South East Asian J. of Mathematics and Mathematical Sciences
Vol. 18, No. 3 (2022), pp. 369-380

DOI: 10.56827/SEAJMMS.2022.1803.31 **ISSN (Online): 2582-0850**

ISSN (Print): 0972-7752

DOUBLE ROMAN DOMINATION NUMBER OF MIDDLE GRAPH

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(**Received:** *Mar. 07, 2022* **Accepted:** *Dec. 26, 2022* **Published:** *Dec. 30, 2022)*

Abstract: For any graph $G(V,\ E)$, a function $f : V(G) \to \{0,\ 1,\ 2,\ 3\}$ is called Double Roman dominating function (DRDF) if the following properties holds,

1. If $f(v) = 0$, then there exist two vertices $v_1, v_2 \in N(v)$ for which $f(v_1) = f(v_2) = 2$ or there exist one vertex $u \in N(v)$ for which $f(u) = 3$.

2. If $f(v) = 1$, then there exist one vertex $u \in N(v)$ for which $f(u) = 2$ or $f(u) = 3$.

The weight of DRDF is the value $w(f) = \sum_{v \in V(G)} f(v)$. The minimum weight among all double Roman dominating function is called double Roman domination number and is denoted by $\gamma_{dR}(G)$. In this article we initiated research on double Roman domination number for middle graphs. We established lower and upper bounds and also we characterize the double Roman domination number of middle graphs. Later we calculated numerical value of double Roman domination number of middle graph of path, cycle, star, double star and friendship graphs.

Keywords and Phrases: Roman Domination, Double Roman Domination, Middle Graph.

2020 Mathematics Subject Classification: 05C69, 05C38.

1. Introduction

Let $G(V,\ E)$ be a simple finite, and undirected graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The cardinality of the vertex set is order and the cardinality of the edge set is size of the graph G. The number of edges incident on the vertex v is called degree of the vertex v and is denoted by $d(v)$. The minimum and maximum degree of G is denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$ respectively. If the degree of each vertex is r then the graph is called r-regular graph i.e. if $\forall v \in V(G), d(v) = r$. For any vertex $v \in V$, the open neighborhood $N(v) = \{u \in V(G) \setminus uv \in E(G)\}$ and the closed neighborhood $N[v] = N(v) \cup v$. A connected acyclic graph is called tree. We denote K_n for complete graph with n vertices, C_n for a cycle of length n, P_n for a path of length n. The line graph $L(G)$ of a given graph G is a graph whose vertex set $V(L(G)) = E(G)$ and two vertices $e_1, e_2 \in V(L(G))$ are adjacent if e_1 and e_2 has common vertex in G. For notation and graph theory terminology we refer [1].

R. A. Beeler et al. introduced the double Roman domination number in order to give a strong protection to an empty location and a location with one legion can be defended by two legions. A function $f : V(G) \to \{0, 1, 2, 3\}$ is said to be double Roman dominating function (DRDF) if the following properties holds,

1. If $\forall v \in V(G)$ with $f(v) = 0$, then there exist two vertices $v_1, v_2 \in N(v)$ such that $f(v_1) = f(v_2) = 2$ or one vertex $u \in N(v)$ such that $f(u) = 3$.

2. If $\forall x \in V(G)$ with $f(x) = 1$, then there exist $y \in N(y)$ such that either $f(y) = 2$ or $f(y) = 3$.

The summation of the function value of double Roman dominating function is called weight of DRDF. The minimum weight among all the DRDF is called double Roman domination number of a given graph G and is denoted by γ_{dR}. The double Roman dominating function with a minimum weight is called γ_{dR}- function of graph G. The DRDF of a graph G partitioned the vertex set by V_0, V_1, V_2, V_3

where $V_i = \{v \in V(G) | f(v) = i\}$. Hence the DRDF f can be represented by the vertex partition (V_0, V_1, V_2, V_3) and clearly weight $w(f) = |V_1| + 2|V_2| + 3|V_3|$.

Let G be a given graph, the middle graph of G denoted by $M(G)$ is the graph obtained by subdividing each edge of G exactly once and joining all the adjacent vertices of G. Precisely the vertex set and the edge set of $M(G)$ is defined as follows,

- $V(M(G)) = V(G) \cup E(G)$.
- Two vertices u and v of $M(G)$ are adjacent if one of the following holds,
  - $u,\ v \in E(G)$ and $u,\ v$ are adjacent in G.
  - $u \in V(G),\ v \in E(G)$, and u, v are incident in G.

If G is a graph of order n and size m then the middle graph $M(G)$ is of order $n+m$ and size $2m + |E(L(G))|$.

