J. of Ramanujan Society of Mathematics and Mathematical Sciences
Vol. 10, No. 1 (2022), pp. 31-38

DOI: 10.56827/JRSMMS.2022.1001.3 **ISSN (Online): 2582-5461**

ISSN (Print): 2319-1023

A NOTE ON ROGERS-RAMANUJAN-SLATER TYPE THETA FUNCTION IDENTITY

Jian Cao, Sama Arjika* and M. P. Chaudhary**

School of Mathematics,
Hangzhou Normal University,
Hangzhou City 311121, Zhejiang Province, People's Republic of CHINA

E-mail : 21caojian@hznu.edu.cn

*Department of Mathematics and Informatics,
University of Agadez, Post Office Box 199, Agadez, NIGER

*International Chair of Mathematical Physics and Applications
(ICMPA-UNESCO Chair),
University of Abomey-Calavi, Post Box 072, Cotonou 50, BENIN

E-mail : rjksama2008@gmail.com

**International Scientific Research and Welfare Organization
(Albert Einstein Chair Professor of Mathematical Sciences)
New Delhi 110018, INDIA

E-mail : dr.m.p.chaudhary@gmail.com

(**Received:** *Sep. 01, 2022* **Accepted:** *Nov. 21, 2022* **Published:** *Dec. 30, 2022)*

Abstract: In this paper, we research theta function identity involving Rogers–Ramanujan identity and establish a Rogers–Ramanujan–Slater type theta function identity related to $G(q)$ and $\varphi(q)$.

Keywords and Phrases: Theta function, Rogers–Ramanujan–Slater identity; Jacobi's triple-product identity.

2020 Mathematics Subject Classification: Primary 05A30, 11B65, 33D15, 33D45; Secondary 33D60, 39A13, 39B32.

1. Introduction and Definitions

Throughout this paper, we refer to [6] for definitions and notations. We also suppose that $0 < q < 1$. For complex numbers a, the q-shifted factorials are defined by

$$(a;q)_0 := 1, \quad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k) \quad \text{and} \quad (a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k), \tag{1}$$

where (see, for example, [6] and [12])

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}.$$

Here, in our present investigation, we are mainly concerned with the homogeneous version of the Cauchy identity or the following q-binomial theorem (see, for example, [6], [12] and [17]):

$$\sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k}\, z^k = \frac{(az;q)_\infty}{(z;q)_\infty} \qquad (|z|<1). \tag{2}$$

Upon further setting $a = 0$, the relation (2) becomes Euler's identity (see, for example, [6]):

$$\sum_{k=0}^{\infty} \frac{z^k}{(q;q)_k} = \frac{1}{(z;q)_\infty} \qquad (|z|<1) \tag{3}$$

and its inverse relation given below [6]:

$$\sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}}}{(q;q)_k} z^k = (z;q)_\infty. \tag{4}$$

Based upon the q-binomial theorem (2) and Heine's transformations, Srivastava *et al.* [15] have considered the function (10) and established a set of two presumably new theta-function identities (see, for details,[15]).

Proposition 1. ([15, Theorem 2.1]) *If* $\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}$, *then*

$$\sum_{n=0}^{\infty} \frac{(a;q^2)_n}{(-q;q)_n} q^n + \varphi(-q) \sum_{n=0}^{\infty} \frac{(a;q^2)_n}{(q;q)_n} q^n = 2\sum_{n=0}^{\infty} (-a)^n q^{n^2} \tag{5}$$

and

$$\sum_{n=0}^{\infty}(-1)^n \frac{q^{\frac{n(n+3)}{2}}}{(q;q)_n(1+q^{n+1})^2} = \frac{\varphi(-q)}{q}\sum_{n=0}^{\infty}\frac{q^n}{1+q^n}, \tag{6}$$

where $\varphi(q)$ is defined in (10).

In fact, Ramanujan (see 10] and [11]) also rediscovered Jacobi's famous triple-product identity which, in Ramanujan's notation, is given by (see [2, p. 35, Entry 19]):

$$f(a,b) = (-a, ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{7}$$

Equivalently, we have [8]:

$$\sum_{n=-\infty}^{\infty} q^{n^2} z^n = (q^2;q^2)_\infty(-zq;q^2)_\infty\left(-\frac{q}{z};q^2\right)_\infty, \quad (|q|<1,\ z\neq 0). \tag{8}$$

As a consequence of (8), we have the following corollary.

