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DETOUR PEBBLING ON CARTESIAN PRODUCT GRAPHS

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Abstract: Given a distribution of pebbles on the vertices of a connected graph G, a pebbling move is defined as the removal of two pebbles from some vertex and the placement of one of those pebbles on an adjacent vertex. The t - *pebbling number* of G is the smallest number, $f_t(G)$ such that from any distribution of $f_t(G)$ pebbles, it is possible to move t pebbles to any specified target vertex by a sequence of pebbling moves. The detour pebbling number of a graph $f^*(G)$ is the smallest number such that from any distribution of $f^*(G)$ pebbles, it is possible to move a pebbles to any specified target vertex by a sequence of pebbling moves using a detour path. In this paper, we find the detour pebbling number for some Cartesian product graphs and also the detour t - pebbling number for those cartesian product graphs.

Keywords and Phrases: t - pebbling number, detour pebbling number, detour t - pebbling number.

2020 Mathematics Subject Classification: 05C99.

1. Introduction

Graph Pebbling has its origin in Number Theory. Chung introduced graph pebbling into literature in [1].

Let G be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. Consider a connected graph with fixed number of pebbles distributed on its vertices. A *pebbling move* consists of the removal of two pebbles from a vertex and placement of one of those pebbles at an adjacent vertex. The *pebbling number of a vertex* v in a graph G is the smallest number $f(G, v)$ such that for every placement of $f(G, v)$ pebbles, it is possible to move a pebble to v by a sequence of pebbling moves. Then the *pebbling number* of G is the smallest number, $f(G)$ such that from any distribution of $f(G)$ pebbles, it is possible to move a pebble to any specified target vertex by a sequence of pebbling moves. Thus $f(G)$ is the maximum value of $f(G, v)$ over all vertices v.

Thus Pebbling number for a graph is the minimum number of required pebbles to reach any target vertex in a graph. Pebbling number is extended to t - pebbling number for making to reach t pebbles to any desired vertex.

The t - *pebbling number of a vertex* v in a graph G is the smallest number $f_t(G, v)$ such that for every placement of $f_t(G, v)$ pebbles, it is possible to move t pebbles to v by a sequence of pebbling moves. Then the t - *pebbling number* of G is the smallest number, $f_t(G)$ such that from any distribution of $f_t(G)$ pebbles, it is possible to move t pebbles to any specified target vertex by a sequence of pebbling moves. Thus $f_t(G)$ is the maximum value of $f_t(G, v)$ over all vertices v. There are many papers with regard to t - pebbling number [3], [4], [9], [10], [11], [12].

Detour pebbling was introduced by Lourdusamy et. al. in [5] using a detour path in any connected graph. For placing t pebbles using detour paths the detour t - pebbling was introduced by Lourdusamy et. al. in [8]. The detour pebbling number and detour t - pebbling number for so many classes of graphs were found on [5], [6], [7], [8].

In this paper, we investigate the detour pebbling number and the detour t - pebbling number for some Cartesian product graphs. In Section 2, we give some preliminary definitions and results which are useful for proving the main results. In Section 3, we find the detour pebbling number for the Cartesian product of path graphs $P_n \square P_n$, path graph and a cycle graph $P_n \square C_n$ and path graph and a complete graph $P_n \square K_n$. Also we calculate the detour t - pebbling numbers for these graphs.

2. Preliminaries

For graph theoretic terminologies, the reader can refer [2].

Definition 2.1. [5] *A detour pebbling number of a vertex v of a graph G is the smallest number $f^*(G, v)$ such that for any placement of $f^*(G, v)$ pebbles on the vertices of G it is possible to move a pebble to v using a detour path by a sequence of pebbling moves. The detour pebbling number of a graph is denoted by $f^*(G)$, is the maximum $f^*(G, v)$ over all the vertices of G.*

Definition 2.2. [8] *A detour t - pebbling number of a vertex v of a graph G is the smallest number $f_t^*(G, v)$ such that for any placement of $f_t^*(G, v)$ pebbles on the vertices of G it is possible to move t pebble to v using a detour path by a sequence of pebbling moves.*

Definition 2.3. [8] *The detour t- pebbling number of a graph is denoted by $f_t^*(G)$, is the maximum $f_t^*(G, v)$ over all the vertices of G.*

Theorem 2.4. [5] *For any path P_n with n vertices, the detour pebbling number is $f^*(P_n) = 2^{n-1}$.*