We noted the following concepts to study the double Roman domination number in the class of middle graph. For $v \in V(G)$, the open neighborhood of vertex v in the middle graph $M(G)$ is defined as $\{e \in E(G) | e \textit{ is incident with } u\}$ and is denoted by $N_M(v)$. For $e \in E(G)$, the open neighborhood of an edge e in the middle graph $M(G)$ is defined as $\{x \in V(G) \cup E(G) |\ x \textit{ is either adjacent or incident with } e\}$ and is denoted by $N_M(e)$. The closed neighborhood of element of G in the middle graph of G is written as $N_M[x] = N_M(x) \cup \{x\}$. The double Roman dominating function of a middle graph of a given graph G is a function $f : V(G) \cup E(G) \to \{0,\ 1,\ 2,\ 3\}$ satisfying the following conditions,

1. For every element $v \in V(G) \cup E(G)$ with $f(v) = 0$ is adjacent to two elements $x, y \in V(G) \cup E(G)$ such that $f(x) = f(y) = 2$ or adjacent to one element $z \in V(G) \cup E(G)$ such that $f(z) = 3$.
2. For every element $v \in V(G) \cup E(G)$ with $f(v) = 1$ is adjacent one element $x \in V(G) \cup E(G)$ such that either $f(x) = 2$ or $f(x) = 3$.

The weight of double Roman dominating function f of a middle graph G is $w(f) = \Sigma_{u \in V \cup E} |f(u)|$. The minimum weight of double Roman dominating function of a middle graph G is called double Roman domination number of $M(G)$ and is denoted by $\gamma^*_{dR}(G)$. Clearly $\gamma^*_{dR}(G) = \gamma_{dR}(M(G))$ for any graph G. A double Roman dominating function of M(G) can be ordered partitioned into $(V_0 \cup E_0,\ V_1 \cup E_1,\ V_2 \cup E_2,\ V_3 \cup E_3)$ where $V_i = \{v \in V(G) \mid f(v) = \ i\}$, and $E_i = \{e \in E(G) \mid f(e) = \ i\}$. Here weight of MRDF is $w(f) = |V_1 \cup E_1| + \ 2|V_2 \cup E_2| + 3 \mid V_3 \cup E_3|$.

2. Results

Proposition 2.1. *For any graph G, $\gamma^*_{dR} = 2\gamma(M(G))$ if and only if $G = \overline{K_n}$.*
Proof. If $G = \overline{K_n}$, then $\gamma(M(G)) =| V(G) |$ and f is a γ^*_{dR}- function of $\overline{K_n}$ and is defined as $f(x) = 2, \forall x \in V(\overline{K_n})$. Hence $\gamma^*_{dR}(G) = 2\gamma(M(G))$.
On the other hand consider $\gamma^*_{dR}(G) = 2\gamma(M(G))$. Let us assume that G contains an edge uv and let f be a γ^*_{dR} function such that $f(uv) = 3,\ f(u) = 0,\ f(v) = 0$ and $f(x) = 2$ otherwise. Obviously $\gamma^*_{dR}(G) \leq 2n - 3$. Now, let S be a dominating set of the middle graph G and S contains uv and all elements other than u and v. Clearly, $\gamma(M(G)) \leq n - 1$. Which is contradiction as $\gamma^*_{dR}(G) = 2\gamma(M(G))$.

Proposition 2.2. *For any γ^*_{dR}- function of a given graph G, no vertex or no edge is assigned by the value 1.*
Proof. Let f be γ^*_{dR}- function of a given graph G. Let us assume that for some $x \in V(G) \cup E(G)$, $f(x) = 1$. Which implies from the definition of double Roman dominating function there exist an element $y \in V(G) \cup E(G)$ such that either $f(y) = 2$ or $f(y) = 3$. If $f(y) = 3$, then we obtain another double Roman dominating function strictly lesser weight than f which can be obtained by reassigning 0 to x. This contradicts our assumption as f is γ^*_{dR}- function of G. If $f(y) = 2$, then we can easily define another double Roman dominating function of middle graph of G, $g : V(G) \cup E(G) \to \{0, 1, 2, 3\}$ such that $g(v) = f(v)\ \forall\ v \in V(G) \cup E(G) - \{x,\ y\}$, $g(x) = 0,\ g(y) = 3$. Clearly weight of both the functions are equal. Hence no vertex or edge need to assign the value 1.

Proposition 2.3. *For any γ^*_{dR}- function of given graph G, no edge is assigned by the value 2 and no vertex is assigned by the value 3.*
Proof. Let f be γ^*_{dR}- function of a given graph G. Suppose that for some $e = uv \in E(G)$, $f(e) = 2$. This implies that the end vertices u and v of e and adjacent edges of u and v can take the function values either 1 or 2. Which leads the function value greater than f.
To prove no vertex is assigned by the value 3, in middle graph clearly the degree of a vertex is lesser than the degree of any incident edge of that vertex. Hence every edge $e = uv$ covers the vertices u, v, $N_M(u)$ and $N_M(v)$. Hence, in the double Roman dominating function no vertex is assigned by the value 3.

