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COMPOSITION OF PATHWAY FRACTIONAL INTEGRAL OPERATOR ON PRODUCT OF SPECIAL FUNCTIONS

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Abstract: In this paper, we study the pathway fractional integral operator colluded with composition of K-Struve function and extended Mittag-Leffler function. The obtained result is expressed in terms of generalized Wright hypergeometric function.

Keywords and Phrases: Pathway fractional integral operator, generalized hypergeometric function, Struve function, K-Struve function, extended Mittag- Leffler function.

2020 Mathematics Subject Classification: 22A33, 33C65, 33C20.

1. Introduction and Definitions

In this paper, let R and C denotes the sets of real and complex numbers, respectively, and also let $R^+(0.\infty)$.

1.1. Pathway fractional integral operator

Nair developed the Pathway fractional integral operator by utilizing Mathai's pathway concept. In this paper, we aim to develop a new fractional integration formula using the generalized K-Wright function [8, 9]. Let $g(x) = L(p, q)$,$\mu \in C, Re(\mu) > 0, p > 0$ and γ is taken as pathway parameter such that $\gamma < 1$ and the pathway fractional integral operator is defined as

$$(P_{(o+)}^{(\mu,\gamma,p)})(x) = x^{\mu} \int_0^{\frac{x}{(p(1-\gamma)}} \left[\frac{p(1-\gamma)t}{x}\right]^{\frac{\mu}{(1-\gamma)}} g(t)dt, \tag{1}$$

where $L(p.q)$ is a set of Lebesgue measurable function defined on (p,q). Basically, the pathway model was studied by Mathai [10] and after that it was further studied by Haubold and Mathai [11]. If we set $\gamma = 0$, $p = 1$ and replacing μ by $\mu - 1$ in equation 1 then we have the following relationship i.e.,

$$(P^{(\mu-1,0,1)}_{(0+)})(x) = \Gamma(\mu)[(I^{\mu}_{0+}g)(x)]$$

where, $I^{\mu}_{(0+)}$ is the left sided Riemann- Liouville fractional integral operator.

1.2. K-Struve Function

Recently, Nisar et al. studied various properties of Struve function and introduced K-Struve function $S^k_{(\alpha,z)}$ is defined by [5, 6, 11, 12, 21]

$$S^k_{\alpha,f}(d) = \sum_{m=0}^{\infty} \frac{(-f)^m}{\Gamma_k(mk+\alpha+\frac{3k}{2})\Gamma(m+\frac{3}{2})}(\frac{d}{2})^{2m+\frac{\alpha}{k}+1} \tag{2}$$

where, $d, \alpha \in C, \alpha > \frac{3k}{2}$. After this the generalized Wright hypergeometric function ${}_a\psi_b(c)$ is given by the series,

$${}_a\psi_b(c) ={}_a \psi_b\left[\begin{matrix}(p_i,\gamma_j)_{1,a}\\(q_i,v_j))_{(1,b)}\end{matrix}|z\right] = \sum_{n=0}^{\infty} \frac{\prod_{i=0}^{a}\Gamma(p_i+\gamma_j k)}{\prod_{j=0}^{b}\Gamma(q_j+v_j k)}\frac{c^n}{n!}$$

where $p_i, q_j \in C$, and real $\gamma_i, \nu_j \in R(i = 1,2,3,\ldots\ldots, a; j = 1,2,\ldots\ldots b)$. Asymptotic behavior of this function for large values of argument $c \in C$ has been studied. In the work of E. M. Wright is has been found that,

$$\sum_{j=1}^{b}\nu_j - \sum_{i=1}^{a}\gamma_j > -1$$

When properties of the generalized Wright function were investigated and proved that ${}_a\psi_b(c), c \in C$ is an entire function under the condition.vspace.2cm

1.3. Extended Mittag-Leffler function

Gosta Mittag-Leffler the Swedish mathematician introduced the termed Gosta Mittag-Leffler function i.e., Mittag-Leffler function, and is defined as [1, 2],

$$E_{\gamma}(d) = \sum_{n=0}^{\infty}\frac{(d)^n}{\Gamma(\gamma n+1)} \qquad (d \in C; R(\gamma) > 0 \tag{3}$$

where Γ is a gamma function, after this Wiman generalized [15] the Mittag-Leffler function as follows,

$$E_{\gamma,\nu}(d) = \sum_{n=0}^{\infty}\frac{(d)^n}{\Gamma(\gamma n+\nu)} \qquad (d \in C, min(R(\gamma)R(\nu)) > 0$$