Theorem 2.5. [8] *For any path P_n with n vertices, the detour t - pebbling number is $f_t^*(P_n) = t2^{n-1}$.*

Theorem 2.6. [5] *For cycles $C_n$with n vertices, the detour pebbling number is $f^*(C_n) = 2^{n-1}$.*

Theorem 2.7. [8] *For any cycle C_n with n vertices, the detour t - pebbling number is $f_t^*(C_n) = t2^{n-1}$.*

Theorem 2.8. [5] *For Complete graph K_n with n vertices $(n > 1)$, the detour pebbling number is $f^*(K_n) = 2^{n-1}$.*

Theorem 2.9. [5] *Let $K_{1,n}$ be an n-star where $n > 1$. The detour pebbling number for the n -star graph is $f^*(K_{1,n}) = n + 2$.*

Theorem 2.10. [5] *For the fan graph F_n with n vertices, the detour pebbling number is $f^*(F_n) = 2^{n-1}$.*

Theorem 2.11. [5] *For the wheel graph W_n with $n+1$ vertices, the detour pebbling number is $f^*(W_n) = 2^n$.*

Theorem 2.12. [6] *Let P_n^2 be the square of path with n vertices. The detour pebbling number $f^*(P_n^2) = 2^{n-1}$.*

Theorem 2.13. [6] *Let $M(P_n)$ be the middle graph path with $2n-1$ vertices. Then the detour pebbling number $f^*(M(P_n)) = 2^{2n-2}$.*

3. Main Results

First we find the detour pebbling number for the Cartesian product of path graphs $P_n \square P_n$, path graph and a cycle graph $P_n \square C_n$ and path graph and a complete graph $P_n \square K_n$.

Theorem 3.1. *The detour pebbling number for $P_3 \square P_3$ is*

$$f^*(P_3 \square P_3) = 256.$$

Proof. Let us label the vertices of the Cartesian product $P_3 \square P_3$ as (u_i, v_j) where $1 \leq i, j \leq 3$. Placing 255 pebbles on the vertex (u_1, v_1) we cannot reach the vertex (u_3, v_3) as a detour path has length 8. Thus $f^*(P_3 \square P_3) \geq 256$.

For the sufficiency, let us consider any distribution of 256 pebbles on the Cartesian product graph. Let (u_1, v_1) be the target vertex. Assume $p((u_1, v_1)) = 0$. In this case from the target vertex we can find many paths containing all the vertices of the graph. Since the length of each path is 8, it is detour. Thus by Theorem 2.4 using 256 pebbles we can move a pebble to the target vertex.

Let (u_1, v_2) be the target vertex. Assume $p((u_1, v_2)) = 0$. Suppose $p((u_1, v_1)) \geq 128$. Then there exists a detour path $P : (u_1, v_1)$, (u_2, v_1), (u_3, v_1), (u_3, v_2), (u_3, v_3), (u_2, v_3), (u_2, v_2), (u_1, v_2) of length 7. By Theorem 2.4, using 128 pebbles we can pebble the target. Otherwise at least 128 pebbles are distributed on the vertices other than the vertex (u_1, v_1). Thus we can find a cycle C containing all the other vertices of length 8. Hence by Theorem 2.6 using 128 pebbles we can move a pebble to (u_1, v_2). By symmetry we can pebble every vertex in the Cartesian product graph $P_3 \square P_3$.

Theorem 3.2. *The detour pebbling number for $P_n \square P_n$ is*

$$f^*(P_n \square P_n) = 2^{n^2-1}.$$

Proof. Let us label the vertices of the Cartesian product $P_n \square P_n$ as (u_i, v_j) where $1 \leq i, j \leq n$. Placing $2^{n^2-1} - 1$ pebbles on the vertex (u_1, v_1) we cannot reach the vertex (u_n, v_n) as a detour path has length $n^2 - 1$. Thus $f^*(P_n \square P_n) \geq 2^{n^2-1}$. For the sufficiency, let us consider any distribution of 2^{n^2-1} pebbles on the Cartesian product graph.

Case 1: Let n be even.

Let (u_1, v_1) be the target vertex. Assume $p((u_1, v_1)) = 0$. In this case from the target, we can find many paths consisting of all vertices in the graph. Since every path contains all the vertices of the graph, the length of those paths is $n^2 - 1$. Thus it will become detour path consists of 2^{n^2-1} pebbles distributed on its vertices. Therefore by Theorem 2.4 using 2^{n^2-1} pebbles we can move a pebble to (u_1, v_1).

