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# NEW FUNCTIONAL RELATION FOR THE DILOGARITHM INVOLVING TWO VARIABLES

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**Abstract:** We establish new functional relations for the dilogarithm involving two variables, which adhere the properties of Polylogarithm. We also considered several closely-related identities such as (for example) polylogarithm (also known as Jonquière's function), Euler dilogarithm function and Clausen's function.

**Keywords and Phrases:** Polylogarithm; Dilogarithm identity or Spence's function; Clausen's function; Definite Integral.

**2020 Mathematics Subject Classification:** 11A05, 11Y16, 68Q25.

## 1. Introduction and Definitions

Alfre Jonquière introduced the concept of polylogarithm (also known as Jonquière's function) (see, for details [6]; [8]; [10]; and [20]), which is a special function represented as $Li_s(z)$ of order $s$ and argument $z$. The polylogarithm reduces to an elementary function such as natural logarithm or rational functions for special values of $s$. The polylogarithm function appears as closed form of integrals such as *Fermi-Dirac integral* and *Bose-Einstein integral* (see [12] and [15]) etc. The polylogarithm of positive integer order arise in the calculation of processes represented by the *Feynman diagrams*; and it is also equivalent to Hurwitz Zeta function (see

[10]).
The polylogarithm function is defined by a power series in $z$, which is also a Dirichlet series in $s$:

$$\mathrm{Li}_s(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^s} = z + \frac{z^2}{2^s} + \frac{z^3}{3^s} + \cdots \qquad |z| < 1. \tag{1}$$

which can be extended to $|z| \geq 1$ by the method of analytic continuation. The particular case ($s = 1$) involves the natural logarithm, $\mathrm{Li}_1(z) = -\log(1-z)$; for $s = 2$ and $s = 3$, and are known as dilogarithm and trilogarithm, respectively (see [15]).
The Euler dilogarithm function (see, for details, [5]; [7] and [9]) is defined as:

$$\mathrm{Li}_2(x) := \sum_{n\geq 1} \frac{x^n}{n^2} = -\int_0^x \frac{log(1-t)}{t} dt, \qquad 0 \leq x \leq 1. \tag{2}$$

is one of the lesser transcendental function. It has many intriguing properties has appeared in various branches of mathematics and physics.

## 2. Main Theorems

In this section, we state and prove a new identity, which adhere the properties of Polylogarithm.

**Theorem 1.** *The following relation holds true;*

$$\sum_{n=1}^{\infty} \frac{b^n}{(1+b)^n} \frac{y^n}{n^2} + \sum_{n=1}^{\infty} (-1)^n \frac{b^n y^n}{n^2} + \sum_{n=1}^{\infty} \frac{b^n}{(1+b)^n} \frac{(1-y)^n}{n^2} + \sum_{n=1}^{\infty} (-1)^n \frac{b^n (1-y)^n}{n^2}$$
$$-\sum_{n=1}^{\infty} \frac{b^n}{(1+b)^n} \frac{1}{n^2} - \sum_{n=1}^{\infty} (-1)^n \frac{b^n}{n^2} = (-1)^n \sum_{n=1}^{\infty} \frac{b^{2n}}{(1+b)^n} \frac{y^n (1-y)^n}{n^2}; \qquad b \neq 0, b \neq y. \tag{3}$$

**Proof.** In order to prove our result, we begin with basic identity;

$$(1+ax)(1+bx) = 1 + (a+b)x + abx^2$$

Consider $a + b = -ab$, after arrangements of terms, we obtain;

$$\frac{1}{(1-\frac{bx}{1+b})} \frac{1}{(1+bx)} = \frac{1}{1+\frac{b^2 x(1-x)}{1+b}}. \tag{4}$$

Applying the properties of logarithm function on both the sides of (4), after simplification we obtain the following series

$$\sum_{n=1}^{\infty}\frac{b^n}{(1+b)^n}\frac{x^n}{n}+\sum_{n=1}^{\infty}\frac{(-1)^n b^n x^n}{n}=\sum_{n=1}^{\infty}\frac{(-1)^n b^{2n}x^n(1-x)^n}{(1+b)^n n}. \tag{5}$$

Multiply the right hand side of (5) with $\frac{(1-2x)}{x(1-x)}$; and after integrating between limits 0 and $y$, we arrived

$$\sum_{n=1}^{\infty}\frac{(-1)^n b^{2n}}{(1+b)^n n}\int_0^y x^{n-1}(1-x)^{n-1}(1-2x)dx=\sum_{n=1}^{\infty}\frac{(-1)^n b^{2n}y^n(1-y)^n}{(1+b)^n n^2}. \tag{6}$$

