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CONNECTION BETWEEN PARTIAL BELL POLYNOMIALS AND $(q; q)_k$; PARTITION FUNCTION, AND CERTAIN q-HYPERGEOMETRIC SERIES

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Abstract: We exhibit a relationship between q-shifted factorial, $(q; q)_n$, and the incomplete exponential Bell polynomials and also evaluate several q-hypergeometric series using the q-version of Petkovsek-WilfZeilberger's algorithm. Finally, we write the partition function $p(n)$ in terms of $Q_m(k)$, the number of partitions of m using (possibly repeated) parts that do not exceed k.

Keywords and Phrases: Partial Bell polynomials, q-analysis, Hessenberg determinant, q-Hypergeometric series, q-Petkovsek-Wilf-Zeilberger's techniques, Partition functions.

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1. Introduction and Preliminaries

The q-shifted factorial is defined by [23, p.139]

$$(a;q)_n = \begin{cases} 1, n=0 \\ (1-a)(1-aq)\left(1-aq^2\right)\ldots\left(1-aq^{n-1}\right), n=1,2,3,\ldots \end{cases} \tag{1}$$

Then by (1), there exists following result

$$\frac{(q^\alpha;q)_n}{(1-q)^n} = [\alpha]_q[\alpha+1]_q[\alpha+2]_q\ldots[\alpha+n-1]_q \forall n=1,2,3,\ldots, \text{ where } [\alpha]_q := \frac{1-q^\alpha}{1-q}. \tag{2}$$

In view of (1), (2), and an infinite product $(a;q)_\infty = \prod_{k=0}^{\infty}\left(1-aq^k\right)$, we have a relation with the q shifted factorial as $(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}$, and then the q-binomial series is given by [28]

$$\frac{(az;q)_\infty}{(z;q)_\infty} = \sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}z^n. \tag{3}$$

Also, the q-exponential function is presented in the form [23, p. 145]

$${}_1\phi_0(0;;q,z) = \sum_{n=0}^{\infty}\frac{z^n}{(q;q)_n} = \frac{1}{(z;q)_\infty}, |z|<1. \tag{4}$$

Recently, Kim et al. [13] investigated some properties and identities for the (exponential) incomplete Bell polynomials or partial Bell polynomials $B_{n,k}\left(x_1,\ldots,x_{n-k+1}\right)$ through a generating function

$$\exp\left[\sum_{i=1}^{\infty}x_i\frac{t^i}{i!}\right] = \sum_{n=k}^{\infty}B_{n,k}\left(x_1,\ldots,x_{n-k+1}\right)\frac{t^n}{n!}, k\geq 0. \tag{5}$$

Here, in this paper we discuss q-analysis of partial Bell Polynomials [27] defined by (5) and evaluate some of its partition functions.

In our recent work (see references of (1)-(4)) by making an appeal to certain allied topics of q analysis and q-calculus ([1], [12]), we exhibit a connection between $(q;q)_n$ and the exponential Bell polynomials, which is motivated by the recent formulae of Malenfant [22] and Jha [11] for the partition function $p(n)$ studied in [21], given in the Section 2. The Section 3 contains proofs of certain q-series via the Petkovsek-Wilf-Zeilberger's method [24] adapted to q-analysis. Finally, the Section 4 shows that it is possible to write the partition function in terms of $Q_m(r)$, that is, the number of partitions of m employing (possibly repeated) parts that do not exceed r.

2. q-shifted factorial, and the incomplete exponential Bell polynomials

In this section on application of q-shifted factorial $(q;q)_n$, found due to (1), we derive incomplete exponential Bell polynomials studied in [27] and Hessenberg determinant found in [26].

