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A STUDY OF AN UNDIRECTED GRAPH ON A FINITE SUBSET OF NATURAL NUMBERS

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Abstract: Let $G_n = (V, E)$ be an undirected simple graph, whose vertex set comprises of the natural numbers which are less than n but not relatively prime to n and two distinct vertices $u, v \in V$ are adjacent if and only if $\gcd(u, v) > 1$. Connectedness, completeness, minimum degree, maximum degree, independence number, domination number and Eulerian property of the graph G_n are studied in this paper.

Keywords and Phrases: Clique, connected graph, complete graph, prime counting function.

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1. Introduction and Preliminaries

Let $G = (V, E)$ be a simple graph where V is the set of vertices and E is the set of edges. For any vertex $u \in V$, the *degree* of a vertex u denoted by $\deg(u)$ is the number of edges incident on u. The maximum (minimum) degree of G is denoted by $\Delta(G)$ ($\delta(G)$). For graph theory terminology and notation in general we follow [8]. Let $n \in \mathbb{N}$ be a composite number. It would be interesting to know the structural properties of the subset of the natural numbers which are less than n but not relatively prime to n. Thus proceeding in that direction, in this paper we define an undirected simple graph $G_n = (V, E)$, whose vertex set $V \subseteq \mathbb{N}$ defined as $V = \{i \in \mathbb{N} | \gcd(i, n) \neq 1, i < n\}$ and the two vertices $i, j \in V$ are adjacent if and only if $\gcd(i, j) > 1$, where $n \in \mathbb{N}$ and n is not a prime number. When studying any collection of newly formulated graphs, properties such as connectedness, completeness, minimum degree, maximum degree, domination number, independence number, Eulerian property etc. are of immediate concern. Hence we study those properties of G_n. Throughout the paper we consider $m \in \mathbb{N}$, $m > 1$, and for a vertex $v \in V$ by $v = a$ we mean that the vertex v is labeled as a and by $a = bc$, we mean a is the product of b and c.

2. Main Results

2.1. Connectedness and Completeness of G_n

A graph is said to be *connected* if there is a path between every pair of distinct vertices. A graph is said to be *complete* if there is an edge between every pair of distinct vertices. The complete graph of order n is denoted by K_n. A *clique* of a graph G is a maximal complete subgraph.

Theorem 2.1. *The graph $G_n = (V, E)$ is disconnected if and only if $n = 2p$, where p is an odd prime. Moreover, the components of G_{2p} are K_{p-1} and K_1.*

Proof. Let $n = 2p$, where p is an odd prime. Then the vertex set $V = \{2, 2 \cdot 2, 3 \cdot 2, \ldots, (p-1) \cdot 2, p\}$. Clearly, the vertex $v = p$ is not adjacent to any vertex $u = i \cdot 2$, for $i = 1, 2, \ldots, p-1$ as $\gcd(p, i \cdot 2) = 1$. Thus G_{2p} is disconnected.

Conversely, let G_n be disconnected. If possible, assume that $n = p_1 p_2 \ldots p_l$, where p_i's are primes (may not be all distinct) for $i = 1, 2, \ldots, l$, $l > 1$. Let $l \geq 3$ and consider two distinct vertices $a, b \in V$ where $a = p_1$, $b = j$. We may get two cases.

Case 1. If p_1 is a factor of j, then the vertices a, b are adjacent.

Case 2. If p_1 is not a factor of j, then for any prime factor q of j, $\gcd(p_1, q) = 1$. Since $l \geq 3$, $p_1 q < n$. This implies that there exists a vertex in V labeled as $p_1 q$. Let $c = p_1 q \in V$. But c is adjacent to both $a = p_1$ and $b = j$, implying the vertices a and b are connected via the vertex c. Thus for $l \geq 3$, G_n is connected. Now suppose $l = 2$. Then $n = p_1 p_2$. Then we may have the two sub-cases, either $p_1 = p_2$ or $p_1 \neq p_2$.

Case i. Let $p_1 = p_2$. Then $n = p_1^2$ and $V = \{p_1, 2 \cdot p_1, \ldots, (p_1 - 1) \cdot p_1\}$. Easily, it can be proved that any two vertices are adjacent.

Case ii. Let $p_1 \neq p_2$ and $p_1 < p_2$. Again, we may have two cases such that either $p_1, p_2 > 2$ or $p_1 = 2, p_2 > 2$. First, let us consider $p_1, p_2 > 2$. Then the vertex set of G_n is $V = \{p_1, 2 \cdot p_1, 3 \cdot p_1, \ldots, (p_2 - 1) \cdot p_1, p_2, 2 \cdot p_2, 3 \cdot p_2, \cdots, (p_1 - 1) \cdot p_2\}$. Let $v_i, v_j \in V$ such that $v_i = i$ and $v_j = j$. Clearly, if $\gcd(i, j) > 1$, then v_i and v_j are adjacent.

Now let $\gcd(i,j)=1$, then i,j must contain distinct prime factors. So, let p_1, p_2 be prime factors of i,j, respectively. But both p_1,p_2 are odd primes. Hence, $2\cdot p_1, 2\cdot p_2 \in V$, which implies that the vertex v_i is adjacent to the vertex labeled as $2\cdot p_1$ and the vertex v_j is adjacent to the vertex labeled as $2\cdot p_2$. Therefore, the vertices v_i and v_j are connected via $2\cdot p_1$ and $2\cdot p_2$.

Finally, we are left with the case when $p_1=2$ and $p_2>2$. In this case, the vertex set $V=\{p_1, 2\cdot p_1, \ldots, (p_2-1)\cdot p_1, p_2\}$ and the vertex $a=p_2$ is isolated as for any $b\in V\setminus\{p_2\}$, the $\gcd(a,b)=1$. Thus G_n is disconnected for $n=2p$.

Moreover, the vertex set $V=V_1\bigcup V_2$, where $V_1=\{2, 2\cdot 2, 3\cdot 2, \ldots, (p-1)\cdot 2\}$, $V_2=\{p\}$ and $V_1\bigcap V_2=\phi$. The total number of vertices in V_1 is $p-1$ and the $p-1$ vertices in V_1 form a clique K_{p-1}. Hence, the components of G_{2p} are K_{p-1} and K_1.

Theorem 2.2. *The graph G_n is complete if and only if $n=p^m$, where p is a prime.*

Proof. Let $n=p^m$, where p is a prime. The vertex set of G_n is $V=\{p, 2\cdot p, 3\cdot p, \ldots, (p^{m-1}-1)\cdot p\}$ and the cardinality of V is $|V|=n-\phi(n)-1=p^{m-1}-1$. Let $a,b\in V$ such that $a=c_1\cdot p$, $b=c_2\cdot p$, where $c_1,c_2\in\{1,2,\ldots,p^{m-1}-1\}, c_1\neq c_2$. Clearly, $\gcd(a,b)\geq p>1$. Thus the vertices a and b are adjacent. Hence, G_n is complete.

Conversely, let G_n be complete. If possible, let $n=p_1^{m_1}p_2^{m_2}\ldots p_k^{m_k}$, where p_i's are distinct primes, $m_i\in\mathbb{N}$ for $i=1,2,\ldots,k$, $k\geq 2$. Consider the vertices $a,b\in V$ where $a=p_1$, $b=p_2$. Then $\gcd(a,b)=1$. So the vertices a and b are non-adjacent. Hence the graph G_n is not complete. Thus we arrive at a contradiction. Thus $n=p^m$.