We can write $\frac{(1-2x)}{x(1-x)}=\frac{1}{x}-\frac{1}{(1-x)}$. Let us multiply by $\frac{1}{x}$ in the left hand side of (5); and then integrate between the limits 0 to $y$, we obtained

$$\sum_{n=1}^{\infty}\frac{b^n}{(1+b)^n}\int_0^y\frac{x^{n-1}}{n}dx+\sum_{n=1}^{\infty}b^n\int_0^y\frac{(-1)^n x^{n-1}}{n}dx=\sum_{n=1}^{\infty}\frac{b^n y^n}{(1+b)^n n^2}+\sum_{n=1}^{\infty}\frac{(-1)^n b^n y^n}{n^2}. \tag{7}$$

When we replace $x\to(1-x)$ in (5), the right hand side is remain the same and we can be write

$$\sum_{n=1}^{\infty}\frac{b^n}{(1+b)^n}\frac{(1-x)^n}{n}+\sum_{n=1}^{\infty}\frac{(-1)^n b^n(1-x)^n}{n}=\sum_{n=1}^{\infty}\frac{(-1)^n b^{2n}x^n(1-x)^n}{(1+b)^n n}. \tag{8}$$

Multiplying left hand side of (8) by $-\frac{1}{(1-x)}$, and integrate between limits 0 to $y$, we obtain

$$-\sum_{n=1}^{\infty}\frac{b^n}{(1+b)^n}\int_0^y\frac{(1-x)^{n-1}}{n}dx-\sum_{n=1}^{\infty}b^n\int_0^y\frac{(-1)^n(1-x)^{n-1}}{n}dx$$

$$=\sum_{n=1}^{\infty}\frac{b^n(1-y)^n}{(1+b)^n n^2}-\sum_{n=1}^{\infty}\frac{b^n}{(1+b)^n n^2}+\sum_{n=1}^{\infty}\frac{(-1)^n(1-y)^n b^n}{n^2}-\sum_{n=1}^{\infty}\frac{(-1)^n b^n}{n^2}. \tag{9}$$

with the help of (6)(7) and (8) we obtained our required identity (3).
We thus have completed our proof of the Theorem 1.

**Note.** The equation (3) can be expressed in terms of the dilogarithm function as follows:

$$\mathrm{Li}_2\left(\frac{by}{1+b}\right)+\mathrm{Li}_2(-by)+\mathrm{Li}_2\left(\frac{b(1-y)}{(1+b)}\right)+\mathrm{Li}_2(-b(1-y))$$

$$-\operatorname{Li}_2\left(\frac{b}{(1+b)}\right)-\operatorname{Li}_2(-b)=\operatorname{Li}_2\left(-\frac{b^2y(1-y)}{(1+b)}\right).$$

## 3. Analysis of Theorem 1.

In order to analyse identity (3) as given in Theorem 1, we split it into parts. First of all we analyse the part $\sum_{n=1}^{\infty}\frac{b^n}{(1+b)^n}\frac{y^n}{n^2}+\sum_{n=1}^{\infty}(-1)^n\frac{b^ny^n}{n^2}$, in this way, multiply it by $y^{\mu-1}$ $(\mu\neq 1)$ and then integrate between the limits 0 and 1, finally we obtain:

$$\begin{aligned}&\int_0^1\sum_{n=1}^{\infty}\frac{b^n}{(1+b)^n}\frac{y^{n+\mu-1}}{n^2}dy+\int_0^1\sum_{n=1}^{\infty}(-1)^nb^n\frac{y^{n+\mu-1}}{n^2}dy\\&=\sum_{n=1}^{\infty}\frac{b^n}{(1+b)^nn^2}\int_0^1y^{n+\mu-1}dy+\sum_{n=1}^{\infty}\frac{(-1)^nb^n}{n^2}\int_0^1y^{n+\mu-1}dy\\&=\sum_{n=1}^{\infty}\frac{b^n}{(1+b)^n}\frac{1}{n^2(n+\mu)}+\sum_{n=1}^{\infty}\frac{(-1)^nb^n}{n^2}\frac{1}{(n+\mu)}\\&=\sum_{n=1}^{\infty}\frac{b^n}{(1+b)^n}\frac{1}{n^3(1+\frac{\mu}{n})}+\sum_{n=1}^{\infty}(-1)^n\frac{b^n}{n^3(1+\frac{\mu}{n})}.\end{aligned}\tag{10}$$