Let (u_1, v_i), where $1 < i < n$ be the target vertex. Assume $p((u_1, v_i)) = 0$. Then there exists a cycle C consisting of all the vertices in the Cartesian product graph.

Since the cycle is spanning, it will be the detour cycle of length n^2 consists of 2^{n^2-1} pebbles distributed on its vertices. Thus by Theorem 2.6, using 2^{n^2-1} pebbles we can move a pebble to the vertex (u_1, v_i), where $1 < i < n$. By symmetry we can pebble all the vertices of the graph.

Case 2: Let n be odd.

Let (u_1, v_1) be the target vertex. Assume $p((u_1, v_1)) = 0$. In this case also from the target, we can find many paths consisting of all the vertices of the graph. Also these detour paths have length $n^2 - 1$ consists of 2^{n^2-1} pebbles distributed on its vertices. Thus by the Theorem 2.4, using 2^{n^2-1} pebbles we can move a pebble to (u_1, v_1). Let (u_1, v_i), where $1 < i < n$ be the target vertex. Assume $p(u_1, v_i) = 0$.

Subcase 2.1: Suppose i is odd.

Then from the target vertex (u_1, v_i), we can find many paths, each containing all the vertices of the graph. Since the paths contain all the vertices of the Cartesian product graph they are detour paths of length $n^2 - 1$ consists of 2^{n^2-1} pebbles on its vertices. Thus by Theorem 2.4, we can pebble the target.

Subcase 2.2: Suppose i is even.

Suppose there exists a cycle C with containing $n^2 - 1$ vertices of the Cartesian product graph including our target vertex, then by Theorem 2.6 using 2^{n^2-2} pebbles we can move a pebble to any vertex of the cycle. Thus we can pebble our target vertex. Suppose not, then we can find a cycle C_1 containing $n^2 - 1$ vertices (including our target vertex) with at most $2^{n^2-2} - 1$ pebbles on its vertices. Thus there exists a vertex say (u, v) which is not on C_1 containing at least $2^{n^2-1} - (2^{n^2-2} - 1) = 2^{n^2-2} + 1$ pebbles. Also the detour distance between (u, v) to our target vertex is at most $n^2 - 2$, we can find a detour path from (u, v) to (u_1, v_i) of length at most $n^2 - 2$ and hence by Theorem 2.4 we can reach the target. By symmetry we can reach all the vertices of the graph.

Theorem 3.3. *The detour pebbling number for $P_3 \square C_3$ is*

$$f^*(P_3 \square C_3) = 256.$$

Proof. Let us label the vertices of the Cartesian product $P_3 \square C_3$ as (u_i, v_j) where $1 \leq i, j \leq 3$. Placing 255 pebbles on the vertex (u_1, v_1) we cannot reach the vertex (u_3, v_3) as a detour path has length 8. Thus $f^*(P_3 \square C_3) \geq 256$.

Now let us consider any distribution of 256 pebbles on the Cartesian product graph. Let (u_i, v_j) be the target vertex, where $1 \leq i, j \leq 3$. Assume $p(u_i, v_j) = 0$. Then we can find a cycle C with starting vertex at (u_i, v_j) containing all the vertices of the Cartesian product graph. Since this cycle is a spanning cycle, it is a detour cycle of length 8 consists of all the 256 pebbles on its vertices. Thus by Theorem 2.6, we can move a pebble to the vertex (u_i, v_j) using 256 pebbles. By symmetry, we are done for any target vertex.

Theorem 3.4. *The detour pebbling number for $P_n \square C_n$ is*

$$f^*(P_n \square C_n) = 2^{n^2-1}.$$

Proof. Let the vertices of the Path graph P_n be $u_1, u_2, ..., u_n$ and the vertices of the Cycle graph C_n be $v_1, v_2, ..., v_n$. Let us label the vertices of the Cartesian product $P_n \square C_n$ as (u_i, v_j) where $1 \leq i, j \leq n$. Placing $2^{n^2-1} - 1$ pebbles on the vertex (u_1, v_1) we cannot reach the vertex (u_n, v_n). Thus $f^*(P_n \square C_n) \geq 2^{n^2-1}$. Now let us consider any distribution of 2^{n^2-1} pebbles on the vertices of the graph $P_n \square C_n$.
Case 1: Let n be even.

