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A NOTE ON AN EQUIVALENT OF THE RIEMANN HYPOTHESIS

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Abstract: In this manuscript we denote by $\sum_\rho$ a sum over the non trivial zeros of Riemann zeta function (or over the zeros of Riemann's xi function), where the zeros of multiplicity k are counted k times. We prove a result that the Riemann Hypothesis is true if and only if

$$\sum_\rho \frac{1}{|\frac{1}{2}-\rho|^4} = \frac{1}{2}\left(\frac{\xi''(\frac{1}{2})}{\xi(\frac{1}{2})}\right)^2 - \frac{1}{6}\left(\frac{\xi^{(4)}(\frac{1}{2})}{\xi(\frac{1}{2})}\right)$$

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1. Introduction and Definitions

The Riemann zeta function, $\zeta(s)$ is defined as the analytic continuation of the Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

which converges in the half plane $\Re(s) > 1$. The Riemann zeta function is a meromorphic function on the whole complex plane, which is holomorphic everywhere except for a simple pole at s = 1 with residue 1. The Riemann Hypothesis states that all the non trivial zeros of the Riemann zeta function lie on the critical line $\Re(s) = \frac{1}{2}$. The Riemann xi function is defined as

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

$\xi(s)$ is an entire function whose zeros are the non trivial zeros of $\zeta(s)$ [2]. Further $\xi(s)$ satisfies the functional equation [2]

$$\xi(s) = \xi(1-s)$$

Equivalents of the Riemann Hypothesis is well studied topic [3]. Recently an equivalent of the Riemann Hypothesis was proved [1] where the authors proved that if $\sum_\rho$ is a sum over the non trivial zeros of $\zeta(s)$ counted with multiplicity, then the Riemann Hypothesis is true if and only if

$$\sum_\rho \frac{1}{\left|\frac{1}{2}-\rho\right|^2} = \frac{\xi''(\frac{1}{2})}{\xi(\frac{1}{2})}$$

In this paper we prove another interesting equivalent of the Riemann Hypothesis connecting the sum over the non trivial zeros of zeta function to the derivatives of Riemann xi function evaluated at $\frac{1}{2}$.

2. Main Theorems

The goal of this short note is to prove the following result.

Theorem 2.1. *If $\sum_\rho$ denotes a sum over the non trivial zeros of Riemann zeta function then the Riemann Hypothesis is true if and only if*

$$\sum_\rho \frac{1}{|\frac{1}{2}-\rho|^4} = \frac{1}{2}\left(\frac{\xi''(\frac{1}{2})}{\xi(\frac{1}{2})}\right)^2 - \frac{1}{6}\left(\frac{\xi^{(4)}(\frac{1}{2})}{\xi(\frac{1}{2})}\right)$$

Proof. Riemann xi function $\xi(s)$ is defined as,

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) \tag{2.1}$$

Riemann xi function is also expressed as the Hadamard product [2, p.4 7]

$$\xi(s) = \xi(0)\prod_\rho\left(1-\frac{s}{\rho}\right) \tag{2.2}$$

where ρ ranges over all the roots ρ of $\xi(\rho) = 0$. Taking logarithmic derivative in both sides of equation (2.2) with s such that $\xi(s) \neq 0$ [2, p. 56]

$$\frac{\xi'(s)}{\xi(s)} = \sum_{\rho} \frac{1}{s-\rho} \tag{2.3}$$

Differentiating both sides of equation (2.3) with respect to s

$$\frac{\xi(s)\xi''(s) - (\xi'(s))^2}{(\xi(s))^2} = -\sum_{\rho} \frac{1}{(s-\rho)^2} \tag{2.4}$$

Now we show that the infinite series in equation (2.4) is uniformly convergent in any disc $|s| \leq K$ and hence can be differentiated termwise.
For this when the terms ρ and $1-\rho$ are paired we have

$$\left|\frac{1}{(s-\rho)^2} + \frac{1}{(s-(1-\rho))^2}\right| = \left|\frac{1}{\left((s-\frac{1}{2}) - (\rho-\frac{1}{2})\right)^2} + \frac{1}{\left((s-\frac{1}{2}) + (\rho-\frac{1}{2})\right)^2}\right| \tag{2.5}$$

So we obtain

$$\left|\frac{1}{(s-\rho)^2} + \frac{1}{(s-(1-\rho))^2}\right| = 2\left|\frac{(s-\frac{1}{2})^2 + (\rho-\frac{1}{2})^2}{\left((s-\frac{1}{2})^2 - (\rho-\frac{1}{2})^2\right)^2}\right| \tag{2.6}$$