Proposition 2.4. *Let $f = (V_0 \cup E_0, V_1 \cup E_1, V_2 \cup E_2, V_3 \cup E_3)$ be a γ^*_{dR} function then,*

1. $V_1 \cup E_1 = \emptyset$

2. $V_2 \cup E_2 \subset V(G)$ *and* $V_3 \cup E_3 \subset E(G)$

3. *No edge joins from* $V_2 \cup E_2$ *to* E_3

4. *The subgraph induced by* $V_2 \cup E_2$ *has a maximum degree 1.*

Proposition 2.5. *For a given graph* G *and* $f = (V_0 \cup E_0, V_1 \cup E_1, V_2 \cup E_2)$ *is a* γ_R^**- function of* G*, then* $\gamma_{dR}^*(G) \leq 2|V_1 \cup E_1| + 3|V_2 \cup E_2|$.
Proof. Let $f = (V_0 \cup E_0, V_1 \cup E_1, V_2 \cup E_2)$ be a γ_R^*- function of G. Now let us define $g : V(G) \cup E(G) \to \{0, 1, 2, 3\}$ as $\forall x \in V_0 \cup E_0$ $g(x) = 0$, $\forall y \in V_1 \cup E_1$ $g(y) = 2$ and $\forall z \in V_2 \cup E_2$ $g(z) = 3$. Thus if $\forall u \in V(G) \cup E(G)$ with $g(u) = 0$ then there exist $v \in N_M(u)$ such that $g(u) = 3$. Hence g is double Roman dominating function of middle graph G and $\gamma_{dR}(G) \leq 2|V_1 \cup E_1| + 3|V_2 \cup E_2|$.

To prove the equality of the above proposition first we would like find the double Roman domination number for middle graph of stars.

Proposition 2.6. *For any star graph* $K_{1,\ n-1}$ *on* $n \geq 3$ *vertices,*

$$\gamma_{dR}^*(K_{1,\ n-1}) = 2n - 1.$$

Proof. Let the vertex set and edge set of star graph is $V(K_{1,\ n-1}) = \{v_0, v_1, \ldots, v_{n-1}\}$ and $E(K_{1,\ n-1}) = \{v_0v_1, v_0v_2, \ldots, v_0v_{n-1}\}$. Now the corresponding vertex and edge set of middle graph of star $K_{1,n-1}$ is $V(M(K_{1,\ n-1})) = V(K_{1,\ n-1}) \cup \mathcal{A}$ where $\mathcal{A} = \{a_i, 1 \leq i \leq n-1\}$ and $E(M(K_{1,\ n-1})) = \{a_1v_1, a_2v_2, \ldots, a_nv_{n-1}\} \cup \mathcal{X} \cup \mathcal{Y}$ where $\mathcal{X} = \{v_0a_i, 1 \leq i \leq n-1\}$, $\mathcal{Y} = \{a_ia_j, 1 \leq i \leq n-1, 1 \leq j \leq n-1\}$. Since the vertex set $V' = \{v_0, a_1, a_2, \ldots, a_{n-1}\}$ forms a complete graph $\gamma_{dr}(\langle V' \rangle) = 3$ (assign 3 to any of one the vertex and remaining all vertices 0). Let us define a γ_{dR}^* function f such that $V_0 \cup E_0 = \{v_0, v_1\} \cup \{e_i, 2 \leq i \leq n-1\}$, $V_1 \cup E_1 = \emptyset$, $V_2 \cup E_2 = \{v_i,\ 2 \leq i \leq n-1\}$ and $V_3 \cup E_3 = \{e_1\}$. Hence

$$\gamma_{dR}^*(K_{1,\ n}) = 2|V_2 \cup E_2| + 3|V_3 \cup E_3| = 2(n-2) + 3 = 2n - 1.$$

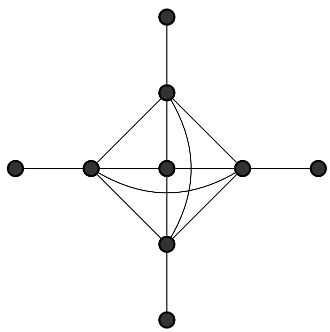

Figure 1: **Middle graph of star** $K_{1,4}$

The equality of the proposition holds for family of stars $K_{1,n-1}$. Let $f = (V_0 \cup E_0, V_1 \cup E_1, V_2 \cup E_2)$ be the middle Roman dominating function of $K_{1,n-1}$. Since $|V_1 \cup E_1| = n-2, V_2 \cup E_2 = 1$. Let $g = (V_0' \cup E_0', V_1' \cup E_1', V_2' \cup E_2', V_3' \cup E_3')$ be the double Roman dominating function of the middle graph of $K_{1,n-1}$, here $V_2' \cup E_2' = n-2$ and $V_3' \cup E_3' = 1$ $\gamma^*_{dR}(K_{1,n-1}) = 2n-1$.