Now, we study the sum of the series $\sum_{n=1}^{\infty}\frac{b^n}{(1+b)^n}\frac{(1-y)^n}{n^2}+\sum_{n=1}^{\infty}(-1)^n\frac{b^n(1-y)^n}{n^2}$, multiply by $y^{\mu-1}$; and integrate with respect to $y$ between limits 0 to 1, we have;

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{b^n}{(1+b)^n}\frac{1}{n^2}\int_0^1y^{\mu-1}(1-y)^ndy+\sum_{n=1}^{\infty}\frac{(-1)^nb^n}{n^2}\int_0^1y^{\mu-1}(1-y)^ndy\\&=\sum_{n=1}^{\infty}\frac{b^n}{(1+b)^n}\frac{1}{n^2}\frac{n!}{\mu(\mu+1)(\mu+2)\cdots(\mu+n)}+\sum_{n=1}^{\infty}\frac{(-1)^nb^n}{n^2}\frac{n!}{\mu(\mu+1)(\mu+2)\cdots(\mu+n)}\\&=\sum_{n=1}^{\infty}\frac{b^n}{(1+b)^n}\frac{1}{n^2}\frac{1}{\mu(1+\mu)(1+\frac{\mu}{2})\cdots(1+\frac{\mu}{n})}+\sum_{n=1}^{\infty}\frac{(-1)^nb^n}{n^2}\frac{1}{\mu(1+\mu)(1+\frac{\mu}{2})\cdots(1+\frac{\mu}{n})}.\end{aligned}\tag{11}$$

Further, we the sums of the series $-\{\sum_{n=1}^{\infty}\frac{b^n}{(1+b)^n}\frac{1}{n^2}+\sum_{n=1}^{\infty}(-1)^n\frac{b^n}{n^2}\}$, multiply by $y^{\mu-1}$; and integrate with respect to $y$ between limits 0 to 1, we have;

$$\begin{aligned}&-\left\{\sum_{n=1}^{\infty}\frac{b^n}{(1+b)^n}\frac{1}{n^2}\int_0^1y^{\mu-1}dy+\sum_{n=1}^{\infty}(-1)^n\frac{b^n}{n^2}\int_0^1y^{\mu-1}dy\right\}\\&\qquad-\left\{\sum_{n=1}^{\infty}\frac{b^n}{(1+b)^n}\frac{1}{\mu.n^2}+\sum_{n=1}^{\infty}(-1)^n\frac{b^n}{\mu.n^2}\right\}.\end{aligned}\tag{12}$$

At last, we study the sum of the series at the right hand side $\sum_{n=1}^{\infty}\frac{b^{2n}}{(1+b)^n}\frac{y^n(1-y)^n}{n^2}$, multiply by $y^{\mu-1}$; and integrate with respect to $y$ between limits 0 to 1, we have;

$$\sum_{n=1}^{\infty}\frac{b^{2n}}{(1+b)^n}\frac{1}{n^2}\int_0^1 y^{n+\mu-1}(1-y)^n dy=\sum_{n=1}^{\infty}\frac{b^{2n}}{(1+b)^n}\frac{1}{n^2}\frac{(\mu+n)!n!}{(\mu+2n+1)}$$

$$=\sum_{n=1}^{\infty}\frac{b^{2n}}{(1+b)^n}\frac{1}{n^2}\frac{n!}{(\mu+n)(\mu+n+1)\cdots(\mu+2n)}$$

$$=\sum_{n=1}^{\infty}\frac{b^{2n}}{(1+b)^n}\frac{1}{n^3}\frac{1}{(1+\frac{\mu}{n})(1+\frac{\mu}{(n+1)})\cdots(1+\frac{\mu}{2n})}. \tag{13}$$