Corollary 1. *For $|q|<1$, we have:*

$$\sum_{n=-\infty}^{\infty} q^{n^2+2nk} = q^{-2k^2+k}\varphi(q). \tag{9}$$

Several q-series identities, which emerge naturally from Jacobi's triple-product identity (7), are worthy of note here (see, for details, [2, pp. 36–37, Entry 22]):

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} = (q^2;q^2)_\infty\{(-q;q^2)_\infty\}^2 = \frac{(-q;q^2)_\infty(q^2;q^2)_\infty}{(q;q^2)_\infty(-q^2;q^2)_\infty}, \tag{10}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}. \tag{11}$$

In [1, Corollary 7. 9, p. 113], Andrews proved that for $|q|<1$

$$G(q) = 1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \frac{1}{(q,q^4;q^5)_\infty}. \tag{12}$$

Rogers–Ramanujan–Slater [7, Eq. (11.2.3)] gave the following relation

$$\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q)_{2n}} = \frac{G(-q)}{(q;q^2)_\infty}. \tag{13}$$

2. Main Theorems

In this section, we establish a Rogers–Ramanujan–Slater type theta function identity.

Theorem 1. *If $\varphi(q)$ and $G(q)$ are defined as in (10) and (12), then the following assertion holds true:*

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q;q)_n}+\varphi(-q)\prod_{n=1}^{\infty}\frac{1}{(1-q^{5n-1})(1-q^{5n-4})}=2G(-q)\varphi(q). \tag{14}$$

Proof of Theorem 1. In the proof of Theorem 1, we assume that an empty product is interpreted to be unity. The left-hand side of (14) equals to

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q;q)_n}+\varphi(-q)\prod_{n=1}^{\infty}\frac{1}{(1-q^{5n-1})(1-q^{5n-4})}\\
&=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q;q)_n}+\frac{(q;q)_\infty}{(-q;q)_\infty}\prod_{n=1}^{\infty}\frac{1}{(1-q^{5n-1})(1-q^{5n-4})} \text{ by (12)}\\
&=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q;q)_n}+\frac{(q;q)_\infty}{(-q;q)_\infty}\sum_{n=0}^{\infty}\frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)}\\
&=\frac{1}{(-q;q)_\infty}\sum_{n=0}^{\infty}q^{n^2}\left\{\frac{(q;q)_\infty}{(q;q)_n}+\frac{(-q;q)_\infty}{(-q;q)_n}\right\}\\
&=\frac{1}{(-q;q)_\infty}\sum_{n=0}^{\infty}q^{n^2}\left\{(q^{1+n};q)_\infty+(-q^{1+n};q)_\infty\right\}\\
&=\frac{1}{(-q;q)_\infty}\sum_{n=-\infty}^{\infty}q^{n^2}\left\{(q^{1+n};q)_\infty+(-q^{1+n};q)_\infty\right\}
\end{aligned} \tag{15}$$

since $(q^{1+n};q)_\infty=0$ when n is a negative integer. Now applying (4), we get:

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q;q)_n}+\varphi(-q)\prod_{n=1}^{\infty}\frac{1}{(1-q^{5n-1})(1-q^{5n-4})}\\
&=\frac{1}{(-q;q)_\infty}\sum_{n=-\infty}^{\infty}q^{n^2}\left\{\sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{k}{2}}(q^{1+n})^k}{(q;q)_k}+\sum_{k=0}^{\infty}\frac{q^{\binom{k}{2}}(q^{1+n})^k}{(q;q)_k}\right\}\\
&=\frac{1}{(-q;q)_\infty}\sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty}\frac{q^{\binom{k}{2}}(q^{1+n})^k}{(q;q)_k}\left\{1+(-1)^k\right\}q^{n^2}\\
&=\frac{2}{(-q;q)_\infty}\sum_{k=0}^{\infty}\frac{q^{k^2+k}}{(q;q)_{2k}}\sum_{n=-\infty}^{\infty}q^{(n^2+2kn+k^2)}
\end{aligned}$$

$$= \frac{2}{(-q;q)_\infty} \sum_{k=0}^{\infty} \frac{q^{k^2+k}}{(q;q)_{2k}} \sum_{n=-\infty}^{\infty} q^{(n+k)^2}$$
$$= \frac{2}{(-q;q)_\infty} \sum_{k=0}^{\infty} \frac{q^{k^2+k}}{(q;q)_{2k}} \sum_{m=-\infty}^{\infty} q^{m^2}. \tag{16}$$

Next, by using (13) and (10), in the right-hand side of (16), we get:

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n} + \varphi(-q) \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-1})(1-q^{5n-4})} = \frac{2}{(-q;q)_\infty} \frac{G(-q)}{(q;q^2)_\infty} \varphi(q) = 2G(-q)\varphi(q)$$

where the identity $(-q;q)_\infty (q;q^2)_\infty \equiv 1$ is used. The proof of Theorem is complete.