Theorem 2.1. *For $|q| < 1$, and $\forall n \geq 1$, there exists following relations among q-shifted factorial, $(q;q)_n$, incomplete exponential Bell polynomials*

$$\frac{1}{(q;q)_n} = \frac{1}{n!}\sum_{k=1}^{n}(-1)^k k! \times B_{n,k}\left(-\frac{1!q^{\binom{1}{2}}}{(q;q)_1}, \frac{2!q^{\binom{2}{2}}}{(q;q)_2}, -\frac{3!q^{\binom{3}{2}}}{(q;q)_3}, \frac{4!q^{\binom{4}{2}}}{(q;q)_4}, \dots, \frac{(-1)^{n-k+1}(n-k+1)!q^{\binom{n-k+1}{2}}}{(q;q)_{n-k+1}}\right). \tag{6}$$

Also, the relation in terms of the Hessenberg determinant is given by

$$\frac{q^{\binom{n}{2}}}{(q:q)_n} = \begin{vmatrix} \frac{1}{(q;q)_1\cdot} & 1 & 0 & 0 & \dots & 0 \\ \frac{1}{(q;q)_2\cdot} & \frac{1}{(q;q)_1} & 1 & 0 & \dots & 0 \\ \frac{1}{(q;q)_3} & \frac{1}{(q;q)_2\cdot} & \frac{1}{(q;q)_1} & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \frac{1}{(q;q)_n} & \frac{1}{(q;q)_{n-1}} & \frac{1}{(q;q)_{n-2}} & \dots & \dots & \frac{1}{(q;q)_1} \end{vmatrix}. \tag{7}$$

Proof. We know the results [12] as

$$(x;q)_n = \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} x^k, \quad \frac{1}{(x;q)_{n+1}} = \sum_{k=0}^{\infty} \begin{bmatrix} n+k \\ k \end{bmatrix}_q x^k \tag{8}$$

Then due to (8), it is immediate that expression

$$\sum_{k=0}^{\infty}\frac{1}{(q;q)_k}x^k = \frac{1}{\sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{k}{2}}}{(q;q)_k}x^k}, \quad (q;q)_0 = 1. \tag{9}$$

Therefore make an appeal to the Eqns. (8) and (9), and the results of ([3], [4], [6], 25]) , to obtain Eqn. (6) as

$$\frac{1}{(q;q)_n} = \frac{1}{n!}\sum_{k=1}^{n}(-1)^k k!$$
$$\times B_{n,k}\left(-\frac{1!q^{\binom{1}{2}}}{(q;q)_1}, \frac{2!q^{\binom{2}{2}}}{(q;q)_2}, -\frac{3!q^{\binom{3}{2}}}{(q;q)_3}, \frac{4!q^{\binom{4}{2}}}{(q;q)_4}, \dots, \frac{(-1)^{n-k+1}(n-k+1)!q^{\binom{n-k+1}{2}}}{(q;q)_{n-k+1}}\right),$$

involving the partial Bell polynomials ([6], [16], [18]-[21], [27]), with the recurrence relation

$$\sum_{k=0}^{n} \frac{(-1)^k\, q^{\binom{k}{2}}}{(q\,;\,q)_k\ \ (q\,;\,q)_{n-k}} = 0, \qquad n \geq 1. \tag{10}$$

We can apply to (10) in the Birmajer-Gil-Weiner's inversion process ([3], [4]) to derive

$$\frac{(-1)^n q^{\binom{n}{2}}}{(q;q)_n} = \frac{1}{n!}\sum_{k=1}^{n}(-1)^k k! B_{n,k}\left(\frac{1!}{(q;q)_1}, \frac{2!}{(q;q)_2}, \frac{3!}{(q;q)_3}, \frac{4!}{(q;q)_4}, \dots, \frac{(n-k+1)!}{(q;q)_{n-k+1}}\right). \tag{11}$$

On the other hand, from ([8], [26]) we find that relations of type (9) are equivalent to the following Hessenberg determinant

$$\frac{(-1)^n q^{\binom{n}{2}}}{(q;q)_n} = (-1)^n \begin{vmatrix} \frac{1}{(q;q)_1} & 1 & 0 & 0 & \cdots & 0 \\ \frac{1}{(q;q)_2} & \frac{1}{(q;q)_1} & 1 & 0 & \cdots & 0 \\ \frac{1}{(q;q)_3} & \frac{1}{(q;q)_2} & \frac{1}{(q;q)_1} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & & 1 \\ \frac{1}{(q;q)_n} & \frac{1}{(q;q)_{n-1}} & \frac{1}{(q;q)_{n-2}} & \cdots & \cdots & \frac{1}{(q;q)_1} \end{vmatrix}. \tag{12}$$

The Eqn. (12) immediately gives the result (7).