Combining identities (10)-(13) together, we have;

$$\sum_{n=1}^{\infty}\frac{b^n}{(1+b)^n}\frac{1}{n^2}\frac{1}{\mu(1+\mu)(1+\frac{\mu}{2})\cdots(1+\frac{\mu}{n})}+\sum_{n=1}^{\infty}\frac{(-1)^n b^n}{n^2}\frac{1}{\mu(1+\mu)(1+\frac{\mu}{2})\cdots(1+\frac{\mu}{n})}$$

$$+\sum_{n=1}^{\infty}\frac{b^n}{(1+b)^n}\frac{1}{n^3(1+\frac{\mu}{n})}+\sum_{n=1}^{\infty}(-1)^n\frac{b^n}{n^3(1+\frac{\mu}{n})}-\sum_{n=1}^{\infty}\frac{b^n}{(1+b)^n}\frac{1}{\mu.n^2}+\sum_{n=1}^{\infty}(-1)^n\frac{b^n}{\mu.n^2}$$

$$=\sum_{n=1}^{\infty}\frac{b^{2n}}{(1+b)^n}\frac{1}{n^3}\frac{1}{(1+\frac{\mu}{n})(1+\frac{\mu}{(n+1)})\cdots(1+\frac{\mu}{2n})}. \tag{14}$$

We can develop (3) into series of $O(\mu^n)$ coefficients, upon analysing each term, we obtain an infinite set of polylogarithm's identities. Further, we obtain a series of $O(\mu^0)$ coefficients, as;

$$\sum_{n=1}^{\infty}(-1)^n\frac{a^{2n}}{(1+a)^n n^3\cdot {}^{2n}c_n}=\sum_{n=1}^{\infty}\frac{a^n}{n^3(1+a)^n}+\sum_{n=1}^{\infty}(-1)^n\frac{a^n}{n^3}$$

$$-\sum_{n=1}^{\infty}(-1)^n\frac{a^n}{n^2}\left\{1+\frac{1}{2}\cdots+\frac{1}{n}\right\}-\sum_{n=1}^{\infty}\frac{a^n}{(1+a)^n n^2}\left\{1+\frac{1}{2}\cdots+\frac{1}{n}\right\}. \tag{16}$$

**4. Further Result to the Theorem 1.**

We begin with our main result (theorem 1) of section 2. Let us multiply in the right hand side of equation (3) by $\frac{1-2y}{y(1-y)}=\frac{1}{y}-\frac{1}{(1-y)}$and then integrate between limits 0 and $t$, we obtain following;

$$\sum_{n=1}^{\infty}\frac{b^{2n}(-1)^n}{(1+b)^n n^2}\int_0^t\frac{y^n(1-y)^n(1-2y)}{y(i-y)}dy=\sum_{n=1}^{\infty}\frac{b^{2n}(-1)^n}{(1+b)^n n^2}\int_0^t y^{n-1}(1-y)^{n-1}(1-2y)dy$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n b^{2n}}{n^2(1+b)^n} \frac{t^n(1-t)^n}{n^3}. \tag{16}$$

Now, we multiply in the left hand side of equation (3) by $\frac{1}{y}$, upon integrating between limits 0 and $t$, we obtain;

$$\sum_{n=1}^{\infty} \frac{b^n}{(1+b)^n n^2} \int_0^t y^{n-1} dy + \sum_{n=1}^{\infty} (-1)^n \frac{b^n}{n^2} \int_0^t y^{n-1} dy + \sum_{n=1}^{\infty} \frac{b^n}{(1+b)^n n^2} \int_0^t \frac{(1-y)^n - 1}{y} dy$$
$$+ \sum_{n=1}^{\infty} \frac{(-1)^n b^n}{n^2} \int_0^t \frac{(1-y)^n - 1}{y} dy$$
$$= \sum_{n=1}^{\infty} \frac{b^n}{(1+b)^n} \frac{t^n}{n^3} + \sum_{n=1}^{\infty} \frac{(-1)^n b^n}{n^3} - \left( \sum_{n=1}^{\infty} \frac{b^n}{(1+b)^n n^2} + \sum_{n=1}^{\infty} \frac{(-1)^n b^n}{n^2} \right) \int_0^t \sum_{i=0}^{(n-1)} (1-y)^i dy$$
$$= \sum_{n=1}^{\infty} \frac{b^n}{(1+b)^n} \frac{t^n}{n^3} + \sum_{n=1}^{\infty} \frac{(-1)^n b^n}{n^3} + \left( \sum_{n=1}^{\infty} \frac{b^n}{(1+b)^n n^2} + \sum_{n=1}^{\infty} \frac{(-1)^n b^n}{n^2} \right) \sum_{i=0}^{n} \frac{(1-t)^i}{i}$$
$$- \left( \sum_{n=1}^{\infty} \frac{b^n}{(1+b)^n n^2} + \sum_{n=1}^{\infty} \frac{(-1)^n b^n}{n^2} \right) \sum_{i=0}^{n} \frac{1}{i}. \tag{17}$$

**Remark:** *In order to establish (17), we use following results;*

1. $\frac{(1-y)^n - 1}{y} = \frac{[(1-y)-1][\sum_{i=0}^{n-1}(1-y)^i]}{y} = -\sum_{i=0}^{n-1}(1-y)^i$;
2. $\sum_{i=0}^{n-1} \int_0^t (1-y)^i dy = -\sum_{i=0}^{n-1} \int_1^{1-t} x^i dx = -\sum_{i=0}^{n-1} [\frac{(1-t)^{i+1}}{1+i} - \frac{1}{1+i}]$.