2.2. Minimum degree, Maximum degree of G_n

Lemma 2.1. *Let $n=p_1^{r_1}p_2^{r_2}\cdots p_{k-1}^{r_{k-1}}p_k^{r_k}$. Then for $a,b\in V$ in G_n, $\deg(a)=\deg(b)$ if the labels of a, b contain same prime factors.*

Proof. Let $a,b\in V$, where $a=p_1^{s_1}p_2^{s_2}\cdots p_j^{s_j}$, $b=p_1^{t_1}p_2^{t_2}\cdots p_j^{t_j}$, p_i's are distinct primes, $s_i,t_i\in\mathbb{N}$ for $i=1,2,\ldots,j$. If possible, let $\deg(a)\neq\deg(b)$. Then either $\deg(a)>\deg(b)$ or $\deg(b)>\deg(a)$. Let $\deg(a)>\deg(b)$. Then there exists a vertex $c\in V$ such that c is adjacent to a but it is not adjacent to b. Then $\gcd(c,a)>1$ and $\gcd(c,b)=1$, which is a contradiction as the prime factors of a and b are same. So our assumption is wrong. Hence $\deg(a)=\deg(b)$.

Theorem 2.3. *Let $n=p_1^{r_1}p_2^{r_2}\cdots p_{k-1}^{r_{k-1}}p_k^{r_k}$, where $p_i<p_{i+1}$ and p_i's are distinct primes for $i\in\{1,2,\ldots,k\}$, then the vertices of the form $p_k^t, t\in\mathbb{N}$ attain the minimum degree of G_n and $\delta(G_n)=\frac{n}{p_k}-2$.*

Proof. In the prime factorization of n, the highest prime is p_k. The vertex $u=p_k$

is adjacent to all the multiples of p_k in V and the number of multiples of p_k up to n is $\left\lfloor \frac{n}{p_k} \right\rfloor$. Thus the degree of $u = p_k$ is $\left\lfloor \frac{n}{p_k} \right\rfloor - 2$ (as $n \notin V$ and u is not adjacent to itself). We claim that the degree of the vertex $u = p_k$ is minimum. Clearly, $\left\lfloor \frac{n}{p_i} \right\rfloor \geq \left\lfloor \frac{n}{p_k} \right\rfloor$ for $i = 1, 2, \ldots, k-1$. Thus the degree of the vertices labeled as p_i for $i = 1, 2, \ldots, k-1$ is greater than or equal to the degree of the vertex $u = p_k$. Now, we consider a vertex which is a multiple of more than one prime. Let us assume $a = p_l p_j \in V$. Then the degree of the vertex a is $\deg(a) = \left\lfloor \frac{n}{p_l} \right\rfloor + \left\lfloor \frac{n}{p_j} \right\rfloor - \left\lfloor \frac{n}{p_l p_j} \right\rfloor = p_1^{r_1} p_2^{r_2} \cdots p_l^{r_l - 1} p_{l+1}^{r_{l+1}} \cdots p_j^{r_j} \cdots p_k^{r_k} + p_1^{r_1} p_2^{r_2} \cdots p_l^{r_l} p_{l+1}^{r_{l+1}} \cdots p_j^{r_j - 1} \cdots p_k^{r_k} - p_1^{r_1} p_2^{r_2} \cdots p_l^{r_l - 1} p_{l+1}^{r_{l+1}} \cdots p_j^{r_j - 1} \cdots p_k^{r_k} =$
$p_1^{r_1} p_2^{r_2} \cdots p_l^{r_l - 1} p_{l+1}^{r_{l+1}} \cdots p_j^{r_j - 1} p_{j+1}^{r_j + 1} \cdots p_k^{r_k} (p_j + p_l - 1)$.

We claim that $p_1^{r_1} p_2^{r_2} \cdots p_l^{r_l - 1} p_{l+1}^{r_{l+1}} \cdots p_j^{r_j - 1} p_{j+1}^{r_j + 1} \cdots p_k^{r_k} (p_j + p_l - 1) > \frac{n}{p_k}$ as
$p_1^{r_1} \cdots p_l^{r_l - 1} p_{l+1}^{r_{l+1}} \cdots p_j^{r_j - 1} p_{j+1}^{r_{j+1}} \cdots p_k^{r_k - 1} (p_j p_k + p_l p_k - p_k - p_l p_j)$
$= p_1^{r_1} \cdots p_l^{r_l - 1} p_{l+1}^{r_{l+1}} \cdots p_j^{r_j - 1} p_{j+1}^{r_{j+1}} \cdots p_k^{r_k - 1} (p_j (p_k - p_l) + p_k (p_l - 1)) > 0$.

Let $w \in V$ be any vertex where the label of w is multiple of $i-1$ primes. That is, $w = p_1 p_2 \cdots p_{i-1}$, p_i's are primes, $i \in \mathbb{N}$, $i < k$ such that $\deg(w) > deg(u = p_k)$.

Now, consider any vertex b where b is a multiple of i primes. So, let $b = p_1 p_2 \ldots p_{i-1} p_i$. Then $\deg(b) \geq \deg(w) + 1 \implies \deg(b) > \deg(u = p_k)$. Hence, the vertex $u = p_k$ is of minimum degree. Moreover, by Lemma 2.1 all the vertices of the label of the form p_k^t, $t \in \mathbb{N}$ are of same degree $\frac{n}{p_k} - 2$.

Theorem 2.4. *Let $n = pq$, $p < q$, where p, q are distinct odd primes. Then*

(i) $\Delta(G_{pq}) = q - 1$, *if* $p = 3$;

(ii) $\Delta(G_{pq}) = \frac{2q+p-5}{2} + \left\lfloor \frac{p-1}{3} \right\rfloor - \left\lfloor \frac{p-1}{6} \right\rfloor$, *if* $p > 3$.

Proof. Let p, q be any two odd primes, where $p < q$. Then the vertex set V of G_{pq} is $V = V_p \bigcup V_q$, where $V_p = \{p, 2 \cdot p, \ldots, (q-1) \cdot p\}$, $V_q = \{q, 2 \cdot q, \ldots, (p-1) \cdot q\}$ and $V_p \bigcap V_q = \phi$. The vertices in V_p, V_q form cliques of size $q-1$, $p-1$, respectively. Let us consider the following two cases:

Case i. Let $p = 3$, $q > 3$. Then, $V = V_3 \bigcup V_q$, where $V_3 = \{3, 2 \cdot 3, 3 \cdot 3, \ldots, (q-1) \cdot 3\}$, $V_q = \{q, 2 \cdot q\}$, $V_3 \bigcap V_q = \phi$. Thus $\deg(v) \geq q - 2$, for all $v \in V_3$. Again any vertex $v \in V_3$ of the form $v = m \cdot 3$, where m is an even integer, is adjacent to the vertex $w = 2 \cdot q \in V_q$ as the $\gcd(v, w) = 2$. So the vertices of the form $v = m \cdot 3 \in V_3$ are of degree $q - 2 + 1 = q - 1$. We claim that $q - 1$ is the maximum possible degree of a vertex in V_3. So, if possible let there exist some vertex $u \in V_3$ such that $\deg(u) = q$. Then the vertex u must be a neighbour of all the vertices in V_3

as well as all the vertices in V_q as the total number of the vertices in the graph G_{3q} is $|V| = |V_3| + |V_q| = q - 1 + 2 = q + 1$. But the vertex u cannot be adjacent to the vertex $q \in V_q$ as $\gcd(u, q) = 1$ and the vertex u is not adjacent to itself. Thus our assumption is wrong. So there is no vertex $u \in V_3$ such that the degree of u is greater than $q - 1$. Again the degree of the vertex $w = 2 \cdot q \in V_q$ is $1 + \frac{q-1}{2} < q - 1$ and the degree of the vertex $q \in V_q$ is 1. Hence $\Delta(G_{3q}) = q - 1$.