There are number of ways in which Mittag-Leffler function E_γ and the extended Mittag-Leffler function $E_{(\gamma,\nu)}$ can be extended and used in various research area [16, 20]. Prabhakar again introduced the another extension of this function $E_{(\gamma,\nu)}$ was introduced by Prabhakar Kumar and is defined as,

$$E^{\xi}_{\gamma,\nu}(d) = \sum_{n=0}^{\infty} \frac{\xi_n}{\Gamma(\gamma n+\nu)} \frac{(d)^n}{n!} \qquad d \in C, min(R(\gamma), R(\nu), R(\xi)) > 0, \tag{4}$$

Again, Shukla and Prajapati defined the new extension of this function i.e.,

$$E^{\xi}_{\gamma,\nu}(d) = \sum_{n=0}^{\infty} \frac{\xi_n}{\Gamma(\gamma n+\nu)} \frac{(d)^n}{n!} \qquad d \in C, min(R(\gamma), R(\nu), R(\xi)) > 0, a \in (0,1) \cup N \tag{5}$$

Then after Salim and Faraj has also given a new extension of this function and Ozarslan and Yilmaz presented this following new extension,

$$E^{\xi;f}_{\gamma,\nu}(d;a) = \sum_{n=0}^{\infty} \frac{B_a(\xi+n, f-\xi)}{B(\xi, f-\xi)} \frac{(f_n)}{\Gamma(\gamma n+\nu)} \frac{(d)_n}{n!} \tag{6}$$

$(d \in C; min(R(\gamma), R(\nu)) > 0; R(z) > R(\xi) > 0; a \in R_0^+$
So, by considering Eq. 6 and Eq. 7 we have concluded by defining a new extension of this function,

$$E^{\xi,f,a}_{\varphi,\nu,\tau}(d;a) = \sum_{n=0}^{\infty} \frac{B_a(\xi+nb, f-\xi)}{B(\xi, f-\xi)} \frac{(f)_{n,b}}{\Gamma(\varphi n+\nu)} \frac{(d)^n}{(\tau)_{nr}} \tag{7}$$

$(b \in R^+; min(R(\varphi), R(\nu), R(\tau) > 0; R(\xi) > 0; a \in R_0^+)$
Where $B_a(s,t)$ is the extended beta function,

$$B_a(u,v) = \int_0^1 t^{u-1}(1-t)^{v-1} e^{\frac{-a}{t(1-t)}} dt \qquad a \in R_0^+; min(R(u), R(v) > 0$$

If $a = 0,$ it reduces to the particular case of well-known beta function

$$B(u,v) = \int_0^1 t^{u-1}(1-t)^{v-1} dt \qquad min(R(u), R(v)) > 0 \tag{8}$$

$$= \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)} \qquad u,v \in C/Z_0^- \tag{9}$$

And $(f)_{nb} = \frac{\Gamma(f+nb)}{\Gamma(f)},$ is the extended pochammer symbol.

2. Preliminary Theorems

In this section Pathway fractional integration of composition of two function i.e., K-Struve function $S^k_{(\alpha,f)}$ and extended Mittag-Leffler function $E^{(\xi,f,a,m)}_{(\varphi,\nu,\tau)}(d;a)$ has been done and then the required result has been converted to generalized K-Wright function. As we know K-Struve function is,

$$S^k_{\alpha,f}(d) = \sum_{m=0}^{\infty} \frac{(-f)^m}{\Gamma_k(mk+\alpha+\frac{3k}{2})\Gamma(m+\frac{3}{2})}(\frac{d}{2})^{2m+\frac{\alpha}{k}+1}$$

And extended Mittag-Leffler function is,

$$E^{\xi,f,a}_{\varphi,\nu,\tau}(d;a) = \sum_{n=0}^{\infty} \frac{B_a(\xi+nb, f-\xi)}{B(\xi, f-\xi)} \frac{(f)_{n,b}}{\Gamma(\varphi n+\nu)} \frac{(d)^n}{(\tau)_{nr}}$$

