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RELATIONS BETWEEN MOCK THETA FUNCTIONS AND COMBINATORIAL PARTITION IDENTITIES

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Abstract: The main object of this paper is to present 6 new interrelationships between mock theta functions and combinatorial partition identities. The results presented in this paper are motivated by some recent works by M.P.Chaudhary [6].

Keywords and Phrases: Mock theta functions, q-product identities, q-series.

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1. Introduction and Definitions

In his last letter to Hardy, dated three months before his death in early 1920, (see [6], pp. 33-34; [8], pp. 354-355; [10], pp. 127-131), Ramanujan gave a list of

17 functions which he called "mock theta functions". He separated these functions into three groups, which were described as four of third order, ten of fifth order, and three of seventh order. Further, the fifth order mock theta functions he divided into two groups. The mock theta functions are functions of a complex variable q, defined by q-series convergent for $|q| < 1$. He stated that they have certain asymptotic properties as q approaches a root of unity, similar to the properties of theta functions.

Throughout this paper, we denote by $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{C}$ the set of positive integers, the set of integers and the set of complex numbers, respectively. We also let

$$\mathbb{N}_0 := \mathbb{N} \cup \{0\} = \{0, 1, 2, \cdots\}.$$

The q-shifted factorial $(a; q)_n$ is defined (for $|q| < 1$) by

$$(a;q)_n := \begin{cases} 1 & (n = 0) \\ \prod_{k=0}^{n-1} (1 - aq^k) & (n \in \mathbb{N}), \end{cases}$$

where $a, q \in \mathbb{C}$ and it is assumed *tacitly* that $a \neq q^{-m}$ $(m \in \mathbb{N}_0)$. We also write

$$(a;q)_\infty := \prod_{n=0}^{\infty} (1 - aq^n) = \prod_{n=1}^{\infty} (1 - aq^{n-1}) \qquad (a, q \in \mathbb{C};\ |q| < 1).$$

It should be noted that, when $a \neq 0$ and $|q| \geqq 1$, the infinite product in the equation (2) diverges. So, whenever $(a; q)_\infty$ is involved in a given formula, the constraint $|q| < 1$ will be *tacitly* assumed to be satisfied.

The following notations are also frequently used in our investigation:

$$(a_1, a_2, a_3, \cdots, a_k; q)_n = (a_1; q)_n (a_2; q)_n (a_3; q)_n \ldots (a_k; q)_n$$

and

$$(a_1, a_2, a_3, \cdots, a_k; q)_\infty = (a_1; q)_\infty (a_2; q)_\infty (a_3; q)_\infty \ldots (a_k; q)_\infty.$$

Ramanujan (see [9] and [10]) defined the general theta function $\mathfrak{f}(a, b)$ as follows:

$$\begin{aligned} \mathfrak{f}(a,b) &= 1 + \sum_{n=1}^{\infty} (ab)^{\frac{n(n-1)}{2}} (a^n + b^n) \\ &= \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} = \mathfrak{f}(b,a) \qquad (|ab| < 1). \end{aligned}$$

where a and b are two complex numbers. The three most important special cases of $\mathfrak{f}(a,b)$ are defined as:

$$\phi(q) = \mathfrak{f}(q,q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty = \frac{(-q;q^2)_\infty (q^2;q^2)_\infty}{(q;q^2)_\infty (-q^2;q^2)_\infty},$$

$$\psi(q) = \mathfrak{f}(q,q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}$$

and

$$f(-q) = \mathfrak{f}(-q,-q^2) = \sum_{n=-\infty}^{+\infty} (-1)^n q^{\frac{n(3n-1)}{2}} = (q;q)_\infty.$$

The last equality is known as *Euler's Pentagonal Number Theorem.* Remarkably, the following q-series identity:

$$(-q;q)_\infty = \frac{1}{(q;q^2)_\infty} = \frac{1}{\chi(-q)}.$$

provides the analytic equivalent form of Euler's famous theorem.