Case ii. Let $p, q > 3$. Then V_q contains vertices of the form $v_q = m_1 \cdot q$ where $m_1 = 1, 2, 3, 4, \ldots, (p-1)$. Consider $v_p \in V_p$ such that $v_p = m_2 \cdot p$, where m_2 is a multiple of 6. Then the vertex v_p is adjacent to all the vertices in V_q which are labeled as multiples of 2, 3. Thus $\deg(m_2 \cdot p) = (q-2) + \left\lfloor \frac{p-1}{2} \right\rfloor + \left\lfloor \frac{p-1}{3} \right\rfloor - \left\lfloor \frac{p-1}{6} \right\rfloor = \frac{2q+p-5}{2} + \left\lfloor \frac{p-1}{3} \right\rfloor - \left\lfloor \frac{p-1}{6} \right\rfloor$. Clearly, $\deg(m_2 \cdot p)$ is maximum as $\left\lfloor \frac{p-1}{2} \right\rfloor > \left\lfloor \frac{p-1}{3} \right\rfloor > \left\lfloor \frac{p-1}{5} \right\rfloor > \cdots > \left\lfloor \frac{p-1}{p_r} \right\rfloor$, where p_r is the prime such that $p_r < p$. Hence, $\Delta(G_{pq}) = \frac{2q+p-5}{2} + \left\lfloor \frac{p-1}{3} \right\rfloor - \left\lfloor \frac{p-1}{6} \right\rfloor$.

Theorem 2.5. *For $n = p_1^{r_1} p_2^{r_2} \cdots p_{k-1}^{r_{k-1}} p_k^{r_k}$, the maximum degree of G_n is $|V| - 1$, if at least one $r_i > 1$ and $r_i \in \mathbb{N}$, where $p_i < p_{i+1}$, p_i's are distinct primes for $i \in \{1, 2, \ldots, k\}$.*

Proof. Let $n = p_1^{r_1} p_2^{r_2} \cdots p_{k-1}^{r_{k-1}} p_k^{r_k}$, where $p_i < p_{i+1}$ and p_i's are distinct primes for $i \in \{1, 2, \ldots, k\}$. We consider the following cases:

Case i. For $i = 1$, $n = p_1^{r_1}$ and by Theorem 2.2, G_n is complete. So $\Delta(G_n) = |V| - 1$.

Case ii. Let $i \geq 2$ and $i \in \{1, 2, \ldots, k\}$. Then $p_1 p_2 \cdots p_i$ is a label of a vertex in G_n as at least one $r_i \geq 1$. Let $u = p_1 p_2 \cdots p_i$. Then u is adjacent to all other vertices in V as the prime factorization of u contains all the prime factors $p_i, i = 1, 2, \cdots, k$ of n. Hence, $\deg(u) = |V| - 1$ which is maximum.

Now we find the maximum degree of G_n where $n = p_1 p_2 \cdots p_k$, $k > 2$. We know the prime-counting function is denoted by $\pi(x)$. It is the number of primes p satisfying $2 \leq p \leq x$. In the following results of this section, we see relations between prime-counting function and the maximum degree of G_n.

Theorem 2.6. *Let $n = p_1 p_2 \cdots p_k$, where p_i's are distinct primes, $i = 1, 2, \ldots, k$.*

- **(a)** *If $p_1 p_2 \cdots p_{k-1} > p_k$, then $\Delta(G_n) \leq |V| - L_p - 3$, where $L_p = \pi(p_1 p_2 \cdots p_{k-1}) - \pi(p_k)$;*
- **(b)** *If $p_1 p_2 \cdots p_{k-1} < p_k$, then $\Delta(G_n) \leq |V| - K_p - 2$, where $K_p = \pi(p_k) - \pi(p_1 p_2 \cdots p_{k-1})$.*

Proof. For any primes p_i, p_j where $p_i < p_j$, $\left\lfloor \frac{n}{p_i} \right\rfloor \geq \left\lfloor \frac{n}{p_j} \right\rfloor$. A vertex $u \in V$ attains the maximum degree if the label of u is product of the maximum number of distinct prime factors q where $q \in \{p_1, p_2, \ldots, p_k\}$. n is a product of k distinct primes and $n \notin V$. So, any vertex $v \in V$ can be multiple of at most $k-1$ distinct primes. There are $C(k, k-1) = k$ vertices whose labels are multiples of $k-1$ distinct primes.

Among the k multiples ($\{p_1p_2\cdots p_{k-1}, p_1p_2\cdots p_{k-2}p_k, \ldots, p_2p_3\cdots p_k\}$) of $k-1$ distinct primes, clearly, the product $p_1p_2\cdots p_{k-1}$ have the maximum number of multiples less than or equal to n in V. Again the number of vertices whose labels are coprime to $p_1p_2\cdots p_{k-1}$ is the least among other $k-1$ multiples of k distinct primes. Assume that $u \in V$ such that the degree of u is maximum. Then $u = m{\cdot}p_1p_2\cdots p_{k-1}$ where $m = q_1q_2\cdots q_t$, q_j's are distinct primes and q_j's are distinct from p_i's too for $j = 1, 2, \ldots, t$, $i = 1, 2, \ldots, k$.

Consider $p_1p_2\cdots p_{k-1} > p_k$. Now, to find the degree of the vertex u, we find the number of vertices non-adjacent to u. We claim, p_k^2 is a label of a vertex in V. If possible, let $p_k^2 \notin V$. Then $p_k^2 > n \implies p_k^2 - p_1p_2\cdots p_k > 0 \implies p_k(p_k - p_1p_2\cdots p_{k-1}) > 0$, which is absurd as $p_1p_2\cdots p_{k-1} > p_k$. Hence, p_k^2 must be a label of a vertex in V. The vertices $u_1 = p_k$, $u_2 = p_k^2$ are not adjacent to u as well as u is not adjacent to itself. As $p_1p_2\cdots p_{k-1} > p_k$, the prime multiples $s \cdot p_k$ of p_k, where s is a prime and $p_1p_2\cdots p_{k-1} > s > p_k$ are labels of some vertices in G_n. The vertices labeled as $s \cdot p_k$, where s is a prime and $p_1p_2\cdots p_{k-1} > s > p_k$ are not adjacent to the vertex u as $\gcd(s \cdot p_k, p_1p_2\cdots p_{k-1}) = 1$. Thus the degree of vertex u cannot exceed $|V| - 3 - L_p$, where $L_p = \pi(p_1p_2\cdots p_{k-1}) - \pi(p_k)$. That is, L_p is the number of primes between p_k and $p_1p_2\cdots p_{k-1}$. Hence $\deg(u) \leq |V| - L_p - 3$.

Now, consider $p_1p_2\cdots p_{k-1} < p_k$. Clearly, the vertex u is not adjacent to the vertex labeled as $u_1 = p_k$ and itself. Again, u is not adjacent to the vertices labeled as $u_j = w \cdot p_k$, w is a prime (not a factor of n, u) and $p_{k-1} < w < p_k$. Moreover, u is not adjacent to the vertices labeled as $x_l = y \cdot p_k$, where y is a prime (not a factor of n, u), such that $p_i < y < p_{i+1}$, where $i \in \{1, 2, \ldots, k-2\}$. Thus the number of vertices non adjacent to u is $2 + |U| + |Y|$, where $U = \{w \cdot p_k : w \text{ is a prime (but not a factor of } n, u \text{ and } p_{k-1} < w < p_k\}$. In other words, $|U| = \pi(p_k) - \pi(p_{k-1})$, $Y = \{y \cdot p_k : y \text{ is a prime (not a factor of } n, u \text{ and } p_i < y < p_{i+1} \text{ for } i = 1, 2, \ldots, k-2\}$. Hence $\deg(u) = |V| - (2 + |U| + |Y|)$, which gives $\Delta(G_n) \leq |V| - K_p - 2$, where $K_p = |U| = \pi(p_k) - \pi(p_{k-1})$.