Now we will combine both the function by replacing m=n=e and getting the required result,

$$[S^k_{\alpha,f}(d)E^{\xi,f,a}_{\varphi,\nu,\tau}(d;a)](d) = \sum_{e=0}^{\infty} \frac{(-f)^e B_a(\xi+eb)(f)_{eb}}{B(\xi,f-\xi)\Gamma(\varphi e+\nu)\Gamma_k(ek+\alpha+\frac{3k}{2})\Gamma(e+\frac{3}{2})(\tau)_{er}(2)^{2e+\frac{\alpha}{k}+1}} \times (d)^{3e+\frac{\alpha}{k}+1}$$

As we know, $\Gamma_k(\varsigma) = k^{\frac{\varsigma}{k}-1}\Gamma(\frac{\varsigma}{k})$, using this above composition can be transformed into,

$$[S^k_{\alpha,f}(d)E^{\xi,f,a}_{\varphi,\nu,\tau}(d;a)](d) = \sum_{e=0}^{\infty} \frac{(-f)^e B_a(\xi+eb)(f)_{e,b}}{B(\xi,f-\xi)\Gamma(\varphi e+\nu)k^{e+\frac{\alpha}{k}+\frac{1}{2}}\Gamma(e+\frac{\alpha}{k})\Gamma(e+\frac{3}{2})(\tau)_{er}(2)^{2e+\frac{\alpha}{k}+1}} \times (d)^{3e+\frac{\alpha}{k}+1}$$

Lemma 2.1. (Agarwal [4])
Let $\mu \in C, R(\mu) > 0, \nu \in C$ and $\gamma < 1$, if $R(\nu) > 0$ and $R(\frac{\mu}{1-\gamma}) > -1$ then,

$$P^{\mu,\gamma,p}_{0+}[t^\nu - 1](x) = \frac{x^{\mu+\nu}\Gamma(\nu)\Gamma(1+\frac{\mu}{1-\gamma})}{[p(1-\gamma)]^\nu\Gamma(1+\frac{\mu}{1-\gamma}+\nu)} \tag{10}$$

The pathway fractional integration of the composition of these two above functions is given by the following result in the next section.

3. Main Theorems

In this section we will show the new result by applying pathway integral operator on the above composition of two function i.e., K-Struve function and extended

Mittag-Leffler function [3] [7] [13] [18] [19].

Theorem 3.1. Let $\mu, \varphi, \alpha, d, \nu, \tau, \xi \in C$ and $\gamma < 1$ be such that $R(\mu), R(\varphi), R(\nu), R(\tau) > 0$, $R(d) > R(\xi) > 0$ with $a \geq 0, r > 0$ and $0 < b \leq r + R(\varphi)$ also $\alpha > \frac{-3}{2}k$ and $R(\frac{\mu}{1-\gamma}) > -1$, then the following formula hold true.

$$P_{0+}^{\mu,\gamma,p}[d^{\varphi-1}(S_{\alpha,f}^{k}(d))(E_{\varphi,\nu,\tau}^{\xi,f,a}(d;a))](z)$$
$$= \frac{d^{\mu+\frac{\alpha}{k}+\varphi+1}B_a(\xi+eb, f-\xi)\Gamma(1+\frac{\mu}{1-\gamma})}{(2)^{\frac{\alpha}{k}+1}(k)^{\frac{\alpha}{k}+\frac{1}{2}}[p(1-\gamma)]^{\frac{\alpha}{k}+\varphi+1}B(\xi, c-\xi)(\tau)_{er}\Gamma(c)}$$
$$\times {}_3\psi_4\left[\begin{matrix} & (\varphi+\frac{\alpha}{k}+1, 3) & (f, b) & (1,1) \\ (\frac{\alpha}{k}+\frac{3}{2}, 1) & (\frac{3}{2}, 1) & (\nu, \varphi) & (\frac{\alpha}{k}+\frac{\mu}{(1-\gamma)}+\varphi+2, 3)\end{matrix}; \left[\frac{-fd^3}{4kp^3(1-\gamma)^3}\right]^e\right]$$