Ramanujan also defined the following function

$$\chi(q) = (-q;q^2)_\infty.$$

We also recall the Rogers-Ramanujan continued fraction $R(q)$ given by

$$\begin{aligned} R(q) &:= q^{\frac{1}{5}} \frac{H(q)}{G(q)} = q^{\frac{1}{5}} \frac{\mathfrak{f}(-q,-q^4)}{\mathfrak{f}(-q^2,-q^3)} = q^{\frac{1}{5}} \frac{(q;q^5)_\infty \, (q^4;q^5)_\infty}{(q^2;q^5)_\infty \, (q^3;q^5)_\infty} \\ &= \frac{q^{\frac{1}{5}}}{1+} \, \frac{q}{1+} \, \frac{q^2}{1+} \, \frac{q^3}{1+} \qquad (|q|<1). \end{aligned}$$

Here $G(q)$ and $H(q)$, which are associated with the widely-investigated Rogers-Ramanujan identities, are defined as follows:

$$\begin{aligned} G(q) &:= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{f(-q^5)}{\mathfrak{f}(-q,-q^4)} \\ &= \frac{1}{(q;q^5)_\infty \, (q^4;q^5)_\infty} = \frac{(q^2;q^5)_\infty \, (q^3;q^5)_\infty \, (q^5;q^5)_\infty}{(q;q)_\infty} \end{aligned} \tag{1}$$

and

$$H(q)=\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(q;q)_n}=\frac{f(-q^5)}{\mathfrak{f}(-q^2,-q^3)}=\frac{1}{(q^2;q^5)_\infty(q^3;q^5)_\infty}=\frac{(q;q^5)_\infty(q^4;q^5)_\infty(q^5;q^5)_\infty}{(q;q)_\infty} \tag{2}$$

and the functions $\mathfrak{f}(a,b)$ and $f(-q)$ are given by the equations (5) and (8), respectively.

Remark 1.1.

- *In the equation (12), the left side can be interpreted as the number of partitions of n whose parts differ by at least 2, and the right side is the number of partitions of n in parts congruent to 1 or 4 modulo 5.*
- *In the equation (13), the left-hand side is the generating series of partitions into n parts such that two adjacent parts differ by is at least 2 and such that the smallest part is at minus 2. The right side $\frac{1}{(q^2;q^5)_\infty(q^3;q^5)_\infty}$ is the generating series of the partitions such that each part is congruent to 2 or 3 modulo 5. Then, the number of partitions of n such that two adjacent shares differ by at least 2 and such that the smallest part is at least 2 is equal to the number of partitions of n such that each part is congruent to 2 or 3 modulo 5.*

Andrews et al. [4], introduces the general family $R(s,t,l,u,v,w)$ as follows:

$$R(s,t,l,u,v,w):=\sum_{n=0}^{\infty}q^{s\binom{n}{2}+tn}r(l,u,v,w:n), \tag{3}$$

where

$$r(l,u,v,w:n):=\sum_{j=0}^{[\frac{n}{u}]}(-1)^j\frac{q^{uv\binom{j}{2}+(w-ul)j}}{(q;q)_{n-uj}(q^{uv};q^{uv})_j}. \tag{4}$$

In the following proposition, we give three particular cases of double q-hypergeometric series R.

Proposition 1.1. *Recently (see* [11], [7]*), following notations have been introduced:*

$$R_\alpha(q):=R(2,1,1,1,2,2)=(-q;q^2)_\infty, \tag{5}$$

$$R_\beta(q):=R(2,2,1,1,2,2)=(-q^2;q^2)_\infty \tag{6}$$

and

$$R_m:=R(m,m,1,1,1,2)=\frac{(q^{2m};q^{2m})_\infty}{(q^m;q^{2m})_\infty},\ (m\in\mathbb{N}^*). \tag{7}$$

2. Third and Fifth Order Mock Theta Functions

Third order mock theta functions are defined as (see [6]):

$$f_3(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q)_n^2}, \tag{8}$$

$$\chi_3(q) := 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{\prod_{m=1}^{n}(1 - q^m + q^{2m})}, \tag{9}$$

$$\omega_3(q) := \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q^2)_{n+1}^2} \tag{10}$$

and

$$\rho_3(q) := \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{\prod_{m=1}^{n}(1 + q^{2m+1} + q^{4m+2})}. \tag{11}$$

Fifth order mock theta functions are defined as (see [12]; [6]):

$$\chi_0(q) = \sum_{n\geq 0}^{\infty} \frac{q^n}{(q^{n+1};q)_n} \tag{12}$$

and

$$\chi_1(q) = \sum_{n\geq 0}^{\infty} \frac{q^n}{(q^{n+1};q)_{n+1}}. \tag{13}$$

Where all symbols and notations are having their usual meaning.