Observation 2.1. *For $n = p_1p_2\cdots p_k$, where p_i's are distinct primes and $p_1p_2\cdots p_{k-1} > p_k$, if $\Delta(G_n)$ of G_n is known, then one can find the maximum number of primes between $p_1p_2\cdots p_{k-1}$ and p_k using the relation $\pi(p_1p_2\cdots p_{k-1}) - \pi(p_k) \leq |V| - \Delta(G_n) - 3$.*

Observation 2.2. *For $n = p_1p_2\cdots p_k$, where p_i's are distinct primes and $p_1p_2\cdots$*

$p_{k-1} < p_k$, if $\Delta(G_n)$ of G_n is known, then one can find the maximum number of primes between p_k and $p_1p_2\cdots p_{k-1}$ using the relation $\pi(p_k) - \pi(p_1p_2\cdots p_{k-1}) \leq |V| - \Delta(G_n) - 2$.

Example 2.1. Consider G_n, where $n = p_1p_2p_3p_4 = 2\cdot 5\cdot 11\cdot 37 = 4070$ and $p_1\cdot p_2\cdot p_3 = 2\cdot 5\cdot 11 > p_4 = 37$. Then the number of vertices in G_n is $|V| = n - \phi(n) - 1 = 2629$. The vertex $u = m\cdot 2\cdot 5\cdot 11 = 2310$, attains the maximum degree where $m = 3\cdot 7$. The number of multiples of 2310 up to 4070 is 1, so there is only one vertex of maximum degree. The vertex u is not adjacent to the vertices labeled as $37, 37^2, 41\cdot 37, 43\cdot 37, 47\cdot 37, 53\cdot 37, 59\cdot 37, 61\cdot 37, 67\cdot 37, 71\cdot 37, 73\cdot 37, 79\cdot 37, 83\cdot 37, 89\cdot 37, 97\cdot 37, 101\cdot 37, 103\cdot 37, 107\cdot 37, 109\cdot 37$ and itself. Thus the degree of u cannot exceed $|V|-3-17 = |V|-3-(\pi(110)-\pi(37)) = 2629-3-17$. Hence $\deg(u) \leq |V| - 3 - L_p = 2609$, where $L_p = \pi(110) - \pi(37)$.

Example 2.2. Consider G_n, where $n = 8827 = p_1\cdot p_2\cdot p_3 = 7\cdot 13\cdot 97$ and $p_1\cdot p_2 = 7\cdot 13 < p_3 = 97$. Then the order of G_n is 1914. The vertex $u = 7\cdot 13\cdot m$, where $m = 2\cdot 3\cdot 5$, that is, $u = 2730 \in V$ is a vertex attaining the maximum degree. The vertex u is not adjacent to the vertices $97, p\cdot 97, q\cdot 97$ and itself, where $p \in U_j$, where $U_j = \{17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89\}$ and $q \in Y_l = \{11\}$. Thus the degree of $u = 2730$ is $\deg(u) = |V|-(2+|U_j|+|Y_l|) = 1914 - (2+18+1) = 1914 - 21 = 1893$.

2.3. Counting principle for the graph $G_{p^mq^m}$

In this section, we study the graph G_n, where $n = p^mq^m$, p, q are distinct primes and $m \in \mathbb{N}$.

The proper divisors of $n = p^mq^m$ are of the form p^i, q^i, p^jq^k, where $i, j, k = 1, 2, \ldots, m$, $p^jq^k < n$. The vertex set V of G_n is $V = V_P \bigcup V_Q \bigcup V_{\bar{P}\bar{Q}}$, where $V_P = V_p \bigcup V_{p^2} \bigcup \ldots \bigcup V_{p^m}$, $V_Q = V_q \bigcup V_{q^2} \bigcup \ldots \bigcup V_{q^m}$, $V_{\bar{P}\bar{Q}} = V_{PQ} \bigcup V_{Pq} \bigcup V_{Pq^2} \bigcup V_{Pq^3} \ldots \bigcup V_{Pq^{m-1}} \bigcup V_{pQ} \bigcup V_{p^2Q} \bigcup \ldots \bigcup V_{p^{m-1}Q}$, $V_{PQ} = V_{pq} \bigcup V_{p^2q^2} \bigcup \ldots \bigcup V_{p^{m-1}q^{m-1}}$, $V_{Pq^i} = V_{p^{i+1}q^i} \bigcup V_{p^{i+2}q^i} \bigcup \ldots \bigcup V_{p^mq^i}, i = 1, 2, \ldots, m-1$, $V_{p^iQ} = V_{p^iq^{i+1}} \bigcup V_{p^iq^{i+2}} \bigcup \ldots \bigcup V_{p^iq^m}, i = 1, 2, \ldots, m-1$ and V_r represents the set of vertices which are multiples of r, where r is a proper divisor of n. That is, $V_r = \{x \in V : \gcd(x, n) = r, r \text{ is a proper divisor of n}\}$. The total number of vertices in $V_{p^{m-i}}, V_{q^{m-i}}$ are $p^{(m-1)-(m-i)}(p-1)q^{m-1}(q-1), p^{m-1}(p-1)q^{(m-1)-(m-i)}(q-1), 1 \leq i < m$, respectively. Again, the total number of vertices in $V_{p^{m-i}q^{m-j}}$ is $p^{(i-1)}(p-1)q^{(j-1)}(q-1), 1 \leq i, j < m$. Further, the total number of vertices in $V_{p^{m-i}q^m}, V_{p^mq^{m-i}}$ is $p^{(i-1)}(p-1), q^{(i-1)}(q-1)$, respectively.

The total number of vertices in V_P is $|V_p| + |V_{p^2}| + \cdots + |V_{p^m}| = p^{m-2}q^{m-1}(p-1)(q-1) + p^{m-3}q^{m-1}(p-1)(q-1) + \cdots + pq^{m-1}(p-1)(q-1) + q^{m-1}(p-1)(q-1) + q^{m-1}(q-1) = q^{m-1}(q-1)[p^{m-2}(p-1) + p^{m-3}(p-1) + \cdots + p(p-1) + (p-

