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On the Projective Flatness of a Finsler space with infinite series (α, β)-metric $\frac{\beta^2}{\beta-\alpha}$

Gauree Shanker and Sruthy Asha Baby
Department of Mathematics and Statistics
Banasthali University, Banasthali
Rajasthan-304022, India
Email:gsp.math.1978@gmail.com, sruthymuthu123@gmail.com

Abstract: The (α, β)-metric is a Finsler metric which is constructed from a Riemannian metric α and a differential 1-form β; it has been sometimes treated in theoritical physics. We consider the projective flatness of Finsler space with (α, β)-metric. Projective Finsler geometry is an important part of Finsler geometry and has been studied by so many differential geometers e.g., Matsumoto, Shen, H. S. Park etc. For two Finsler spaces (M, F) and $(M, \bar{F})$ on a common underlying manifold M, we say (M, F) is projective to $(M, \bar{F})$ (or Finsler metric F is projectively related to Finsler metric $\bar{F}$) if any geodesic of (M, F) coincides with a geodesic of $(M, \bar{F})$ as a set of points and vice-versa. The purpose of this paper, is to find the conditions for Finsler space with infinite series (α, β)-metric -$\frac{\beta^2}{\beta-\alpha}$ to be projectively flat.

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1. Introduction

Let $F^n = (M^n, L)$ be an n-dimensional Finsler space, that is, an n-dimensional differentiable manifold M^n equipped with a fundamental function L(x,y). The concept of an (α, β)- metric $L(\alpha, \beta)$ was introduced in 1972 by M. Matsumoto [6]. A Finsler metric L(x,y) is called an (α, β)-metric $L(\alpha, \beta)$, if L is positively homogenous function of α and β of degree one, where $\alpha^2 = a_{ij}(x)y^i y^j$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a one form on M^n. The most interesting examples of (α, β) metric are the Randers metric, Kropina metric and Matsumoto metric.

A Finsler space $F^n = (M^n, L)$ is called a locally Minkowski space[8], if M^n is covered by coordinate neighborhood system (x^i) in each of which L is a function of y^i only and the geodesics can be represented by $n-1$ linear equations of coordinates. A Finsler space $F^n = (M^n, L)$ is called projectively flat, if F^n is projective to a

locally Minkowski space. In fact, a projectively flat Finsler space is Finsler space with rectilinear extremals, which was originally suggested by Hilbert.

The condition for a Finsler space with (α, β)-metric to be projectively flat was first studied by L. Berwald in tensorial form and completed by M. Matsumoto [7]. Aikou, Hashiguchi and Yamauchi [1] and Hashiguchi, Ichijyo [4] gave interesting results on the projective flatness of Matsumato space. A locally Minskowski space with (α, β)-metric is called flat-parallel [2], if α is locally flat and β is parallel with respect to α.

The purpose of the present paper is to consider the projective flatness of Finsler space with infinite series (α, β)-metric.Throughout the present paper we use the terminology and notations in Matsumoto monograph's [8].

2. Preliminaries

Let $F^n = (M^n, L(\alpha, \beta))$ be an n-dimensional Finsler space with an (α, β)-metric and $R^n = (M^n, \alpha)$, the associated Riemannian space, where $\alpha^2 = a_{ij}(x)y^i y^j$. Let γ^i_{jk} be the Christoffel symbols with respect to α and denoted by the semicolon (;), the covarient differentiation with respect to γ^i_{jk}. From the differential 1- form $\beta(x, y) = b_i(x)y^i$, we define

$$2r_{ij} = b_{i;j} + b_{j;i}\ ,\ 2s_{ij} = b_{i;j} - b_{j;i},\ s^i_j = a^{ir}s_{rj},$$

$$s_i = b^r s_{ri},\ \gamma_{jhk} = a_{hr}\gamma^r_{jk},\ b^2 = a^{rs}b_r b_s. \tag{1}$$

Next, we consider the Berwald connection $B\Gamma = \left(G^i_{jk}, G^i_j, 0\right)$ of the Finsler space with (α, β) metric $L(\alpha, \beta)$. It is well known that

$G^i_{jk} = \dot{\partial}_k G^i_j, 2G^i = G^i_0 (= G^i_r y^r), G^i_j = \dot{\partial}_j G^i$.