Combinatorial Interpretation (see [1])

- $q\chi_1(q)$ is the generating function for partitions in which no part is as large as twice the smallest part.
- $\chi_0(q)$ is the generating function for partitions with unique smallest part and the largest part at most twice the smallest part.

In [5], following identities have been recorded:

$$\begin{aligned}\rho_3(q) + \rho_3(-q) &+ \frac{1}{2}\left(\omega_3(q) + \omega_3(-q)\right) \\ &= \frac{3(q^{12};q^{12})_\infty(-q^{12};q^{24})_\infty^2(q^{24};q^{24})_\infty}{(q^2;q^2)_\infty(q^6;q^{12})_\infty}.\end{aligned} \tag{14}$$

$$(q^2;q^2)_\infty(\rho_3(q)+\frac{1}{2}\omega_3(q))=\frac{3}{2}\frac{(q^6;q^6)^2_\infty}{(q^3;q^6)^2_\infty}. \tag{15}$$

$$\begin{aligned} B(z;q) &= \sum_{-\infty}^{\infty}\frac{(-1)^n q^{n(n+1)/4}}{1+zq^{(n+1)/2)}} \\ &= \frac{(q^{1/2},q)_\infty(q^2,zq,z^{-1}q;q^2)_\infty}{(-zq^{1/2},-z^{-1/2}q^{1/2})_\infty}. \end{aligned} \tag{16}$$

$$A(-q^2;q^8)=\sum_{-\infty}^{\infty}\frac{(-1)^n q^{4n(n+1)}}{1-q^{(8n+2)}}. \tag{17}$$

$$A(-q^2;q^8)=\frac{(q^8;q^8)^2_\infty}{(q^2;q^4)_\infty}. \tag{18}$$

$$V_1(q)-V_1(-q)=2q\frac{(-q^4;q^4)_\infty}{(q^4;q^4)_\infty}\sum_{-\infty}^{\infty}\frac{(-1)^n q^{4n(n+1)}}{1-q^{(8n+2)}}. \tag{19}$$

$$U_0(q)+2U_1(q)=(-q;q^2)^3_\infty(q^2;q^2)_\infty(q^2;q^4)_\infty. \tag{20}$$

$$U_0(-q)+2u_1(-q)=2\frac{(q;q^2)_\infty}{q^2;q^2)_\infty}. \tag{21}$$

3. Main Results

We first state our main results as follows.

Theorem 3.1. *Each of the following inter-relations between mock theta functions and combinatorial partition identities holds true:*

$$\begin{aligned} \rho_3(q)+\rho_3(-q) &+ \frac{1}{2}\left(\omega_3(q)+\omega_3(-q)\right) \\ &= 3R(6,6,1,1,1,2)R^2_\alpha(q^6)\frac{(q^{24};q^{24})_\infty}{(q^2;q^2)_\infty}. \end{aligned} \tag{22}$$

$$(q^2;q^2)_\infty(\rho_3(q)+\frac{1}{2}\omega_3(q))=\frac{3}{2}R^2(3,3,1,1,1,2). \tag{23}$$

$$V_1(q)-V_1(-q)=2qR_\beta(q)R^2_\beta(q^2)(q^8;q^8)_\infty. \tag{24}$$

$$V_1(q)-V_1(-q)=2qR_\beta(q^2)\frac{(q^8;q^8)_\infty}{(q^2;q^2)}. \tag{25}$$

$$U_0(q)+2U_1(q)=R_\alpha(q)\phi(q)(q^2;q^4)_\infty. \tag{26}$$

$$U_0(-q) + 2U_1(-q) = \frac{2B(q^{-1}; q^2)}{R_1}. \tag{27}$$

Proof of Theorem 3.1. In the proof of Theorem 1, first of all, in order to prove our first assertion (22), we apply the identity (14) into identity (5) and further apply identity (7), after computation by means of use of little algebra, we obtain (22). Now, we have to attempt for proof of our second assertion (23), use the identity (15) into identity (7), after computation by means of use of little algebra, we obtain (23). Similarly, with help of identities (19) and (6), and by use of little algebra, we obtain (24).
Further, we can prove other three identities (25)-(27) on the similar techniques, which are easy and left for the readers.
We thus have completed our proof of the Theorem 3.1.

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