$1)+1] = p^{m-1}q^{m-1}(q-1)$. Similarly, we find the total number of vertices in V_Q is $|V_q| + |V_{q^2}| + \cdots + |V_{q^m}|$ is $p^{m-1}q^{m-1}(p-1)$.
Again, $V_{\bar{P}\bar{Q}} = V_{PQ} \bigcup V_{Pq} \bigcup V_{Pq^2} \bigcup V_{Pq^3} \bigcup \cdots \bigcup V_{Pq^{m-1}} \bigcup V_{pQ} \bigcup V_{p^2Q} \bigcup \ldots \bigcup V_{p^{m-1}Q}$.
The total number of vertices in $V_{\bar{P}\bar{Q}}$ is $|V_{PQ}|+|V_{Pq}|+|V_{Pq^2}|+\cdots+|V_{Pq^{m-1}}|+|V_{pQ}|+|V_{p^2Q}| + \cdots + |V_{p^{m-1}Q}| = \sum_{i=0}^{m-2} p^i q^i (p-1)(q-1) + (p-1)(q-1)\sum_{i=0}^{m-3} p^i q^{m-2} + q^{m-2}(q-1)+\sum_{i=0}^{m-4} p^i q^{m-3}(p-1)(q-1)+q^{m-3}(q-1)+\sum_{i=0}^{m-5} p^i q^{m-4}(p-1)(q-1)+ q^{m-4}(q-1)+\cdots+\sum_{i=0}^{1-1} q(p-1)(q-1)+q(q-1)+(q-1)+\sum_{i=0}^{m-3} p^{m-2}q^i(p-1)(q-1) +p^{m-1}(p-1)+\sum_{i=0}^{m-4} p^{m-3}q^i(p-1)(q-1)+p^{m-3}(p-1)+\cdots+p(p-1)(q-1)+ p(p-1)+(p-1) = (p-1)(q-1)[\sum_{i=0}^{m-2} p^i q^i + \sum_{i=0}^{m-3} p^i q^{m-2} + \sum_{i=0}^{m-4} p^i q^{m-3} + \sum_{i=0}^{m-5} p^i q^{m-4} + \cdots + q + \sum_{i=0}^{m-3} p^{m-2}q^i + \sum_{i=0}^{m-4} p^{m-3}q^i + \cdots + p] + (q-1)[q^{m-2} + q^{m-3} + q^{m-4} + \cdots + q + 1] + (p-1)[p^{m-2} + p^{m-3} + \cdots + p + 1]= (pq-p-q+ 1)[\sum_{i=0}^{m-2} p^i q^i + \sum_{i=0}^{m-3} p^i q^{m-2} + \sum_{i=0}^{m-4} p^i q^{m-3} + \sum_{i=0}^{m-5} p^i q^{m-4} + \cdots + \sum_{i=0}^{1} p^i q^2 + \sum_{i=0}^{0} p^i q+\sum_{i=0}^{m-3} p^{m-2}q^i+\sum_{i=0}^{m-4} p^{m-3}q^i+\sum_{i=0}^{m-5} p^{m-4}q^i+\cdots+\sum_{i=0}^{1} p^2q^i+\sum_{i=0}^{0} pq^i]+ [q^{m-1}+q^{m-2}+\cdots+q^2+q-q^{m-2}-q^{m-3}-\cdots-q-1]+[p^{m-1}+p^{m-2}+\cdots+p^2+p-p^{m-2}- p^{m-3}-\cdots-p-1]= [\sum_{i=0}^{m-2} p^{i+1}q^{i+1}+\sum_{i=0}^{m-3} p^{i+1}q^{m-2+1}+\sum_{i=0}^{m-4} p^{i+1}q^{m-3+1}+\ldots+ \sum_{i=0}^{1} p^{i+1}q^3+\sum_{i=0}^{0} p^{i+1}q^2+\sum_{i=0}^{m-3} p^{m-2+1}q^{i+1}+\sum_{i=0}^{m-4} p^{m-3+1}q^{i+1}+\sum_{i=0}^{m-5} p^{m-4+1}q^{i+1} +\cdots+\sum_{i=0}^{1} p^3q^{i+1}+\sum_{i=0}^{0} p^2q^{i+1}-\sum_{i=0}^{m-2} p^{i+1}q^i-\sum_{i=0}^{m-3} p^{i+1}q^{m-2}-\sum_{i=0}^{m-4} p^{i+1}q^{m-3}- \cdots-\sum_{i=0}^{1} p^{i+1}q^2-\sum_{i=0}^{0} p^{i+1}q-\sum_{i=0}^{m-3} p^{m-2+1}q^i-\sum i = 0_{m-4}^{p^{m-3+1}q^i}-\sum_{i=0}^{m-5} p^{m-4+1}q^i- \cdots-\sum_{i=0}^{1} p^3q^i-\sum_{i=0}^{0} p^2q^i-\sum_{i=0}^{m-2} p^i q^{i+1}-\sum_{i=0}^{m-3} p^i q^{m-2+1}-\sum_{i=0}^{m-4} p^i q^{m-3+1}- \cdots-\sum_{i=0}^{1} p^i q^{2+1}-\sum_{i=0}^{0} p^i q^{1+1}-\sum_{i=0}^{m-3} p^{m-2}q^{i+1}- \sum_{i=0}^{m-4} p^{m-3}q^{i+1}-\sum_{i=0}^{m-5} p^{m-4}q^{i+1}-\cdots-\sum_{i=0}^{1} p^2q^{i+1}-\sum_{i=0}^{0} pq^{i+1}+\sum_{i=0}^{m-2} p^i q^i + \sum_{i=0}^{m-3} p^i q^{m-2}+\sum_{i=0}^{m-4} p^i q^{m-3}+\sum_{i=0}^{m-5} p^i q^{m-4}+\cdots+\sum_{i=0}^{1} p^i q^2+\sum_{i=0}^{0} p^i q+\sum_{i=0}^{m-3} p^{m-2}q^i +\sum_{i=0}^{m-4} p^{m-3}q^i+\sum_{i=0}^{m-5} p^{m-4}q^i+\cdots+\sum_{i=0}^{1} p^2q^i+\sum_{i=0}^{0} pq^i]+[q^{m-1}-1]+[p^{m-1}-1] = p^{m-1}q^{m-1}-p^{m-1}-q^{m-1}+1+q^{m-1}-1+p^{m-1}-1 = p^{m-1}q^{m-1}-1$.

Example 2.3. Consider the graph $G_n = (V, E)$, where $n = 216 = 2^3 \cdot 3^3$, $p = 2, q = 3, m = 3$. The vertices of the graph G_{216} are the natural numbers which are less than n but not relatively prime to n. Thus the order of the graph G_{216} is $|V| = 216 - \phi(216) - 1 = 143$. The vertices are $V = \{$2,3,4,6,8,9,10,12,14,15,16,18,20,21, 22,24,26,27,28,30,32,33,34, 36,38,39,40,42,44, 45,46,48,50,51,52,54,56,57,58,60, 62, 63,64,66,68,69,70,72,74,75,76,78,80,81,82,84,86, 87,88,90,92,93,94,96,98,99,100,102, 104,105,106,108,110,111,112,114,116, 117,118, 120,122,123,124,126,128, 129,130,132, 134,135,136,138, 140,141,142,144,146,147,148,150,152,153,154,156,158,159, 160,162, 164,165,166,168,170,171,172,174,176,177,178, 180,182,183,184, 186,188,189,190,192, 194,195,196,198, 200,201,202,204,206,207,208, 210, 212,213, 214$\}$.

The proper divisors of 216 are $2, 2^2, 2^3, 3, 3^2, 3^3, 6 = 2 \cdot 3, 36 = 2^2 \cdot 3^2, 12 = 2^2 \cdot 3, 24 = 2^3 \cdot 3, 18 = 2 \cdot 3^2, 54 = 2 \cdot 3^3, 72 = 2^3 \cdot 3^2, 108 = 2^2 \cdot 3^3$.

The vertex set of G_{216} can be represented as $V = V_P \bigcup V_Q \bigcup V_{\bar{P}\bar{Q}}$, where $V_P =$

$V_2 \bigcup V_{2^2} \bigcup V_{2^3}$, $V_Q = V_3 \bigcup V_{3^2} \bigcup V_{3^3}$, $V_{\bar{P}\bar{Q}} = V_{PQ} \bigcup V_{Pq} \bigcup V_{Pq^2} \bigcup V_{pQ} \bigcup V_{p^2Q}$, $V_{PQ} = V_{2 \cdot 3} \bigcup V_{2^2 . 3^2} = V_6 \bigcup V_{36}$, $V_{Pq} = V_{2^2 \cdot 3} \bigcup V_{2^3 \cdot 3} = V_{12} \bigcup V_{24}$, $V_{Pq^2} = V_{2^3 . 3^2} = V_{72}$, $V_{pQ} = V_{2 \cdot 3^2} \bigcup V_{2 \cdot 3^3} = V_{18} \bigcup V_{54}$, $V_{p^2Q} = V_{2^2 . 3^3} = V_{108}$. Thus, $V_{\bar{P}\bar{Q}} = V_6 \bigcup V_{36} \bigcup V_{12} \bigcup V_{24} \bigcup V_{72} \bigcup V_{18} \bigcup V_{54} \bigcup V_{108}$.