The expressions (10), (11) and (12) were inspired by the formulae of Malenfant [22] and Jha [11] for the partition function ([16], [18]-[21]).

Theorem 2.2. *For all $k \geq 1$, there exists an inequality*

$$\left|\frac{(-1)^k (a;q)_k}{(q;q)_k}\right| \leq |(q;q)_k| \sum_{n=k}^{\infty} \left|\frac{(a;q)_{n-k}\left(aq^{n-k};q\right)_k}{(q;q)_{n-k}\left(q^{n-k+1};q\right)_k}\right|$$
$$\times \left| B_{n,k}\left(-\frac{1!q^{\binom{1}{2}}}{(q;q)_1}, \frac{2!q^{\binom{2}{2}}}{(q;q)_2}, -\frac{3!q^{\binom{3}{2}}}{(q;q)_3}, \frac{4!q^{\binom{4}{2}}}{(q;q)_4}, \dots, \frac{(-1)^{n-k+1}(n-k+1)!q^{\binom{n-k+1}{2}}}{(q;q)_{n-k+1}}\right)\right|. \tag{13}$$

Proof. Consider (10) and find that

$$\frac{1}{(q;q)_n} = \sum_{k=1}^{n} (-1)^k (1-q)^{n-k} \frac{(q;q)_k}{(q;q)_n}$$
$$\times B_{n,k}\left(-\frac{1!q^{\binom{1}{2}}}{(q\ ;\ q)_1}, \frac{2!q^{\binom{2}{2}}}{(q\ ;\ q)_2}, -\frac{3!q^{\binom{3}{2}}}{(q\ ;\ q)_3}, \frac{4!q^{\binom{1}{2}}}{(q;q)_4}, \dots, \frac{(-1)^{n-k+1}(n-k+1)!q^{\binom{n-k+1}{2}}}{(q;q)_{n-k+1}}\right). \tag{14}$$

Again then for all $a, z, q \in \mathbb{C}, |q| < 1, |z| < 1$, consider q-binomial series in the following form [23, p. 142] and use the result (14), we get

$$\frac{\left(a\frac{z}{(1-q)};q\right)_\infty - \left(\frac{z}{(1-q)};q\right)_\infty}{\left(\frac{z}{(1-q)};q\right)_\infty} = \sum_{n=1}^{\infty} \frac{(a;q)_n}{(q;q)_n}\left(\frac{z}{(1-q)}\right)^n.$$
$$\Rightarrow \sum_{n=1}^{\infty} \frac{(a;q)_n}{(q;q)_n}\left(\frac{z}{(1-q)}\right)^n$$
$$= \sum_{n=1}^{\infty}\sum_{k=1}^{n} (-1)^k \frac{(q;q)_k}{(q;q)_n} \frac{(a;q)_n z^n}{(1-q)^k}$$
$$\times B_{n,k}\left(-\frac{1!q^{\binom{1}{2}}}{(q;q)_1}, \frac{2!q^{\binom{2}{2}}}{(q;q)_2}, -\frac{3!q^{\binom{3}{2}}}{(q;q)_3}, \frac{4!q^{\binom{4}{2}}}{(q;q)_4}, \dots, \frac{(-1)^{n-k+1}(n-k+1)!q^{\binom{n-k+1}{2}}}{(q;q)_{n-k+1}}\right).$$
$$= \sum_{n=1}^{\infty}\left(-\frac{z}{(1-q)}\right)^k (q;q)_k \sum_{k=1}^{\infty} \frac{(a;q)_{n+k}}{(q;q)_{n+k}} z^n$$
$$\times B_{n+k,k}\left(-\frac{1!\ q^{\binom{1}{2}}}{(q\ ;\ q)_1}, \frac{2!\ q^{\binom{2}{2}}}{(q\ ;\ q)_2}, -\frac{3!\ q^{\binom{3}{2}}}{(q\ ;\ q)_3}, \frac{4!\ q^{\binom{4}{2}}}{(q\ ;\ q)_4}, \dots, \frac{(-1)^{n+1}\ (n+1)!\ q^{\binom{n+1}{2}}}{(q\ ;\ q)_{n+1}}\right). \tag{15}$$