We shall denote the homogenous polynomials in (y^i) of degree r by hp(r) for brevity and the subscript 0 means contraction by y^i, for instance; $t_o = t_i y^i$.
Now the following Matsumato's theorem [7] is well known:

Theorem 2.1 *A Finsler space $F^n = (M^n, L)$ with an (α, β)-metric $L(\alpha, \beta)$ is projectively flat if and only if for any point of space M, there exist local coordinate neighborhoods containing the point such that γ^i_{jk} satisfies:*

$$\frac{1}{2}\Big(\gamma^i_{oo} - \frac{\gamma_{ooo}y^i}{\alpha^2}\Big) + \Big(\frac{\alpha L_\beta}{L_\alpha}\Big)s^i_o + \Big(\frac{L_{\alpha\alpha}}{L_\alpha}\Big)(C + \frac{\alpha r_{oo}}{2\beta})\Big(\frac{\alpha^2 b^i}{\beta} - y^i\Big) = 0, \tag{2}$$

where a subscript 0 means a contraction by y^i, $L_\alpha = \frac{\partial L}{\partial \alpha}$, $L_\beta = \frac{\partial L}{\partial \beta}$, $L_{\alpha\alpha} = \frac{\partial L_\alpha}{\partial \alpha}$, $L_{\beta\beta} = \frac{\partial L_\beta}{\partial \beta}$ and C is given by

$$C+\Big(\frac{\alpha^2 L_\beta}{\beta L_\alpha}\Big)s_o+\Big(\frac{\alpha L_{\alpha\alpha}}{\beta^2 L_\alpha}\Big)(\alpha^2b^2-\beta^2)\Big(C+\frac{\alpha r_{00}}{2\beta}\Big)=0. \tag{3}$$

By the homogeneity of L, we know that $\alpha^2 L_{\alpha\alpha}=\beta^2 L_{\beta\beta}$, *so the formula (3) can be rewritten in the following form:*

$$\Big\{1+\Big(\frac{L_{\beta\beta}}{\alpha L_\alpha}\Big)(\alpha^2b^2-\beta^2)\Big\}\Big(C+\frac{\alpha r_{00}}{2\beta}\Big)=\Big(\frac{\alpha}{2\beta}\Big)\{r_{00}-\frac{2\alpha L_\beta}{L_\alpha}s_o\Big\}. \tag{4}$$

If $1+\frac{L_{\beta\beta}}{\alpha L_\alpha}(\alpha^2b^2-\alpha^2)\neq 0$, then we can eliminate $(C+\frac{\alpha r_{oo}}{2\beta})$ in (2) and it is written as the form:

$$\begin{aligned}&\Big\{1+\Big(\frac{L_{\beta\beta}}{\alpha L_\alpha}\Big)(\alpha^2b^2-\beta^2)\Big\}\Big\{\frac{1}{2}\Big(\gamma^i_{oo}-\frac{\gamma_{ooo}y^i}{\alpha^2}\Big)+\Big(\frac{\alpha L_\beta}{L_\alpha}\Big)s^i_0\Big\}+\Big(\frac{L_{\alpha\alpha}}{L_\alpha}\Big)\Big(\frac{\alpha}{2\beta}\Big)\\&\Big\{r_{oo}-\Big(\frac{2\alpha L_\beta}{L_\alpha}\Big)s_o\Big\}\Big(\frac{\alpha^2b^i}{\beta}-y^i\Big)=0\end{aligned} \tag{5}$$

Thus we have

Theorem 2.2 *If* $1+\Big(\frac{L_{\beta\beta}}{\alpha L_\alpha}\Big)(\alpha^2b^2-\beta^2)\neq 0$, *then a Finsler space* F^n *with an* (α,β)*-metric is projectively flat if and only if (5) is satisfied.*

It is known [3] that if α^2 contains β as a factor, then the dimension is equal to two and b^2 =0. Throughout this paper, we assume that the dimension is more than two and $b^2\neq 0$, that is, $\alpha^2\ncong 0(\mathrm{mod}\ \beta)$.

3. Projectively Flat Finsler space with infinite series (α,β)-metric

Let us consider the r-series (α,β)-metric

$$F(\alpha,\beta)=\beta\sum_{k=0}^{r}\left(\frac{\alpha}{\beta}\right)^k,$$

where we assume $\alpha<\beta$.
If r=1, then $F(\alpha,\beta)=\alpha+\beta$ is a Randers metric.
If r=2, then $F(\alpha,\beta)=\alpha+\beta+\frac{\alpha^2}{\beta}$.

If r=3, then $F(\alpha,\beta)=\alpha+\beta+\frac{\alpha^2}{\beta}+\frac{\alpha^3}{\beta^2}$.

. . .

If $r=\infty$, then the above metric is expressed in the form

$$F(\alpha,\beta)=\frac{\beta^2}{\beta-\alpha}. \tag{6}$$

It is called infinite series (α,β)-metric and the space equipped with this metric is called the Finsler space with infinite series (α,β)-metric.