By V_x, we mean $V_x = \{x \in V \,|\, \gcd(x, 216) = x\}$, where x is a proper divisor of $n = 216$.

Thus $V_2 = \{2, 10, 14, 22, 26, 34, 38, 46, 50, 58, 62, 70, 74, 82, 86, 94, 98, 106, 110, 118, 122, 130, 134, 142, 146, 154, 158, 166, 170, 178, 182, 190, 194, 202, 206, 214\}$,
$V_4 = \{4, 20, 28, 44, 52, 68, 76, 92, 100, 116, 124, 140, 148, 164, 173, 188, 196, 212\}$,
$V_8 = \{8, 16, 32, 40, 56, 64, 80, 88, 104, 112, 128, 136, 152, 160, 176, 184, 200, 208\}$
$V_3 = \{3, 15, 21, 33, 39, 51, 57, 69, 75, 87, 93, 105, 111, 123, 129, 141, 147, 159, 165, 177, 183, 195, 201, 213, \}$, $V_9 = \{9, 45, 63, 99, 117, 153, 207, 171\}$,
$V_{27} = \{27, 81, 135, 189\}$
$V_6 = \{6, 30, 42, 66, 78, 102, 114, 138, 150, 174, 186, 210\}$, $V_{36} = \{36, 180\}$
$V_{12} = \{12, 60, 84, 132, 156, 204\}$, $V_{24} = \{24, 48, 96, 120, 168, 192\}$
$V_{72} = \{72, 144\}$, $V_{18} = \{18, 90, 126, 198\}$,
$V_{54} = \{54, 162\}$, $V_{108} = \{108\}$

The cardinalities of V_2, V_4, V_8, V_3, V_9, V_{27}, V_6, V_{36}, V_{12}, V_{24}, V_{72}, V_{18}, V_{54} and V_{108} are 36, 18, 18, 24, 8, 4, 12, 2, 6, 6, 2, 4, 2, 1, respectively.

The total number of vertices in $V_{p^{m-i}}$ is $p^{i-1}q^{m-1}(p-1)(q-1)$, where $1 \leq i < m$. Thus to find the cardinality of the set V_2, take $i = 2$, as $V_{2^{3-2}} = V_2$, and $|V_2| = p^{2-1}q^{3-1}(p-1)(q-1) = 2^{2-1}3^{3-1}(2-1)(3-1) = 36$. Similarly, $|V_4| = |V_{2^2}| = |V_{2^{3-1}}| = p^{1-1}q^{3-1}(p-1)(q-1) = 18$, $|V_3| = |V_{3^{3-2}}| = p^{m-1}q^{i-1}(p-1)(q-1) = 2^2 \cdot 3^1 \cdot 2 = 24$, $|V_9| = p^{3-1}q^{1-1}(p-1)(q-1) = 8$, the cardinality of the set $V_{27} = V_{3^3}$ is $p^{m-1}(p-1) = 4$ and cardinality of V_8 is $|V_8| = q^{m-1}(q-1) = 18$.

The cardinality of a set of the form $V_{p^{m-i}q^{m-j}}$ is $p^{i-1}q^{j-1}(p-1)(q-1)$, where $1 \leq i, j < m$. We find, the cardinality of $V_6 = V_{2 \cdot 3} = V_{2^{m-2}3^{m-2}} = 2^{2-1}3^{2-1}(2-1)(3-1) = 2 \cdot 3 \cdot 1 \cdot 2 = 12$ and the cardinality of $V_{36} = V_{2^2 3^2} = 2^{1-1}3^{1-1}(2-1)(3-1) = 2$. Similarly, we find that the number of vertices in V_{12} is 6 by taking $i = 1, j = 2$, and the cardinality of V_{18} is 4, by taking $i = 2, j = 1$, the cardinality of the set $V_{54} = V_{2 \cdot 3^3}$ which is of the form $V_{p^{m-i}q^m} = V_{p^{m-2}q^m}$ is $p^{i-1}(p-1) = 2$, as $i = 2$, $|V_{24}| = |V_{2^3 3^{3-2}}| = 3^{2-1}(3-1) = 6$, $|V_{72}| = |V_{2^3 3^2}| = |V_{2^3 3^{3-1}}| = 3^{1-1}(3-1) = 2$, $|V_{108}| = |V_{2^2 . 3^3}| = |V_{2^{3-1} . 3^3}| = 2^{1-1}(2-1) = 1$.

The total number of vertices in V_P is $|V_2| + |V_4| + |V_8| = 72 = 2^{3-1}3^{3-1}(3-1)$. Similarly, $|V_Q| = |V_3| + |V_9| + |V_{27}| = 36 = 2^{3-1}3^{3-1}(2-1)$ and $|V_{PQ}| = |V_6| + |V_{36}| + |V_{12}| + |V_{24}| + |V_{72}| + |V_{18}| + |V_{54}| + |V_{108}| = 35 = 2^{3-1}3^{3-1} - 1$.

The study of cliques in the theory of graphs is significant. The concept of clique has been applied on the probabilistic method, sociometry, computer vision, economics, signal transmission theory, coding theory etc. [1]

Theorem 2.7. *Let $n = p^m q^m$, where p, q are distinct primes. Then the graph $G_{p^m q^m}$ contains two maximal cliques and the clique number of $G_{p^m q^m}$ is $p^{m-1}q^m - 1$.*
Proof. For $G_{p^m q^m}$, where p, q are distinct primes, following the above discussion the vertex set V of $G_{p^m q^m}$ can be represented as $V = V_P \bigcup V_Q \bigcup V_{\bar{P}\bar{Q}}$. Let $u_1, u_2 \in V_P$, then the vertices u_1, u_2 are adjacent as $\gcd(u_1, u_2) \geq p$ and the vertices in V_P form the complete subgraph $g_1 = K_{p^{m-1}q^{m-1}(q-1)}$ as the total number of vertices in V_P is $p^{m-1}q^{m-1}(q-1)$. Similarly, the vertices in V_Q form the complete subgraph $g_3 = K_{p^{m-1}q^{m-1}(p-1)}$ and the vertices in $V_{\bar{P}\bar{Q}}$ form the complete subgraph $g_2 = K_{p^{m-1}q^{m-1}-1}$.

The vertices in V_P are adjacent to all the vertices in $V_{\bar{P}\bar{Q}}$ as $u_1 \in V_P, u_2 \in V_{\bar{P}\bar{Q}}$, $\gcd(u_1, u_2) \geq p$ and the vertices in V_Q are adjacent to the vertices in $V_{\bar{P}\bar{Q}}$ as the $\gcd(u_3, u_4) \geq q, u_3 \in V_{\bar{P}\bar{Q}}, u_4 \in V_Q$ but there exists at least one pair of vertices $u_1 = p \in V_P$ and $u_2 = q \in V_Q$ such that u_1 and u_2 are not adjacent. Thus the set of vertices $V_{C_1} = V_P \bigcup V_{\bar{P}\bar{Q}}$ form the clique C_1. Similarly, the set of vertices $V_{C_2} = V_Q \bigcup V_{\bar{P}\bar{Q}}$ form the clique C_2.

Let $u_i \in V$ such that $u_i \in V \setminus V_{C_1}$. Then $u_i \in V_Q$, the vertex u_i cannot be adjacent to the vertex $p \in V_{C_1}$ as $\gcd(u_i, p) = 1$. Thus both the cliques C_1 and C_2 are maximal.