$$2|V_1 \cup E_1| + 3|V_2 \cup E_2| = 2(n-2) + 3(1) = \gamma^*_{dR}(K_{1,n-1}).$$

Proposition 2.7. *Let G be a graph. Then $\gamma^*_{dR}(G) \leq 2\gamma^*_R(G)$.*
Proof. Let $f = (V_0 \cup E_0, V_1 \cup E_1, V_2 \cup E_2)$ be a γ^*_R function of graph G. Since $\gamma^*_R = |V_1 \cup E_1| + 2|V_2 \cup E_2|$ by proposition 2.2, $\gamma^*_{dR}(G) \leq 2|V_1 \cup E_1| + 3|V_2 \cup E_2| = \gamma^*_R(G) + |V_1 \cup E_1| + |V_2 \cup E_2| \leq 2\gamma^*_R(G)$.

Proposition 2.8. *For any graph G, $\gamma^*_R(G) < \gamma^*_{dR}(G)$.*

Theorem 2.9. *For any graph G, $\gamma^*_{dR}(G) - 3 \leq \gamma^*_{dR}(G+e) \leq \gamma^*_{dR}(G) + 3$.*
Proof. Let f be a γ^*_{dR}- function of G and $e = uv$. Obviously we can extend f to double RDF of middle graph of $G + e$ by assigning $f(e) = 3$ and $f(u) = f(v) = 0$. Hence $\gamma^*_{dR}(G+e) \leq \gamma^*_{dR}(G) + 3$.
Let f be a $\gamma^*_{dR}(G+e)$- function and $e = uv$. Now here arises two cases.
Case 1: If $f(e) = 0$, then the function $f : V(G) \cup E(G) \to \{0, 1, 2, 3\}$ is a double RDF of middle graph G. This implies $\gamma^*_{dR}(G) - 3 \leq \gamma^*_{dR}(G) \leq \gamma^*_{dR}(G+e)$.
Case 2: If $f(e) \neq 0$ obviously $f(e)$ must be equal to 3. Now let us define a function $g : V(G) \cup E(G) \to \{0, 1, 2, 3\}$ such that $g(u) = g(v) = 0$ and $g(x) = f(x)$ $\forall x \in V(G) \cup E(G)$. Now g is a double RDF of middle graph with weight $\gamma^*_{dR}(G+e) + f(e)$. Thus $\gamma^*_{dR}(G) - 3 \leq \gamma^*_{dR}(G) - f(e) \leq \gamma^*_{dR}(G+e)$.

Theorem 2.10. *For any graph G with maximum degree $\Delta \geq 2$,*

$$\gamma^*_{dR}(G) - \Delta(G) + 2 \leq \gamma^*_{dR}(G \setminus v) \leq \gamma^*_{dR}(G).$$

Proof. Let f be a γ^*_{dR} function of $G \setminus v$. We define a function $g : V(G) \cup E(G) \to \{0, 1, 2, 3\}$ such that $g(v) = 3$, $g(e_i) = 0$ for all $e_i \in N_M(v)$ and $g(u) = f(u)$ otherwise. Clearly function g is a DRDF of middle graph of G with weight at most $\gamma^*_{dR}(G) - \Delta(G) + 2$. Hence $\gamma^*_{dR}(G) - \Delta(G) + 2 \leq \gamma^*_{dR}(G \setminus v)$.
Now we need to prove that $\gamma^*_{dR}(G \setminus v) \leq \gamma^*_{dR}(G)$. Let f be a γ^*_{dR} function of G. Let us define a function $h : V(G \setminus v) \cup E(G \setminus v) \to \{0, 1, 2, 3\}$ by $h(u) = 1$ if $u \in N_M(v)$ and $h(w) = f(w)$ otherwise. Clearly h is double RDF of middle graph of G with a weight lesser than the weight of f.

Proposition 2.11. *For any graph G, $\gamma^*_{dR}(G) = 3$ if and only if $G = P_2$.*
Proof. Let $G = P_2$. Clearly, $\gamma^*_{dR}(P_2) = 3$. On the other hand, let us assume that

$G \neq P_2$ then G must be either $\overline{K_2}$ or $n \geq 3$. In both the cases $\gamma^*_{dR}(G) \geq 4$ and which contradicts our assumption. Hence $G = P_2$.