We consider the Finsler space L with infinite series (α,β)-metric (6). We can easily find that

$$\begin{aligned} L_\alpha &= \frac{\beta^2}{(\beta-\alpha)^2},\\ L_\beta &= \frac{\beta(\beta-2\alpha)}{(\beta-\alpha)^2},\\ L_{\alpha\alpha} &= \frac{2\beta^2}{(\beta-\alpha)^3},\\ L_{\beta\beta} &= \frac{2\alpha^2}{(\beta-\alpha)^3}. \end{aligned} \tag{7}$$

From (6) and (7), we have the following identities.

$$\begin{aligned} L_\alpha &= \frac{L}{(\beta-\alpha)},\\ L_\beta &= \frac{(\beta-2\alpha)L}{\beta(\beta-\alpha)},\\ L_{\alpha\alpha} &= \frac{2L}{(\beta-\alpha)^2},\\ L_{\beta\beta} &= \frac{2\alpha^2 L}{\beta^2(\beta-\alpha)^2}. \end{aligned} \tag{8}$$

Now, equation (3) can be written as

$$\Big\{1+\Big(\frac{\alpha L_{\alpha\alpha}}{\beta^2 L_\alpha}\Big)(\alpha^2b^2-\beta^2)\Big\}\Big(C+\frac{\alpha r_{00}}{2\beta}\Big)=\frac{\alpha}{2\beta}r_{00}-\Big(\frac{\alpha^2 L_\beta}{\beta L_\alpha}\Big)s_0, \tag{9}$$

Now, using (7) in (9), we get

$$2\{\beta^3 - 3\alpha\beta^2 + 2\alpha^3 b^2\}\Big(C + \frac{\alpha r_{00}}{2\beta}\Big) = (\beta - \alpha)[\alpha r_{00}\beta - 2s_0\alpha^2(\beta - 2\alpha)] \tag{10}$$

Using this value in (2) with the help of (7), we get

$$\begin{aligned}&\frac{1}{2}\Big(\gamma^i_{oo} - \frac{\gamma_{ooo}y^i}{\alpha^2}\Big) + \Big(\frac{\alpha(\beta - 2\alpha)}{\beta}\Big)s^i_o + \Big(\frac{\alpha r_{00}\beta - 2s_0\alpha^2(\beta - 2\alpha)}{\beta^3 - 3\alpha\beta^2 + 2\alpha^3 b^2}\Big)\\ &\Big(\frac{\alpha^2 b^i}{\beta} - y^i\Big) = 0\end{aligned} \tag{11}$$

which gives

$$\begin{aligned}&\{\beta(\alpha^2\gamma^i_{00} - \gamma_{000}y^i) + 2\alpha^3(\beta - 2\alpha)s^i_0\}(\beta^3 - 3\alpha\beta^2 + 2\alpha^3 b^2) + 2\alpha^3(r_{00}\beta -\\ &2s_0\alpha(\beta - 2\alpha))(\alpha^2 b^i - \beta y^i) = 0\end{aligned}$$

which can also be re-arranged as follows

$$\begin{aligned}&\alpha\{\alpha^6(-8b^2 s^i_0 + 8b^i s_0) + \alpha^4(12\beta^2 s^i_0 + 2\beta b^2\gamma^i_{00} + 2r_{00}b^i\beta - 8s_0\beta y^i) +\\ &\alpha^2(2\beta^4 s^i_0 - 3\beta^3\gamma^i_{00} - 2\beta b^2\gamma_{000}y^i - 2r_{00}\beta^2 y^i) + 3\beta^3\gamma_{000}y^i\} + \alpha^6(4b^2\beta s^i_0\\ &-4b^i s_0\beta) + \alpha^4(4s_0\beta^2 y^i - 4\beta^3 s^i_0 - 6\beta^3 s^i_0) + \alpha^2\beta^4\gamma^i_{00} - \beta^4\gamma_{000}y^i = 0\end{aligned} \tag{12}$$

which is of the type $P^i\alpha + Q^i = 0$, where

$$\begin{aligned}P^i \;=\;& \alpha^6(-8b^2 s^i_0 + 8b^i s_0) + \alpha^4(12\beta^2 s^i_0 + 2\beta b^2\gamma^i_{00} + 2r_{00}b^i\beta - 8s_0\beta y^i) +\\ &\alpha^2(2\beta^4 s^i_0 - 3\beta^3\gamma^i_{00} - 2\beta b^2\gamma_{000}y^i - 2r_{00}\beta^2 y^i) + 3\beta^3\gamma_{000}y^i\end{aligned} \tag{13}$$

$$\begin{aligned}Q^i \;=\;& \alpha^6(4b^2\beta s^i_0 - 4b^i s_0\beta) + \alpha^4(4s_0\beta^2 y^i - 4\beta^3 s^i_0 - 6\beta^3 s^i_0) + \alpha^2(\beta^4\gamma^i_{00})\\ &-\beta^4\gamma_{000}y^i\end{aligned} \tag{14}$$

From (13) and (14), we observe that both P^i and Q^i are rational function in (x^i, y^i) and α is irrational function in (x^i, y^i).
Hence $P^i\alpha + Q^i = 0$ will be satisfied if and only if $P^i = 0$ and $Q^i = 0$.