Again the number of vertices in V_{C_1} is $p^{m-1}q^{m-1}(q-1) + p^{m-1}q^{m-1} - 1 = p^{m-1}q^m - 1$ and the total number of vertices in V_{C_2} is $p^{m-1}q^{m-1}(p-1) + p^{m-1}q^{m-1} - 1 = p^m q^{m-1} - 1$. Hence the clique number of $G_{p^m q^m}$ is $p^{m-1}q^m - 1$ since $p^{m-1}q^m - 1 > p^m q^{m-1} - 1$.

2.4. Independence number and Domination number of G_n

A set of vertices in G is *independent* if no two of them are adjacent. The largest number of vertices in such a set is called the *independence number* of G.

A set D of vertices in $G = (V, E)$ is called a *dominating set* of $G = (V, E)$ if every vertex in $V \setminus D$ is adjacent to some vertex in D. The minimum cardinality among the dominating sets of G is called the *domination number* of G. The domination number of G is denoted by $\gamma(G)$. The concept of clique domination was first studied by Cozzens and Kelleher in [3]. A *clique dominating set* is a dominating set that induces a complete graph [3]. A *clique dominating graph* is a graph that contains a dominating clique. The clique domination number of G is denoted by $\gamma_{cl}(G)$. An independent dominating set of G is a set that is both dominating and independent in G [5]. The *independent domination number* of G is the minimum size of an independent dominating set. The *independent domination number* of G is denoted by $\gamma_i(G)$.

Independent set has wide range of applications in optimization theory, signal transmission analysis, classification theory, economics, scheduling and biomedical engineering etc. The concept of dominating sets have been used extensively in the applications of communication theory, social networks, game theory etc. Clique domination of various graphs was studied in [4,6]. In [7], the authors determined the independent domination number of the graph $G_{m,n}$.

In this section, we study the independence number $\alpha(G_n)$, domination number $\gamma(G_n)$, clique domination number $\gamma_{cl}(G_n)$, independent domination number $\gamma_i(G_n)$ of G_n.

Theorem 2.8. *Let $n = p_1^{r_1} p_2^{r_2} \cdots p_{k-1}^{r_{k-1}} p_k^{r_k}$, where p_i's are distinct primes for $i \in \{1, 2, \dots, k\}$, then $\alpha(G_n)$ is k. Further, the set of distinct prime factors $\{p_1, p_2, \dots, p_k\}$ form a maximal independent set in G_n.*

Proof. Consider the set $I_s = \{v_i = p_i : v_i \in V, i = 1, 2, \dots, k\}$, where p_i's are distinct primes. I_s form the independent set of G_n as $\gcd(p_i, p_j) = 1$. If possible, let $w \in V$ such that w is a composite number, then $w = p_{t_1} p_{t_2} \cdots p_{t_r}$, where $p_{t_i} \in \{p_1, p_2, \dots, p_k\}$. Then the vertex w is adjacent to at least one vertex labeled as p_i, $i = 1, 2, \dots, k$. Thus $w \notin I_s$. Hence the independence number of G_n is $\alpha(G_n) = k$ and $\{p_1, p_2, \dots, p_k\}$ is a maximal independent set.

Theorem 2.9. *Let $n = p_1^{r_1} p_2^{r_2} \cdots p_{k-1}^{r_{k-1}} p_k^{r_k}$, where p_i's are distinct primes, $k_i \in \mathbb{N}$, $i = 1, 2, \dots, k$, then $\gamma(G_n) = \gamma_{cl}(G_n) = 1$, if at least one $r_i > 1$.*

Proof. Let $n = p_1^{r_1} p_2^{r_2} \cdots p_{k-1}^{r_{k-1}} p_k^{r_k}$, where at least one $r_i > 1$, $i \in \{1, 2, \dots, k\}$. Then the domination number $\gamma(G_n) = 1$ as the vertex $v = p_1 p_2 \cdots p_k \in V$ is adjacent to all other the vertices in G_n. It is easy to see that the clique domination number $\gamma_{cl}(G_n) = 1$.

Theorem 2.10. *Let $n = p_1 p_2 \cdots p_{k-1} p_k$ where p_i's are distinct primes and $i \in \{1, 2, \dots, k\}$, then $\gamma(G_n) = \gamma_{cl}(G_n) = \gamma_i(G_n) = 2$.*

Proof. The integer $n = p_1 p_2 \cdots p_{k-1} p_k$ may be a product of two distinct primes or more than two distinct primes. Thus we consider the following two cases:

Case i. For $k = 2$, $n = p_1 p_2$. Then the vertex set V is disjoint union of V_{p_1}, V_{p_2}, where $V_{p_1} = \{x \in V : \gcd(x, n) = p_1\}$ and $V_{p_2} = \{y \in V : \gcd(y, n) = p_2\}$. Let $v_p \in V_{p_1}$ and $v_q \in V_{p_2}$. Then $\{v_p, v_q\}$ form a dominating set. It is clear that $\{v_p, v_q\}$ is an independent dominating set if $\gcd(v_p, v_q) = 1$ and $\{v_p, v_q\}$ is a clique dominating set if $\gcd(v_p, v_q) \neq 1$. Thus $\gamma(G_n) = \gamma_i(G_n) = \gamma_{cl}(G_n) = 2$.

Case ii. Let $n = p_1 p_2 \cdots p_k$. Then the vertex $v = p_1 p_2 \cdots p_{k-1}$ is adjacent to all the vertices u_i, where the label of u_i is multiple of the primes p_i, $i \in \{1, 2, \dots, k-1\}$. Now, consider the vertex $w = m \cdot p_k$, $m \in \mathbb{N}$. Then w is adjacent to all the vertices labeled as multiples of p_k. Thus the set of vertices $\{v, w\} \subseteq V$ form a dominating set and the domination number $\gamma(G_n) = 2$.

As $w = m \cdot p_k$, where $m \in \mathbb{N}$ and the following subcases may arise.

Case A. Let m be a multiple of primes q_j, where $q_j \in \{p_1, p_2, \ldots, p_{k-1}\}$. Then the vertices v and w are adjacent as $\gcd(v, w) > 1$ and v, w induce a complete subgraph, which implies $\{v, w\}$ is a clique dominating set of size two. Thus $\gamma_{cl}(G_n) = 2$.

Case B. Let $m \in \{1, s_1, s_2, \ldots, s_t\}$, where s_l is a prime distinct from $p_1, p_2, \ldots, p_k$ for $l = 1, 2, \ldots, t$. More precisely, $q_l < p_k$, if $p_1 p_2 \cdots p_{k-1} < p_k$ and $p_k < q_l < p_1 p_2 \cdots p_{k-1}$, if $p_k < p_1 p_2 \cdots p_{k-1}$. In these cases, the vertices v, w are non-adjacent as $\gcd(v, w) = 1$. Thus the dominating set $\{v, w\}$ form an independent dominating set. Hence $\gamma_i(G_n) = 2$.

Proposition 2.1. [2] *D is an independent dominating set in graph G if and only if D is a maximal independent set in G.*

Proposition 2.2. *For $n = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$, where at least one $r_i > 1$, then the independent dominating set of G_n is $\{p_1, p_2, \ldots, p_k\}$ and the independent domination number of G_n is the number of distinct primes present in the prime factorization of n.*

Proof. As $\{p_1, p_2, \ldots, p_k\}$ is a maximal independent set in G_n, it is an independent dominating set in G_n of cardinality k by Proposition 2.1, where k is the number of distinct prime factors of n.

2.5. Eulerian property of G_n

In this section we characterize the values of n for which the graphs G_n are Eulerian. It is well known [8] that a simple connected graph G is Eulerian if and only if every vertex of G is of even degree.

