{aligned}$$

Therefore $\lim_{n\to+\infty} \psi(d(T_\lambda x^*, x_{n+1})) = 0$
which implies that $\psi(\lim_{n\to+\infty} d(T_\lambda x^*, x_{n+1})) = 0$ so that $\psi(d(T_\lambda x^*, x^*)) = 0$.
Hence $d(T_\lambda x^*, x^*) = 0$. Therefore $T_\lambda x^* = x^*$.
Now suppose that x^*, x_1^* are two distinct fixed points of T_λ. Then
$0 < \psi(d(x^*, x_1^*)) = \psi(d(T_\lambda x^*, T_\lambda x_1^*)) \leq \psi(d(x^*, x_1^*)) - \varphi_\lambda(d(x^*, x_1^*))$
which implies that $\varphi_\lambda(d(x^*, x_1^*)) \leq 0$ so that $\varphi_\lambda(d(x^*, x_1^*)) = 0$.
Hence, we have $d(x^*, x_1^*) = 0$.
Therefore $x^* = x_1^*$.
Therefore T_λ has a unique fixed point x^* so that $Fix(T_\lambda) = \{x^*\}$.
Now, by Lemma 2, we have $Fix(T) = \{x^*\}$.

This completes the proof of the theorem.

Remark 2. *In the proof of Theorem 1, Dutta and Choudhury [10] assumed the continuity of φ, where as, in Theorem 3, we did not use the continuity of φ_λ.*

By choosing $\psi(t) = t, t \geq 0$ in Theorem 3, we have the following corollary.

Corollary 1. *Let (X, d, W) be a complete convex metric space. If $T : X \to X$ is an enriched φ_λ-weakly contractive map of X, then T has a unique fixed point in X.*

Corollary 2. *Let (X, d) be a complete metric space. If $T : X \to X$ is a weakly* *contractive map, then T has a unique fixed point in X.*

Proof. By choosing ψ as the identity mapping and $\lambda = 0$, we get the conclusion.

Remark 3. *From Remark 1 (i), clearly Theorem 2 follows as a corollary to Corollary 1.*

The following is an example in support of Theorem 3.

Example 6. Let $T : [0,1] \to [0,1]$ be defined by $Tx = x - \frac{x}{2+x},\ \ if\ 0 \le x \le 1$. To avoid cumbersome computations, we choose $\psi(t) = t, t \ge 0$. Let $\lambda \in (0,1)$. We choose

$$\varphi_\lambda(t) = \begin{cases} \frac{2}{9}\lambda t, & if\ 0 \le t \le 1, \\ \frac{2}{9}, & if\ \ t > 1. \end{cases}$$

Then, for each $\lambda \in (0,1)$, clearly φ_λ is nondecreasing for $t \ge 0$ and $\varphi_\lambda(t) = 0$ if and only if $t = 0$.

Here we note that φ_λ is not continuous on $[0, +\infty)$. Let $x, y \in [0,1]$.

We consider,

$$\begin{aligned} \psi(d(W(x,Tx,\lambda), W(y,Ty,\lambda))) &= d(W(x,Tx,\lambda), W(y,Ty,\lambda)) \\ &= d((1-\lambda)x + \lambda Tx, (1-\lambda)y + \lambda Ty) \\ &= |(1-\lambda)x + \lambda(x - \tfrac{x}{2+x}) - (1-\lambda)y - \lambda(y - \tfrac{y}{2+y})| \\ &= |(1-\lambda)(x-y) + \lambda(x-y) - \lambda(\tfrac{x}{2+x} - \tfrac{y}{2+y})| \\ &= |(x-y) - \tfrac{2\lambda(x-y)}{(2+x)(2+y)}| \\ &= \begin{cases} (x-y) - \frac{2\lambda(x-y)}{(2+x)(2+y)} \text{ if } x > y \\ (y-x) - \frac{2\lambda(y-x)}{(2+x)(2+y)} \text{ if } x < y \end{cases} \\ &\le \begin{cases} (x-y) - \frac{2\lambda}{9}(x-y) \text{ if } x > y \\ (y-x) - \frac{2\lambda}{9}(y-x) \text{ if } x < y \end{cases} \\ &= \psi(d(x,y)) - \varphi_\lambda(d(x,y)). \end{aligned}$$

Hence T satisfies the inequality (11). So T satisfies all the hypotheses of Theorem 3 and T has a unique fixed point 0.

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