Then on changing the order of the summation in (15) to get

$$\sum_{k=1}^{\infty}\frac{(a;q)_k}{(q;q)_k}\left(\frac{z}{1-q}\right)^k=\sum_{k=1}^{\infty}(q;q)_k\left(-\frac{z}{(1-q)}\right)^k\sum_{n=1}^{\infty}\frac{(a;q)_n\,(aq^n;q)_k}{(q;q)_n\,(q^{n+1};q)_k}z^n$$
$$\times B_{n+k,k}\left(-\frac{1!q^{\binom{1}{2}}}{(q;q)_1},\frac{2!q^{\binom{2}{2}}}{(q;q)_2},-\frac{3!q^{\binom{3}{2}}}{(q;q)_3},\frac{4!q^{\binom{4}{2}}}{(q;q)_4},\ldots,\frac{(-1)^{n+1}(n+1)!q^{\binom{n+1}{2}}}{(q;q)_{n+1}}\right).$$
$$\Rightarrow\sum_{k=1}^{\infty}\frac{(a;q)_k}{(q;q)_k}\left(\frac{z}{1-q}\right)^k=\sum_{k=1}^{\infty}(q;q)_k\left(-\frac{z}{(1-q)}\right)^k\sum_{n=k}^{\infty}\frac{(a;q)_{n-k}\left(aq^{n-k};q\right)_k}{(q;q)_{n-k}\left(q^{n-k+1};q\right)_k}z^{n-k}.$$

$$\times B_{n,k}\left(-\frac{1!q^{\binom{1}{2}}}{(q;q)_1},\frac{2!\ q^{\binom{2}{2}}}{(q;q)_2},-\frac{3!q^{\binom{3}{2}}}{(q;\ q)_3},\frac{4!q^{\binom{4}{2}}}{(q;q)_4},\ldots,\frac{(-1)^{n-k+1}\ (n-k+1)!q^{\binom{n-k+1}{2}}}{(q;q)_{n-k+1}}\right)\quad(16)$$

Finally, on equating the coefficients of $\left(-\frac{z}{(1-q)}\right)^k$, in both of the sides of the result (16), we obtain the equality given by

$$\frac{(-1)^k(a;q)_k}{(q;q)_k}=(q;q)_k\sum_{n=k}^{\infty}\frac{(a;q)_{n-k}\left(aq^{n-k};q\right)_k}{(q;q)_{n-k}\left(q^{n-k+1};q\right)_k}z^{n-k}$$
$$\times B_{n,k}\left(-\frac{1!q^{\binom{1}{2}}}{(q;q)_1},\frac{2!q^{\binom{2}{2}}}{(q;q)_2},-\frac{3!q^{\binom{3}{2}}}{(q;q)_3},\frac{4!q^{\binom{4}{2}}}{(q;q)_4},\ldots,\frac{(-1)^{n-k+1}(n-k+1)!q^{\binom{n-k+1}{2}}}{(q;q)_{n-k+1}}\right).\quad(17)$$

By the Eqn. (17), we obtain the inequality (13).

3.*q*-Hypergeometric series

In this section, we derive various results pertaining to the q-hypergeometric series ([11], [28]). In [11], the following q-series are studied

$$A\equiv\sum_{k=0}^{n}\left[\begin{array}{c}n\\k\end{array}\right]_q(-t)^kq^{\binom{k}{2}}\frac{(b;q)_k}{(bt;q)_k}=\frac{(t;q)_n}{(bt;q)_n},\quad(18)$$
$$B\equiv\sum_{n=0}^{\infty}\frac{(a;q)_n}{(e;q)_n(q;q)_n}\left(-\frac{e}{a}\right)^nq^{\binom{n}{2}}=\frac{\left(\frac{e}{a};q\right)_\infty}{(e;q)_\infty},\quad(19)$$
$$C\equiv\sum_{j=0}^{\infty}\frac{\left(b-q^k\right)\left(b-q^{k+1}\right)\cdots\left(b-q^{k+j-1}\right)}{(q;q)_j}t^j=\frac{\left(tq^k;q\right)_\infty}{(bt;q)_\infty},\quad(20)$$

$$D \equiv \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}}}{(q;q)_k (bx;q)_k} (b-a)(bq-a)\cdots\left(bq^{k-1}-a\right) = \frac{(ax;q)_\infty}{(bx;q)_\infty}, \tag{21}$$

here we show these identities by using the Petkovsek-Wilf-Zeilberger's method ([2], [9], [10], [14], [15], [17], [18]-[21], [24]) adapted to q-analysis ([3], [4], [11], [12], [25]).