3. Concluding Remarks and Observations

The present investigation was motivated by several recent developments dealing essentially with theta-function identities and combinatorial partition-theoretic identities. We have established a Rogers–Ramanujan–Slater type theta function identity related to $G(q)$ and $\varphi(q)$.

A view to further motivating researches involving theta-function identities and combinatorial partition theoretic identities, we have chosen to indicate rather briefly a number of recent developments on the subject-matter of this article. The list of citations, which we have included in this article, is believed to be potentially useful for indicating some of the directions for further researches and related developments on the subject-matter which we have dealt with here. In particular, we have cited the recent works by Chaudhary *et al.* (see [3] to [6]) and Srivastava *et al.* (see [14] to [15]).

Acknowledgements

The research work of M. P. Chaudhary is supported through a major research project of National Board of Higher Mathematics (NBHM) of the Department of Atomic Energy (DAE), Government of India by its sanction letter Ref. No. 02011/12/2020 NBHM(R.P.)/R D II/7867, dated 19th October 2020.

References

[1] Andrews G. E., The Theory of Partitions, Addison-Wesley, Reading, MA, (1976); reissued: Cambridge University Press, Cambridge, 1998.

[2] Berndt, B. C., Ramanujan's Notebooks- Part III, Springer-Verlag, Berlin, Heidelberg and New York, 1991.

[3] Chaudhary, M. P., Generalization of Ramanujan's identities in terms of q-products and continued fractions, Global J. Sci. Front. Res. Math. Decision Sci., 12 (2012), 53-60.

[4] Chaudhary, M. P., Salilew G. A. and Choi J., Five relationships between continued fraction identities, q-product identities and combinatorial partition identities, Far East J. Math. Sci., 102 (2017), 855-863.

[5] Chaudhary, M. P., Salah Uddin and Choi J., Certain relationships between q-product identities, combinatorial partition identities and continued-fraction identities, Far East J. Math. Sci., 101 (2017), 973-982.

[6] Gasper G. and Rahman M., Basic Hypergeometric Series (with a Foreword by Richard Askey), Encyclopedia of Mathematics and Its Applications, Vol. 35, Cambridge University Press, Cambridge, New York, Port Chester, Melbourne and Sydney, 1990; Second edition, Encyclopedia of Mathematics and Its Applications, Vol. 96, Cambridge University Press, Cambridge, London and New York, 2004.

[7] George E. Andrews, Bruce C. Berndt, Ramanujan's Lost Notebook, Springers Part I, 2005.

[8] Jacobi, C. G. J., Fundamenta Nova Theoriae Functionum Ellipticarum (Regiomonti, Sumtibus Fratrum Bornträger, Königsberg, Germany, 1829; Reprinted in Gesammelte Mathematische Werke 1 (1829), 497–538), American Mathematical Society, Providence, Rhode Island, (1969), 97-239.

[9] Koekock, R. and Swarttouw, R. F., The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue, Report No. 98-17, Delft University of Technology, Delft, The Netherlands, 1998.

[10] Ramanujan, S., Notebooks, Tata Institute of Fundamental Research, Bombay, Vols. 1 and 2, 1957.

[11] Ramanujan, S., The Lost Notebook and Other Unpublished Papers, Narosa Publishing House, New Delhi, 1988.

[12] Slater, L. J., Generalized Hypergeometric Functions, Cambridge University Press, Cambridge, London and New York, 1966.

[13] Srivastava, H. M., Chaudhary, M. P. and Chaudhary, S., A family of theta-function identities related to Jacobi's triple-product identity, Russian J. Math. Phys., 27 (2020), 139-144.

[14] Srivastava, H. M., Srivastava, R., Chaudhary, M. P. and Uddin, S., A family of theta- function identities based upon combinatorial partition identities and related to Jacobi's triple-product identity, Mathematics, 8 (6) (2020), Article ID 918, 1-14.

[15] Srivastava, H. M., Chaudhary, M. P., and Wakene, F. K., A Family of Theta-Function Identities Based Upon q-Binomial Theorem and Heine's Transformations, Montes Taurus J. Pure Appl. Math., 2 (2) (2020), 1-6.

[16] Srivastava, H. M., Certain q-polynomial expansions for functions of several variables I and II, IMA J. Appl. Math., 30 (1983), 315-323; ibid. 33 (1984), 205-209.

[17] Srivastava, H. M. and Karlsson, P. W., Multiple Gaussian Hypergeometric Series, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1985.