From $Q^i = 0$, we observe that $\beta^4\gamma_{000}y^i$ has a factor α^2. Therefore, there exists function $\lambda_i(x)$ such that

$$\gamma_{000} = \alpha^2\lambda_0. \tag{15}$$

Using (15) in $P^i = 0$, we observe that $2b^i\beta r_{00}$ has a factor α^2. Therefore, there exist a function $\mu(x)$ such that

$$r_{00} = \alpha^2\mu(x). \tag{16}$$

Using (15) and (16) in $Q^i = 0$, we get

$$\alpha^4(4b^2s_0^i - 4b^is_0) + \alpha^2(4s_0\beta y^i - 10\beta^2 s_0^i) + \beta^3\gamma_{00}^i - \beta^3\lambda_0 y^i = 0. \tag{17}$$

From (17), we observe that $\beta^3(\gamma_{00}^i - \lambda_0 y^i)$ has a factor α^2. Therefore there exist a function $\upsilon^i(x)$ such that

$$\beta^3(\gamma_{00}^i - \lambda_0 y^i) = \alpha^2\upsilon^i(x) \tag{18}$$

Transvecting (18) by $y_i = a_{ir}y^r$ and using (15), we get $\upsilon_0 = 0$, which gives $\upsilon^i = 0$. Using this in (18), we get

$$\gamma_{00}^i = \lambda_0 y^i. \tag{19}$$

This equation gives,

$$\gamma_{jk}^i y^j y^k = \lambda_0 y^i.$$
$$or \quad \gamma_{jk}^i y^j y^k = \lambda_s y^s y^i. \tag{20}$$

Differentiating (20) with respect to y^l, we get

$$\gamma_{jk}^i y^j\delta_l^k = \lambda_s y^i\delta_l^s + \lambda_s y^s\delta_l^i.$$

$$\gamma_{jl}^i y^j = \lambda_l y^i + \lambda_s y^s\delta_l^i. \tag{21}$$

Differentiating (21) with respect to y^m, we get

$$\gamma_{jk}^i = \lambda_k\delta_j^i + \lambda_j\delta_k^i.$$

This equation shows the projective flatness of the associated Riemannian space of Finsler space with metric $\frac{\beta^2}{\beta-\alpha}$. Putting (19) in $P^i = 0$, we get

$$s_0^i = \frac{-2b^i\alpha^2 s_0}{3\beta^2} \tag{22}$$

Contracting (22) by y_i and using $s_0^i y_i = 0$, we get $s_0 = 0$. Using (15) and (19) in (13), we get

$$\begin{aligned} &s_0^i(2\alpha^2\beta^4 + 12\beta^2\alpha^4 - 8b^2\alpha^6) + s_0(8b^i\alpha^6 - 8\alpha^4 s_0\beta y^i) - \alpha^4\beta b^2\lambda_0 y^i \\ &+2r_{00}\beta\alpha^2(\alpha^2 b^i - \beta y^i) = 0 \\ \\ &s_0^i(2\beta^4 + 12\beta^2\alpha^2 - 8b^2\alpha^4) + s_0(8b^i\alpha^4 - 8\alpha^2 s_0\beta y^i) - \alpha^2\beta b^2\lambda_0 y^i + \\ &2r_{00}\beta(\alpha^2 b^i - \beta y^i) = 0 \end{aligned} \tag{23}$$

Since $s_0^i = 0$, $s_{ij} = 0$ (from (1)).
From (22), we get

$$\beta r_{00}(\alpha^2 b^i - \beta y^i) = 0 \tag{24}$$

$$r_{00} = 0 \tag{25}$$

Since $r_{00} = 0$, we get $r_{ij} = 0$ ($r_{00} = r_{ij}b^i b^j$) (from (1)). Hence, we get $b_{i;j} = 0$.
Conversely, assume that the associated Riemannian space of Finsler space with metric $\frac{\beta^2}{\beta-\alpha}$ is projectively flat and $b_{i;j} = 0$. Then, since $2s_{ij} = b_{i;j} - b_{j;i}$ and $2r_{ij} = b_{i;j} + b_{j;i}$, we get $r_{ij} = 0$ and $s_{ij} = 0$ which gives $r_{00} = 0$, $s_0^i = 0$, $s_0 = 0$.
Using all these results with equation (19), we see that equation (12) is satisfied identically. Therefore, the Finsler space with infinite series (α, β)-metric $\frac{\beta^2}{\beta-\alpha}$ is projectively flat.

Theorem 3.1 *A Finsler space with infinite series (α, β)-metric $F = \frac{\beta^2}{\beta-\alpha}$ is projectively flat if and only if the associated Riemannian space is projectively flat and $b_{i;j} = 0$.*

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