In fact we have

$$A = \sum_{k=0}^{\infty} r_k, \quad \frac{r_{k+1}}{r_k} = \frac{\left(1-q^{-n}q^k\right)\left(1-bq^k\right)}{\left(1-btq^k\right)\left(1-q^{k+1}\right)} tq^n.$$

That is

$$A = {}_2F_1\left(q^{-n}, b; bt; q, tq^n\right), \tag{22}$$

but we have the Heine's q-Gauss summation formula [12] for this q-hypergeometric function ([7], [28])

$$ {}_2F_1\left(a, b; c; q, \frac{c}{ab}\right) = \frac{\left(\frac{c}{a};q\right)_\infty \left(\frac{c}{b};q\right)_\infty}{(c;q)_\infty \left(\frac{c}{ab};q\right)_\infty}, \tag{23}$$

also obtained by Jacobi and Ramanujan; then from (22) and (23)

$$A = \frac{(btq^n;q)_\infty (t;q)_\infty}{(tq^n;q)_\infty (bt;q)_\infty} = \frac{(t;q)_n}{(bt;q)_n}, \quad \text{q.e.d.}$$

Similarly, we get

$$B = \sum_{n=0}^{\infty} s_n, \quad \frac{s_{n+1}}{s_n} = \frac{(1-aq^n)\left(1-\frac{1}{Q}q^n\right)}{(1-eq^n)\left(1-q^{n+1}\right)} \frac{Qe}{a}.$$

That is

$$B = {}_2F_1\left(a, \frac{1}{Q}; e; q, \frac{Qe}{a}\right), \tag{24}$$

and in the final step we will apply $\lim_{Q\to 0}$; with (23) and (24):

$$B = \frac{\left(\frac{e}{a};q\right)_\infty (Qe;q)_\infty}{(e;q)_\infty \left(\frac{Qe}{a};q\right)_\infty} \underset{Q\to 0}{\rightarrow} \frac{\left(\frac{e}{a};q\right)_\infty}{(e;q)_\infty}, \quad q\cdot e.d$$

From (20), we obtain

$$C = \sum_{j=0}^{\infty} u_j, \quad \frac{u_{j+1}}{u_j} = \frac{\left(1-\frac{q^k}{b}q^j\right)}{\left(1-q^{j+1}\right)} bt.$$

Therefore

$$C = {}_1F_0\left(\frac{q^k}{b}; q, bt\right), \tag{25}$$

but we know the following result due to [12] given by

$${}_1F_0(a; q, z) = \frac{(az; q)_\infty}{(z; q)_\infty}, \tag{26}$$

hence (25) and (26) imply (20), q.e.d.

From (21), we find

$$D = \sum_{k=0}^{\infty} v_k, \quad \frac{v_{k+1}}{v_k} = \frac{\left(1 - \frac{1}{Q}q^k\right)\left(1 - \frac{b}{a}q^k\right)}{(1 - bxq^k)} Qax.$$

Therefore

$$D = {}_2F_1\left(\frac{1}{Q}, \frac{b}{a}; bx; q, Qax\right)$$

where we can apply (23) to obtain

$$D = \frac{(Qbx; q)_\infty (ax; q)_\infty}{(Qax; q)_\infty (bx; q)_\infty} \underset{Q \to 0}{\longrightarrow} \frac{(ax; q)_\infty}{(bx; q)_\infty}, \quad \text{q.e.d.}$$

The identity (21) was deduced by Cauchy and Ramanujan. The property (19) is a particular case of the Andrews formula ([1], [12])

$$\sum_{n=0}^{\infty} \frac{(a; q)_n (b; q)_n}{(e; q)_n (ax; q)_n (q; q)_n} \left(-\frac{ex}{b}\right)^n q^{\binom{n}{2}} = \frac{(x; q)_\infty}{(ax; q)_\infty} {}_2F_1\left(a, \frac{e}{b}; e; q, x\right), \tag{27}$$

for $x = \frac{b}{a}$.