Theorem 2.12. *Let G by a connected graph of order $n \geq 3$ and size m. Then $\gamma^*_{dR}(G) = 2n - 1$ if and only if $\Delta(M(G)) = m + n - 1$.*

Proof. Let $f = (V_0 \cup E_0, V_1 \cup E_1, V_2 \cup E_2, V_3 \cup E_3)$ be a γ^*_{dR} function of given graph G and $V_1 \cup E_1 = \emptyset$ (by Theorem). By the definition double Roman dominating function of middle graph $\gamma^*_{dR}(G) = 2|V_2 \cup E_2| + 3|V_3 \cup E_3| \geq 3$. If $\Delta(M(G)) = m + n - 1$ then clearly it must be an edge $e = uv$ of G and degree of e in $M(G)$ is maximum i.e., $\Delta(M(G)) = d(e) = m + n - 1$. Since it covers vertices u, v and all edges of G, $\gamma^*_{dR}(G) = 3 + 2(n-2) = 2n-1$. On the other hand let $\gamma^*_{dR}(G) = 2n-1$ implies $2|V_2 \cup E_2| + 3|V_3 \cup E_3| = 2n - 1$.

Case 1: If $|V_3 \cup E_3| = 1$ then for any $e = uv$ such that $V_3 \cup E_3 = \{e\}$. Clearly, $2|V_2 \cup E_2| + 3|V_3 \cup E_3| = 2|V_2 \cup E_2| + 3 = 2(n-2) + 3$. Implies $|V_2 \cup E_2| = n - 2$ which means $\forall x \in V(G) - \{u, v\}$ such that x is adjacent to either u or v. Hence $\Delta(M(G)) = d(e) = m + n - 1$.

Case 2: If $|V_3 \cup E_3| > 1$ then there exist at least two edges e_1 and e_2 such that $f(e_1) = f(e_2) = 3$. Suppose e_1 and e_2 are adjacent edges then $\Delta(M(G)) \leq m+n-2$ and if they are non adjacent edges then $\Delta(M(G)) \leq m + n - 3$, which contradicts our assumption.

3. Double Roman domination and domination number of Middle graph

In this section we obtained upper and lower bounds for double Roman domination number of middle graphs in terms of domination number of middle graph.

Proposition 3.1. *For any graph G, $2\gamma(M(D)) \leq \gamma^*_{dR}(G) \leq 3\gamma(M(D))$.*

Proof. Let $f = (V_0 \cup E_0, V_2 \cup E_2, V_3 \cup E_3)$ be a γ^*_{dR} function of G and let S be a γ set of middle graph of G. Clearly $f = (\emptyset, \emptyset, S)$ is double Roman dominating function of middle graph of G which implies $\gamma^*_{dR}(G) \leq 3\gamma(M(D))$.

Next to prove the upper bound, $S = (V_1 \cup E_1) \cup (V_2 \cup E_2)$ is a dominating set of $M(G)$ and hence $\gamma(M(G)) \leq |V_2 \cup E_2| + |V_3 \cup E_3|$. From proposition 2.2, $\gamma^*_{dR}(G) = 2|V_2 \cup E_2| + 3|V_3 \cup E_3| \geq 2(|V_2 \cup E_2| + |V_3 \cup E_3|) \geq 2\gamma(M(G))$.

Theorem 3.2. *Let T be a tree of order $n \geq 2$ then,*

$$\gamma^*_{dR}(T) \leq 3(n-1).$$

Proof. Let S is dominating set of $M(T)$. From [2] $S \subseteq E(T)$, implies $|S| \leq |E(T)| = n - 1$. From proposition,

$$\gamma^*_{dR}(T) \leq 3\gamma(M(D)) \leq 3(n-1)$$

Theorem 3.3. *Let T be a tree of order $n \geq 2$ then,*

$$\gamma^*_{dR}(T) \geq 3|leaf(T)|$$

where $leaf(T) = \{v \in V(G)|d(v) = 1\}$.

Theorem 3.4. *If G be any graph of order n and size m, then*

$$\gamma^*_{dR}(G) \leq 2(m+n) - 2\Delta(M(G)) + 1.$$

Theorem 3.5. *For any tree T of order $n \geq 3$, $\gamma^*_{dR} \leq 3\lceil \frac{n}{3} \rceil$.*

Proof. We prove the result by induction on n. If $T = P_3$ obviously $\gamma^*_{dR}(P_3) = 3 \leq 3\lceil \frac{n}{3} \rceil$.

Now consider tree T of order $n \geq 4$. Let us assume the result is holds good for all the tree T' of order $n' < n$ i.e., $\gamma^*_{dR}(T') \leq 3\lceil \frac{n'}{3} \rceil$. Let $f : V(T') \cup E(T') \to \{0, 1, 2, 3\}$ be a γ^*_{dR}- function of T'. If $T = K_{1,n-1}$ is a star then clearly $\gamma^*_{dR}(T) = 2n + 1 \leq 3\lceil \frac{n}{2} \rceil$.