Take ρ sufficiently large such that $\left|\rho - \frac{1}{2}\right|^2 \geq 2\left|s-\frac{1}{2}\right|^2$ which can be done since $\left|s-\frac{1}{2}\right|^2 \leq (K+\frac{1}{2})^2$. Then we have by equation (2.6)

$$\left|\frac{1}{(s-\rho)^2} + \frac{1}{(s-(1-\rho))^2}\right| \leq 8\left(\frac{(K+\frac{1}{2})^2 + |\frac{1}{2}-\rho|^2}{|\frac{1}{2}-\rho|^4}\right) \tag{2.7}$$

So we have

$$\left|\frac{1}{(s-\rho)^2} + \frac{1}{(s-(1-\rho))^2}\right| \leq 8\left(\frac{(K+\frac{1}{2})^2}{|\frac{1}{2}-\rho|^4} + \frac{1}{|\frac{1}{2}-\rho|^2}\right) \tag{2.8}$$

for all sufficiently large ρ once K is fixed. Since we know that [2, p. 42] for any given $\epsilon > 0$ the series

$$\sum_{\rho} \frac{1}{|\frac{1}{2}-\rho|^{1+\epsilon}}$$

converges where ρ are the zeros of $\xi(s)$. So the series in equation (2.4) converges uniformly and hence we can differentiate termwise. Differentiating both sides of equation (4) with respect to s

$$\frac{\xi^{(3)}(s)}{\xi(s)}+2\left(\frac{\xi'(s)}{\xi(s)}\right)^3-\frac{3\,\xi'(s)\xi''(s)}{(\xi(s))^2}=2\sum_{\rho}\frac{1}{(s-\rho)^3} \tag{2.9}$$

Differentiating both sides of equation (2.9) with respect to s

$$\frac{\xi^{(4)}(s)}{\xi(s)}-3\left(\frac{\xi''(s)}{\xi(s)}\right)^2-6\left(\frac{\xi'(s)}{\xi(s)}\right)^4-\frac{4\,\xi^{(3)}(s)\xi'(s)}{(\xi(s))^2}+\frac{12\,(\xi'(s))^2\xi''(s)}{(\xi(s))^3}=-6\sum_{\rho}\frac{1}{(s-\rho)^4} \tag{2.10}$$

Lemma 2.1. *We have*

$$\xi'\left(\frac{1}{2}\right)=0$$

Proof. Since the Riemann xi function, $\xi(s)$ satisfies the functional equation

$$\xi(s)=\xi(1-s)$$

We have since $\xi(s)$ is entire, so by differentiating the above equation with respect to s

$$\xi'(s)=-\xi'(1-s) \tag{2.11}$$

Now putting $s=\frac{1}{2}$ in equation (11), we have

$$\xi'\left(\frac{1}{2}\right)=-\xi'\left(\frac{1}{2}\right)$$

So we get,

$$\xi'\left(\frac{1}{2}\right)=0$$

We have a more general result that the odd order derivatives of the Riemann xi function at $s=\frac{1}{2}$ vanishes [4, p. 528]. This completes the proof of Lemma.
Putting $s=\frac{1}{2}$ in equation (2.10) and using $\xi'(\frac{1}{2})=0$ and $\xi^3(\frac{1}{2})=0$ [4, p. 528] we have,

$$\frac{\xi^{(4)}(\frac{1}{2})}{\xi(\frac{1}{2})}-3\left(\frac{\xi''(\frac{1}{2})}{\xi(\frac{1}{2})}\right)^2=-6\sum_{\rho}\frac{1}{(\frac{1}{2}-\rho)^4} \tag{2.12}$$

Taking real parts of both sides of equation (2.12) and noting that $\xi^{(4)}(\frac{1}{2})$, $\xi''(\frac{1}{2})$ and $\xi(\frac{1}{2})$ are real [5], we have

$$\frac{\xi^{(4)}(\frac{1}{2})}{\xi(\frac{1}{2})} - 3\left(\frac{\xi''(\frac{1}{2})}{\xi(\frac{1}{2})}\right)^2 = -3\left[\sum_\rho \frac{1}{(\frac{1}{2}-\rho)^4} + \sum_\rho \frac{1}{(\frac{1}{2}-\overline{\rho})^4}\right] \tag{2.13}$$