Lemma 2.2. *The graph G_n is Eulerian if $n = 2^m$, where $m \in \mathbb{N}$.*

Proof. Let $n = 2^m$, where $m \in \mathbb{N}$. For $m = 1$, G_n is a null graph. Hence G_n is Eulerian.

For $m = 2$, G_4 is a graph with isolated vertex, which implies G_n is Eulerian.

For $m > 2$, $n = 2^m$ and by Theorem 2.2, G_n is complete, where the order of G_n is $2^{m-1} - 1$ and each vertex is of degree $2^{m-1} - 2$, which is even. Thus G_{2^m}, $m \in \mathbb{N}$, is Eulerian.

Lemma 2.3. *The graph G_n is non-Eulerian if $n = p^m$, where p is an odd prime and $m > 1 \in \mathbb{N}$.*

Proof. Let $n = p^m$, where p is an odd prime and $m > 1 \in \mathbb{N}$. Then by Theorem 2.2, the order of G_n is $p^{m-1} - 1$ and G_n is complete. So the degree of each vertex is $p^{m-1} - 2$, which is always an odd integer as p^{m-1} is an odd integer. Hence the result follows.

Observation 2.3. *For $n = 12$, G_n is Eulerian.*
Proof. Easily it can be seen that G_{12} is Eulerian from Figure 1.

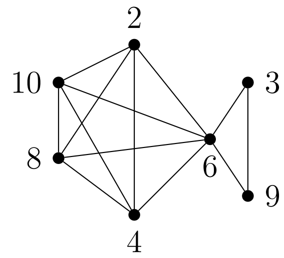

Figure 1: G_{12}

Lemma 2.4. *For $n = 2p_1^{r_1}p_2^{r_2}\cdots p_k^{r_k}$, where p_i's are distinct odd primes, $i \in \{1, 2, \ldots, k\}$, $r_i \in \mathbb{N}$, G_n is non-Eulerian.*
Proof. Let $n = 2p_1^{r_1}p_2^{r_2}\cdots p_k^{r_k}$, where p_i's are distinct odd primes. Since n is even, so $u = 2$ is a vertex of G_n. The number of multiples of 2 up to n is $\frac{n}{2}$, but $n \notin V$, so the number of multiples of 2 up to $n-1$ is $\frac{n}{2}-1$ and the vertex $u = 2$ is not adjacent to itself, thus the degree of the vertex $u = 2 \in V$ is $\frac{n}{2} - 2 = p_1^{r_1}p_2^{r_2}\cdots p_k^{r_k} - 2$, which is an odd integer as $p_1^{r_1}p_2^{r_2}\cdots p_k^{r_k}$ is an odd integer. Hence G_n is non-Eulerian.

Lemma 2.5. *The graph G_n is non-Eulerian if $n = p_1^{r_1}p_2^{r_2}\cdots p_k^{r_k}$, $i = 1, 2, \ldots, k$, $r_i \in \mathbb{N}$, where each p_i is a distinct odd prime.*
Proof. Let $n = p_1^{r_1}p_2^{r_2}\cdots p_k^{r_k}$, where p_i's are distinct odd primes, $r_i \in \mathbb{N}$, $i = 1, 2, \ldots, k$. Consider the following cases.
Case i. Let $n = p_1p_2\cdots p_k$. Then by Theorem 2.3, the minimum degree of G_n is $p_1p_2\cdots p_{k-1} - 2$, which is odd. Thus G_n is non-Eulerian.
Case ii. Let $n = p_1^{r_1}p_2^{r_2}\cdots p_k^{r_k}$, where at least one $r_i > 1$. Then the order of G_n is $|V| = n - \phi(n) - 1 = n - (\phi(n) + 1)$, where the cardinality of V is even as n is odd, $\phi(n)$ is even and $\phi(n) + 1$ is odd. Again, by Theorem 2.5, the maximum degree of G_n is $|V| - 1$, which is an odd integer. Hence G_n is non-Eulerian.

Lemma 2.6. *Let $n = 2^m p_1^{r_1}p_2^{r_2}\cdots p_k^{r_k}$, where p_i's are distinct odd primes and $m > 1$, $r_i \in \mathbb{N}$ for $i \in \{1, 2, \ldots, k\}$. Then G_n is non-Eulerian.*
Proof. Consider $n = 2^m p_1^{r_1}p_2^{r_2}\cdots p_k^{r_k}$, where p_i's are distinct odd primes. Let p_1 be the smallest odd prime in the prime factorization of n. For an odd integer t, the total number of vertices of the form $u = 2t \in V$ is $\frac{n}{2} - 1$. Let p_1 be the smallest odd prime in the prime factorization of n and let p be the highest prime among the primes less than or equal to $\frac{n}{p_1}$. It is clear that $u_1 = 2p \in V$ and the degree of the vertex $u_1 = 2p$ is at least $\frac{n}{2} - 2$. That is, $\deg(u_1) \geq \frac{n}{2} - 2$, as u_1 is adjacent to

all the vertices labeled as $u = 2t$, where t is an odd integer except itself. Again the vertex $u_1 = 2p$ is adjacent to the vertex $u_2 = pp_1$ as the $\gcd(u_1, u_2) = p > 1$. We show that the degree of $u_1 = 2p$ is $\deg(u_1) = \frac{n}{2} - 2 + 1 = \frac{n}{2} - 1$, which is odd as $\frac{n}{2}$ is even for $m > 1$. The vertex $u_1 = 2p$ cannot be adjacent to any other vertex labeled as an odd integer less than n as $pp_1 < n < pp_i$, where $i \in \{2, 3, \ldots, k\}$ and p_i's are odd primes greater than p_1. Thus G_n is non-Eulerian.

Example 2.4. Let $n = 700 = 2^2 \cdot 5^2 \cdot 7$. Then in G_{700}, the total number of vertices of the form $u = 2t$, where t is an odd integer is $\frac{n}{2} - 1 = 349$, (-1 is coming as $n \notin V$). In the prime factorization of n, 5 is the smallest odd prime. Thus we find $\frac{n}{5} = 140$ and the highest prime less than 140 is 139. Clearly, $278 = 2 \cdot 139 \in V$. Now, we find the degree of $u_1 = 278 = 2 * 139$. The vertex u_1 is adjacent to all the vertices (except itself) labeled as even integer in G_n and u_1 is adjacent to the vertex labeled as $u_2 = 5 * 139 = 695$. Thus the degree of u_1 is $\deg(u_1) = \deg(278) = \deg(2 * 139) = \frac{n}{2} - 1 - 1 + 1 = \frac{n}{2} - 1 = 349$, which is odd. Hence the graph G_{700} is non-Eulerian.

Theorem 2.11. *Let $n = p^m$, where p is a prime. Then G_n is Eulerian if and only if $p = 2$ or $n = 12$.*

Proof. The proof follows immediately from Lemma 2.2, Lemma 2.3, Lemma 2.4, Lemma 2.5, Lemma 2.6 and Observation 2.3.

3. Conclusion

In this paper we have defined and studied an undirected graph G_n for $n \in \mathbb{N}$ and $n > 1$, whose vertex set comprises of the natural numbers which are less than n but not relatively prime to n, where n is not a prime number and two distinct vertices are adjacent if and only if the labels of the vertices are not coprime. We have studied connectedness, completeness, minimum degree, maximum degree of G_n. Interestingly we have observed relations between the maximum degree of G_n and the prime counting function. For $n = p_1 p_2 \cdots p_k$, where p_i's are distinct primes, if we know the maximum degree of G_n, then we can estimate the number of primes between $n = p_1 p_2 \cdots p_{k-1}$ and p_k due to Observation 2.1 and Observation 2.2. We have studied the independence number, domination number, clique domination number, independent domination number and Eulerian property of G_n.

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