Further, multiply the left hand side of equation (3) by $-\frac{1}{1-y}$, and integrate between limits 0 and $t$, we obtain;

$$-\sum_{n=1}^{\infty} \frac{b^n}{(1+b)^n} \int_0^t \frac{y^n}{1-y} dy - \sum_{n=1}^{\infty} (-1)^n \frac{b^n}{n^2} \int_0^t \frac{y^n}{1-y} dy - \sum_{n=1}^{\infty} \frac{b^n}{(1+b)^n n^2} \int_0^t (1-y)^{n-1} dy$$
$$-\sum_{n=1}^{\infty} \frac{(-1)^n b^n}{n^2} \int_0^t (1-y)^{n-1} dy + \sum_{n=1}^{\infty} \frac{b^n}{(1+b)^n n^2} \int_0^t \frac{1}{1-y} dy + \sum_{n=1}^{\infty} (-1)^n \frac{b^n}{n^2} \int_0^t \frac{1}{1-y} dy$$
$$= \sum_{n=1}^{\infty} \frac{b^n}{(1+b)^n n^2} \sum_{i=1}^{n} \frac{t^i}{i} + \sum_{n=1}^{\infty} (-1)^n \frac{b^n}{n^2} \sum_{i=1}^{n} \frac{t^i}{i} + \sum_{n=1}^{\infty} \frac{b^n (1-t)^n}{(1+b)^n n^3} + \sum_{n=1}^{\infty} \frac{(-1)^n b^n (1-t)^n}{n^3}$$
$$-\sum_{n=1}^{\infty} \frac{b^n}{(1+b)^n n^3} - \sum_{n=1}^{\infty} (-1)^n \frac{b^n}{n^3}. \tag{18}$$

using $\int_0^t \frac{y^n}{1-y}dy = \int_0^t \sum_{i=n}^{\infty} y^i dy = Ln\frac{1}{1-t} - \sum_{i=1}^{n} \frac{t^i}{i}$.

Now combining (16), (17) and (18), we obtain following result corresponding to the main result (3) of our theorem 1;

$$\sum_{n=1}^{\infty} \frac{b^n t^n}{(1+b)^n n^3} + \sum_{n=1}^{\infty} \frac{(-1)^n b^n t^n}{n^3} + \left(\sum_{n=1}^{\infty} \frac{b^n}{(1+b)^n n^2} + \sum_{n=1}^{\infty} \frac{(-1)^n b^n}{n^2}\right) \sum_{i=1}^{n} \frac{(1-t)^i}{i}$$
$$-\left(\sum_{n=1}^{\infty} \frac{b^n}{(1+b)^n} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{(-1)^n b^n}{n^2}\right) \sum_{i=1}^{n} \frac{1}{i} + \left(\sum_{n=1}^{\infty} \frac{b^n}{(1+b)^n} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{(-1)^n b^n}{n^2}\right) \sum_{i=1}^{n} \frac{t^i}{i}$$
$$+\sum_{n=1}^{\infty} \frac{b^n (1-t)^n}{(1+b)^n n^3} + \sum_{n=1}^{\infty} \frac{(-1)^n b^n (1-t)^n}{n^3} - \sum_{n=1}^{\infty} \frac{b^n}{(1+b)^n n^3} - \sum_{n=1}^{\infty} \frac{(-1)^n b^n}{n^3}$$
$$= \sum_{n=1}^{\infty} \frac{(-1)^n b^{2n}}{(1+b)^n} \frac{t^n (1-t)^n}{n^3}. \tag{19}$$

## 5. Concluding Remarks and Observations

The present investigation was motivated by several recent developments dealing essentially with dilogarithm and polylogarithm (for example see [3] and [16]). Here, in this article, we have established presumably a set of new identities. We have also considered several closely-related identities such as (for example) polylogarithm (also known as Jonquière's function) and Euler dilogarithm function etc. We have chosen to indicate rather briefly a number of recent developments on the subject-matter of this article.

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