Theorem 3.6. *For any graph G of order n and size m,*

$$5 \leq \gamma^*_{dR}(G) + \gamma^*_{dR}(\overline{G}) \leq n(n+1)/2 + 3.$$

Proof. Suppose G is a graph of order 2 then $\gamma^*_{dR}(G) \geq 3$. The equality holds, only when $M(G)$ has a dominating vertex. Since, the graph G and its complement $\overline{G}$ does not contain domination vertices, we have $\gamma^*_{dR}(G) + \gamma^*_{dR}(\overline{G}) \geq 5$. Equality holds if and only if G or $\overline{G}$ has an edge $e = uv$ with $d(u) + d(v) = \ n - 1$ and its complement has an edge $e' = xy$ with $d(x) + d(y) = \ n - 2$.

Now we claim that $\gamma^*_R(G) + \gamma^*_R(\overline{G}) \leq m + n + 3$. From the above proposition 2.8

$$\begin{aligned}\gamma^*_R(G) + \gamma^*_R(\overline{G}) &\leq (n + m - \Delta(G) + \ 1) + (n(n-1)/2 - m - \Delta(\overline{G}) + 1)\\ &= n + n(n-1)/2 - \Delta(G) + \delta(G) + 3\\ &\leq n(n+1)/2 + 3.\end{aligned}$$

Definition 3.1. *A double star graph $S_{1,n,n}$ is obtained from the star graph $K_{1,n}$ by replacing every edge with a path of length 2.*

Proposition 3.7. *For any double star graph $S_{1,n,n}$ on $2n + 1$ vertices with $n \geq 2$,*

$$\gamma^*_{dR}(S_{1,n,n}) = 3(n+1).$$

Proof. For notation, let us consider $V(S_{1,n,n}) = \{v_0, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$ and $E(S_{1,n,n}) = \{v_0v_i, v_iu_i, | \ 1 \leq i \leq n\}$. The vertex and edge set of middle graph of $S_{1,n,n}$ is $V(M(S_{1,n,n})) = V(S_{1,n,n}) \cup \mathcal{A}$, Where $\mathcal{A} = \{v'_i, u'_i \mid 1 \leq i \leq n\}$ and $E(S_{1,n,n}) = \{v_0v'_i, v'_iv_i, v_iu'_i, u'_iu_i, v'_iu'_i \mid 1 \leq i \leq n\} \cup E(K_{n+1})$, where K_{n+1} is complete graph with vertex set $V(K_{n+1}) = \{v_0, v'_1, v'_2, \ldots, v'_n\}$.

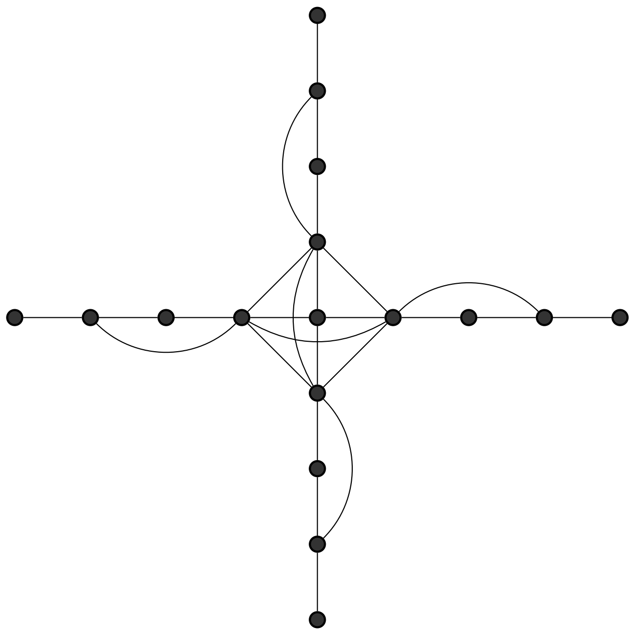

Figure 2: **Middle graph of double star** $S_{1,4,4}$

If $f : V(M(S_{1,n,n})) \rightarrow \{0, 1, 2, 3\}$ is DRD function of middle graph of $S_{1,n,n}$ with minimum weight. Since, middle graph of $S_{1,n,n}$ contains n independent P_3 and a complete graph K_{n+1}. Hence, $\gamma^*_{dR}(S_{1,n,n}) = 3n + 3 = 3(n + 1)$.

Proposition 3.8. *For a path of length* $n \geq 2$, $\gamma^*_{dR}(P_n) = n + 1$.