Proof. By changing the integration and summation orders and using the pathway operator from lemma 2.1, we obtain

$$P_{0+}^{\mu,\gamma,p}[d^{\varphi-1}(S_{\alpha,f}^{k}(d)(E_{\varphi,\nu,\tau}^{\xi,f,a}(d;a))](z)$$
$$= P_{0+}^{\mu,\gamma,p}\left[d^{\varphi-1}\sum_{e=0}^{\infty}\frac{(-f)^e B_a(\xi+eb, f-\varphi)(f)_{e,b}}{k^{e+\frac{\alpha}{k}+\frac{1}{2}}B(\xi, f-\xi)\Gamma(e+\frac{\alpha}{k}+\frac{3}{2})\Gamma(e+\frac{3}{2})\Gamma(\varphi e+\nu)(\tau)_{er}(2)^{2e+\frac{\alpha}{k}+1}} \times d^{3e+\frac{\alpha}{k}+1}\right](d)$$
$$= \sum_{e=0}^{\infty}\frac{(-f)^e B_a(\xi+eb, f-\xi)(f)_{e,b}}{k^{e+\frac{\alpha}{k}+\frac{1}{2}}B(\xi, f-\xi)\Gamma(e+\frac{\alpha}{k})\Gamma(e+\frac{3}{2})\Gamma(\varphi e+\nu)(\tau)_{er}(2)^{2e+\frac{\alpha}{k}+1}} \times P_{0+}^{\mu,\gamma,p}(d^{3e+\frac{\alpha}{k}+1+\varphi-1})(d)$$

Using Eq. 9 we get,

$$= \sum \frac{(-f)^e B_a(\xi+eb, f-\xi)(f)_{e,b}d^{\mu+3e+\frac{\alpha}{k}\varphi+1}\Gamma(3e+\frac{\alpha}{k}+\varphi+1)\Gamma(1+\frac{\mu}{1-\gamma})}{(k^{e+\frac{\alpha}{k}+\frac{1}{2}}B(\xi, f-\xi)(2)^{2e+\frac{\alpha}{k}+1}\Gamma(e+\frac{\alpha}{k}+\frac{3}{2})\Gamma(e+\frac{3}{2})\Gamma(\varphi e+\nu)(\tau)_{er}}$$
$$\times[p(1-\gamma)]^{3e+\frac{\alpha}{k}+\varphi+1}\Gamma(1+\frac{\mu}{1-\mu}+3e+\frac{\alpha}{k}+\varphi+1))$$
$$= \frac{d^{\mu+\frac{\alpha}{k}+\varphi+1}B_a(\xi+eb, f-\xi)(f)_{e,b}\Gamma(1+\frac{\mu}{1-\gamma})}{(2)^{\frac{\alpha}{k}+1}k^{\frac{\alpha}{k}+\frac{1}{2}}[p(1-\gamma)]^{\frac{\alpha}{k}+\varphi+1}B(\xi, f-\xi)(\tau)_{er}}$$
$$\times \sum_{e=0}^{\infty}\frac{\Gamma(3e+\frac{\alpha}{k}+\varphi+1)}{\Gamma(e+\frac{\alpha}{k}+\frac{3}{2})\Gamma(e+\frac{3}{2})\Gamma(\varphi e+\nu)\Gamma(3e+\frac{\alpha}{k}+\frac{\mu}{1-\gamma}+\varphi)}\left[\frac{-fd^3}{4kp^3(1-\gamma)^3}\right]^e$$

Hence, we get the desired result.

3.1. Special Case:

1. If we take $e = 0, a = 1, \tau = 1$ and $\nu = 1$ in Eq.10 we get the required result reduces to the known result of [14] Eq. 8

$$\frac{d^{\mu+\frac{\alpha}{k}+\varphi+1}\Gamma(1+\frac{\mu}{1-\gamma})}{(2)^{\frac{\alpha}{k}+1}(k)^{\frac{\alpha}{k}+\frac{1}{2}}[p(1-\gamma)]^{\frac{\alpha}{k}+\varphi+1}}$$
$$\times{}_2\psi_3\left[\begin{matrix} & (\varphi+\frac{\alpha}{k}+1,2) & (1,1) \\ (\frac{\alpha}{k}+\frac{3}{2},1) & (\frac{3}{2},1) & (\frac{\alpha}{k}+\frac{\mu}{(1-\gamma)}+\varphi+2,2)\end{matrix};\left[\frac{-fd^2}{4p^2(1-\gamma)^e}\right]^e\right]$$

4. Conclusion

The theory of the pathway fractional integral operator can be utilized to construct multiple integral formulas by applying the pathway integral operator to the composition of the K-Struve function and extended Mittag-Leffler function. This work also uses a generalized hypergeometric function to express the derived.

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