So above equation gives

$$\left(\frac{\xi''(\frac{1}{2})}{\xi(\frac{1}{2})}\right)^2 - \frac{\xi^{(4)}(\frac{1}{2})}{3\ \xi(\frac{1}{2})} = \sum_\rho \frac{(\frac{1}{2}-\rho)^4 + (\frac{1}{2}-\overline{\rho})^4}{|\frac{1}{2}-\rho|^8} \tag{2.14}$$

So we have

$$\left(\frac{\xi''(\frac{1}{2})}{\xi(\frac{1}{2})}\right)^2 - \frac{\xi^{(4)}(\frac{1}{2})}{3\ \xi(\frac{1}{2})} = \sum_\rho \frac{[(\frac{1}{2}-\rho)^2 - (\frac{1}{2}-\overline{\rho})^2]^2 + 2|\frac{1}{2}-\rho|^4}{|\frac{1}{2}-\rho|^8} \tag{2.15}$$

So we get

$$\left(\frac{\xi''(\frac{1}{2})}{\xi(\frac{1}{2})}\right)^2 - \frac{\xi^{(4)}(\frac{1}{2})}{3\ \xi(\frac{1}{2})} = \sum_\rho \frac{[(-2i\Im(\rho)(1-2\Re(\rho)))^2]^2 + 2|\frac{1}{2}-\rho|^4}{|\frac{1}{2}-\rho|^8} \tag{2.16}$$

Which gives

$$\left(\frac{\xi''(\frac{1}{2})}{\xi(\frac{1}{2})}\right)^2 - \frac{\xi^{(4)}(\frac{1}{2})}{3\ \xi(\frac{1}{2})} = \sum_\rho \frac{-4(\Im(\rho))^2(1-2\Re(\rho))^2 + 2|\frac{1}{2}-\rho|^4}{|\frac{1}{2}-\rho|^8} \tag{2.17}$$

Above equation can be rewritten as

$$\sum_\rho \frac{4(\Im(\rho))^2(1-2\Re(\rho))^2}{|\frac{1}{2}-\rho|^8} = 2\sum_\rho \frac{1}{|\frac{1}{2}-\rho|^4} + \frac{\xi^{(4)}(\frac{1}{2})}{3\ \xi(\frac{1}{2})} - \left(\frac{\xi''(\frac{1}{2})}{\xi(\frac{1}{2})}\right)^2 \tag{2.18}$$

Now we know that [2, p. 42] for any given $\epsilon > 0$ the series

$$\sum_\rho \frac{1}{|\frac{1}{2}-\rho|^{1+\epsilon}}$$

converges where ρ are the zeros of $\xi(s)$. We prove that Riemann Hypothesis is true if and only if

$$\sum_\rho \frac{1}{|\frac{1}{2}-\rho|^4} = \frac{1}{2}\left(\frac{\xi''(\frac{1}{2})}{\xi(\frac{1}{2})}\right)^2 - \frac{1}{6}\left(\frac{\xi^{(4)}(\frac{1}{2})}{\xi(\frac{1}{2})}\right) \tag{2.19}$$

Assume Riemann Hypothesis is true then we have $\Re(\rho) = \frac{1}{2}$ for all ρ and hence by equation (2.18) we get the result. Conversely suppose that equation (2.19) holds then by equation (2.18) we have

$$\sum_{\rho} \frac{(\Im(\rho))^2(1-2\Re(\rho))^2}{|\frac{1}{2}-\rho|^8} = 0 \tag{2.20}$$

Write

$$S = \sum_{\rho} \frac{(\Im(\rho))^2(1-2\Re(\rho))^2}{|\frac{1}{2}-\rho|^8}$$

Since the non trivial zeros of Riemann zeta function are countable, assume there exists at least four non trivial zeros off the critical line ρ_k, $\overline{\rho_k}$, $1-\rho_k$ and $1-\overline{\rho_k}$ such that $1-2\Re(\rho_k) \neq 0$ .Then for $n \geq k$, let the partial sum be S_n.
Since we know that the zeta function has no non trivial zeros on the real line so we have $\Im(\rho_k) \neq 0$, which gives $S_n \geq \frac{(\Im(\rho_k))^2(1-2\Re(\rho_k))^2}{|\frac{1}{2}-\rho_k|^8} > 0$
Thus

$$\lim_{n\to\infty} S_n = S > 0$$

which contradicts the fact that $S = 0$. So, $\Re(\rho_n) = \frac{1}{2}$ for all n.

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