Proof. Let $P_n = v_1v_2v_3 \dots v_n$ be a path of length n. The middle graph of path P_n divides each edge $v_1v_2, v_2v_3 \dots v_{n-1}v_n$ by adding new vertices $x_1, x_2 \dots x_{n-1}$ respectively and which forms a path $P_{2n-1} = v_1x_1v_2x_2v_3x_3 \dots x_{n-1}v_n$ of length $2n - 1$. Hence the middle graph $M(P_n) = P_{2n-1} + Q_{n-1}$ where $Q_{n-1} = x_1x_2 \dots x_{n-1}$.
Now let us prove the result by induction on n. One can easily obtain $\gamma^*_{dR}(P_3) = 4, \gamma^*_{dR}(P_4) = 5, \gamma^*_{dR}(P_5) = 6$.
Let us assume the result holds for $n = m$. We claim that $\gamma^*_{dR}(P_{m+1}) = m + 2$. Consider a double Roman dominating function $f : V(M(P_n)) \rightarrow \{0, 1, 2, 3\}$ with a vertex partition $V_0 \cup E_0 = \{x_1, v_2, v_3, x_3, v_4, x_4 \dots\}$, $V_1 \cup E_1 = \emptyset$, $V_2 \cup E_2 = \{v_1, v_4, v_8, \dots, \}$ and $V_3 \cup E_3 = \{x_2, x_5, x_8, \dots\}$. Then, the double Roman domination number is $\gamma^*_{dR}(P_{m+1}) = \gamma^*_{dR}(P(m)) + f(v_{m+1}) + f(x_m)$. Suppose $x_{m-1} \in V_2 \cup E_2$, then clearly $f(x_m) = 0$ and $f(v_{m+1}) = 1$. On the other hand if $x_{m-1} \in V_0 \cup E_0$ then obviously $f(v_m) = 1$. Now assign $f(x_m) = 2$ $f(v_m) = 0$ and $f(v_{m+1}) = 0$. Hence the proof.

In figure 3, the vertices assigned with 0, 2 and 3 are shaded by white, gray and black colors respectively.

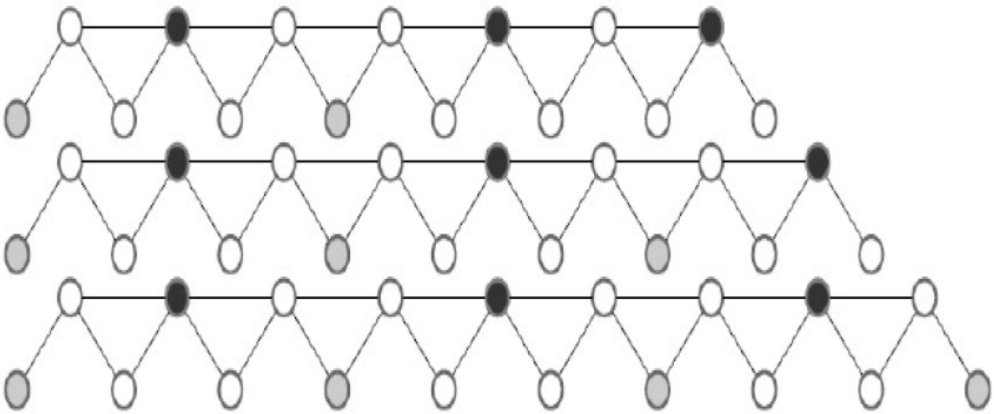

Figure 3: γ^*_{dR} function for path P_7, P_8 and P_9

Proposition 3.9. *For a cycle of length $n \geq 3$, $\gamma^*_{dR}(C_n) = n$.*

Proof. Let $C_n = v_1v_2\ldots v_nv_1$ be a cycle of length n. Let f be a γ^*_R function of a path P_n with minimum weight. Clearly the $E(M(C_n)) = E(M(P_n)) \cup \{v_1x_n, x_nv_n\}$. Here, we get two cycles $C_{2n} = v_1x_1v_2x_2\ldots v_nx_n$ and $C_n = x_1x_2\ldots x_n$. Hence the middle graph $M(C_n) = C_{2n} \cup C_n$. Now let f be a γ^*_{dR} function for the path P_n and we can extend this function for the cycle C_n say $g : V(M(C_n)) \to \{0, 1, 2, 3\}$. Here arises two cases.

Case 1: If $f(x_{n-1}) = 0$ then $f(v_n) = 2$. Now the double Roman dominating function is defined as $g(x_n) = 3,\ g(v_1) = g(v_n) = 0, and g(v) = f(v)\ \forall v \in V(M(C_n))$.

Case 2: If $f(x_{n-1}) = 3$ then $f(v_n) = 0$ and the double Roman domination number is obtained by reassigning 0 to v_1 and 3 to x_n, then and C_n is a closed path of length n. Hence $\gamma^*_R(C_n) = n$.