4. Partition function $p(n)$ in terms of $Q_m(k)$

In this section, we derive various results of partition function $p(n)$ and the $Q_m(n)$, where $Q_m(n)$ is the number of partitions of m employing (possibly repeated) parts that do not exceed n.

If $p(n)$ is the partition function and $Q_m(n)$ is the number of partitions of m employing (possibly repeated) parts that do not exceed n, then there exists the following relations due to [12]

$$\sum_{n=0}^{\infty} p(n) q^n = \frac{1}{(q; q)_\infty} = \sum_{n=0}^{\infty} \frac{q^n}{(q; q)_n} \tag{28}$$

$$\frac{1}{(q;q)_n} = \frac{1}{(1-q)\,(1-q^2)\cdots(1-q^n)} = \sum_{m=0}^{\infty} Q_m(n) q^m. \tag{29}$$

Now in (28) and (29), we verify some interesting properties given by
$Q_r(0) = 0,\ Q_0(n) = Q_r(1) = Q_1(n) = 1;\ Q_m(m) = 1 + Q_m(m-1) = 2 + Q_m(m-2)$, $m \geq 2$

$$Q_m(n) = p(m),\ n \geq m,\ Q_{2r}(r) = Q_{2r}(r-1) + p(r),\ r \geq 1. \tag{30}$$

The application of (29) into (28) implies the expressions

$$\begin{aligned} p(n) &= \sum_{k=0}^{n} Q_{n-k}(k), \quad n \geq 1 \\ &= 3 + \sum_{k=2}^{n-2} Q_{n-k}(k), \quad n \geq 3 \end{aligned} \tag{31}$$

which allow us to consider the case $n = 2\lambda$ as

$$\begin{aligned} p(n) &= 3 + \lambda + \sum_{k=3}^{\lambda-1} Q_{2\lambda-k}(k) + \sum_{r=2}^{\lambda} p(r), \quad \lambda \geq 3, \\ &= 1 + \lambda + \sum_{k=3}^{\lambda-2} Q_{2\lambda-k}(k) + \sum_{r=2}^{\lambda+1} p(r), \quad \lambda \geq 4 \end{aligned} \tag{32}$$

and the case $n = 2\beta + 1$:

$$p(n) = 2 + \beta + \sum_{k=3}^{\beta-1} Q_{2\beta+1-k}(k) + \sum_{r=2}^{\beta+1} p(r), \quad \beta \geq 3. \tag{33}$$

5. Concluding Remarks

In this article we focused on the incomplete exponential Bell polynomials, q-hypergeometric series using the q version of Petkovsek-Wilf-Zeilberger's algorithm [24] and the partition function $p(n)$ in terms of $Q_m(k)$, the number of partitions of m using (possibly repeated) parts that do not exceed k. The recent literature includes works which have taken into consideration central complete and incomplete Bell polynomials [13] and the partial r-Bell polynomials [25].Note that r-partial partial r-Bell polynomials generalize the classical partial Bell polynomials by coinciding with them when $r = 0$, by assigning a different set of weights to the blocks containing the r smallest elements of a partition no two of which are allowed to belong to the same block.

In our next paper, we propose to study the partial r-Bell polynomials from a combinatorial standpoint and derive several new formulas. The explicit closed-form formula for $B_{n,k}$ given by (5) is particularly useful to obtain the connections with the well-known and widely studied Davey-Stewartson system of equations. In several cases, these extend previous formulas for the partial Bell polynomials which follow by taking $r = 0$.For appropriate choices of the indeterminates, the partial r-Bell polynomials reduce to some special combinatorial sequences (see [25]) including unsigned r-Stirling number of the first kind, r-Stirling number of the second kind, r-Whitney number of the second kind and r-Lah number.

Our arguments as suggested by M. Shattuck [25] largely combinatorial, may be provided alternatively, by using proofs of these formulas by algebraic methods. In this way some general identities valid for arbitrary values of the parameters as well as formulas for some specific evaluations may be established..Thee results may extend known formulas for the partial Bell polynomials given in this paper.

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