Corollary 3.10. *For any wheel graph W_n where $n \geq 3$, $\gamma^*_R(W_n) = n - 1$.*

Proposition 3.11. *For any complete bipartite graph $K_{m,n}$ with $1 \leq m \leq n$,*

$$\gamma^*_{dR}(K_{m,n}) = m + 2n.$$

Proof. Let us assume that, $V(K_{m,n}) = \{v_1, v_2, \ldots v_m, u_1, u_2, \ldots, u_n\}$ and $E(K_{m,n}) = \{v_iu_j = x_{ij}|\}$. Clearly $V(M(K_{m,n})) = V(K_{m,n}) \cup E(K_{m,n})$ and $E(M(K_{m,n})) = E(K_m) \cup \{v_ix_{ij}|1 \leq i \leq m \ and\ 1 \leq j \leq n\} \cup \{u_jx_{ij}|1 \leq i \leq m \ and\ 1 \leq j \leq n\}$ where $\mathcal{X}_\rangle = \{\}$. Let us define $f : V(K_{m,n}) \cup E(K_{m,n}) \to \{0, 1, 2, 3\}$ with minimum weight by $f(x_{ij}) = 3\ for\ i = j$, $f(N_M(x_{ij})) = 0$ and $f(u_j) = 2$ for $n - m \leq j \leq n$. Hence $\gamma^*_{dR}(K_{m,n}) = 3m + 2(n - m) = m + 2n$.

Definition 3.2. *The friendship graph F_n of order $2n+1$ is obtained by joining n copies of the cycle C_3 with a common vertex.*

Proposition 3.12. *Let F_n be a friendship graph with $n \geq 2$ then,*

$$\gamma^*_{dR}(F_n) = 3n + 2.$$

Proof. Let vertex set $V(F_n) = \{v_0, v_1, \dots v_{2n}\}$ and edge set $E(F_n) = \{e_i = v_0v_i | 1 \leq i \leq 2n\} \cup \{v_1v_2,\ v_3v_4, \dots, v_{2n-1}v_{2n}\}$. Clearly the middle graph of F_n form a complete graph K_{2n+1} with the vertex set $v_0 \cup \{e_i = v_0v_i | 1 \leq i \leq 2n\}$. Hence,

$$\begin{aligned}\gamma^*_{dR}(F_n) &= \gamma_{dR}(K_{2n+1}) + (n-1)\gamma_{dR}(C_3) + 2 \\ &= 3 + (n-1)3 + 2 = 3n + 2.\end{aligned}$$

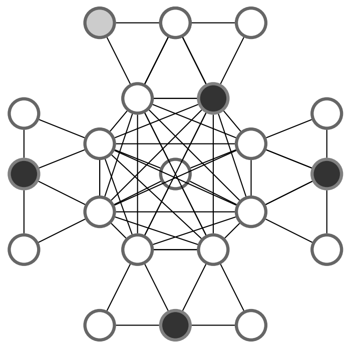

Figure 4: γ^*_{dR} **function for the friendship graph F_4.**

Definition 3.3. [5] *The corona graph $G \circ K_1$ also denoted by $cor(G)$, of a graph G is the graph of order $2 \mid V(G) \mid$ obtained by adding a pendent edge to each vertex of G. The 2-corona $G \circ P_2$ of G is the graph of order $3 \mid V(G) \mid$ obtained by adding a path of length 2 to each vertex of G.*

Theorem 3.13. *For any connected graph G of order $n \geq 2$,*

$$\gamma^*_{dR}(G \circ K_1) \leq 3n.$$

Proof. Let G be a graph with vertex set $V(G) = \{v_1, \dots, v_n\}$. Clearly $V(G \circ K_1) = \{v_1, \dots v_{2n}\}$ and $E(G \circ K_1) = \{v_1v_{n+1}, \dots v_nv_{2n}\} \cup E(G)$. Then $V(M(G \circ K_1)) = V(G \circ K_1) \cup \mathcal{M} \cup \mathcal{A}$, Where $\mathcal{M} = \{m_{i(n+i)} \mid 1 \leq i \leq n\}$ and $\mathcal{A} = \cup\{a_{ij} \mid v_iv_j \in E(G)\}$.
The MRDF $(\mathcal{A}, \emptyset, \emptyset, \mathcal{M})$ has weight $3 \mid \mathcal{M} \mid = 3n$, and hence $\gamma^*_R(G \circ K_1) \leq 3n$.

4. Conclusion

In this paper we established bounds for double Roman domination number of middle graph. And also calculated the exact value of double Roman domination number of middle graph of path, Cycle, Star, Double star etc. Furthermore, characterization of double Roman domination of middle graph is studied.

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