Let (u_i, v_j) be the target vertex, where $1 \leq i, j \leq n$. Then there exists a cycle C from (u_i, v_j) containing all the vertices of the graph. Thus the cycle is a spanning cycle and hence detour cycle of length n^2 consists of 2^{n^2-1} pebbles distributed on its vertices. By Theorem 2.6, using 2^{n^2-1} pebbles we can move a pebble to the target vertex. By symmetry, we can reach any vertex of the Cartesian product graph.
Case 2: Let n be odd.

Let (u_i, v_j) be the target vertex, where $1 \leq i, j \leq n$. From (u_i, v_j) we can find a path containing all the vertices of the graph. Also since this path is a spanning path, it is a detour path of length $n^2 - 1$ consists of 2^{n^2-1} pebbles distributed on its vertices. Thus by Theorem 2.4, we can move a pebble to the target vertex.

Hence by symmetry, we can reach any vertex of the Cartesian product graph.

Theorem 3.5. *The detour pebbling number for $P_n \square K_n$ is*

$$f^*(P_n \square K_n) = 2^{n^2-1}.$$

Proof. Let the vertices of the Path graph P_n be $u_1, u_2, ..., u_n$ and the vertices of the Complete graph K_n be $v_1, v_2, ..., v_n$. Let us label the vertices of the Cartesian product $P_n \square K_n$ as (u_i, v_j) where $1 \leq i, j \leq n$. Placing $2^{n^2-1} - 1$ pebbles on the vertex (u_1, v_1) we cannot reach the vertex (u_n, v_n). Thus $f^*(P_n \square K_n) \geq 2^{n^2-1}$.

Now let us consider any distribution of 2^{n^2-1} pebbles on the vertices of the graph $P_n \square C_n$. Let (u_i, v_j) be the target vertex, where $1 \leq i, j \leq n$. Then we can find a cycle C with starting vertex at (u_i, v_j) containing all the vertices of the Cartesian product graph. Since this cycle is a spanning cycle, it is a detour cycle of length n^2 consists of all the 2^{n^2-1} pebbles on its vertices. Thus by Theorem 2.6, we can move a pebble to the vertex (u_i, v_j) using 2^{n^2-1} pebbles. By symmetry, we are done for any target vertex.

Now let us compute the detour t - pebbling number for the Cartesian product graphs.

Theorem 3.6. *The detour t - pebbling number for $P_n \Box P_n$ is*

$$f_t^*(P_n \Box P_n) = t2^{n^2-1}$$

Proof. Placing $t2^{n^2-1} - 1$ pebbles on (u_1, v_1) we cannot move t pebbles to (u_n, v_n). Thus $f^*(P_n \Box P_n) \geq t2^{n^2-1}$. Let us consider any distribution of $t2^{n^2-1}$ pebbles on the Cartesian product graph. We prove this part by induction on t. For $t = 1$, the result follows from Theorem 3.2. We assume that the result is true for $2 \leq k < t$.

Let (u, v) be the target vertex. The Cartesian product graph contains at least 2^{n^2} pebbles. Since the detour distance between the target vertex and any vertex of the Cartesian product graph is at most $n^2 - 1$, we can move one pebble to the target using maximum of 2^{n^2-1} pebbles. Thus the number of remaining pebbles is at least

$$t2^{n^2-1} - 2^{n^2-1} = (t-1)2^{n^2-1} = f_{t-1}^*(P_n \Box P_n).$$

Hence by induction we can move $(t-1)$ additional pebbles to the target. Thus the any target vertex receives $1 + (t-1) = t$ pebbles.

Theorem 3.7. *The detour t- pebbling number for $P_n \Box C_n$ is*

$$f_t^*(P_n \Box C_n) = t2^{n^2-1}$$

Proof. Placing $t2^{n^2-1} - 1$ pebbles on (u_1, v_1) we cannot move t pebbles to (u_n, v_n). Thus $f^*(P_n \Box C_n) \geq t2^{n^2-1}$. Let us consider any distribution of $t2^{n^2-1}$ pebbles on the Cartesian product graph. We prove this part by induction on t. For $t = 1$, the result follows from Theorem 3.4. We assume that the result is true for $2 \leq k < t$.

Let (u_i, v_j) be the target vertex, where $u_i \in P_n$ and $v_j \in C_n$. Since the Cartesian product graph $P_n \Box C_n$ contains at least 2^{n^2} pebbles we can move a pebble to the target vertex from any vertex at a cost of at most 2^{n^2-1} pebbles as the distance between the target vertex and any vertex is at most $n^2 - 1$. Thus we are left with at least

$$t2^{n^2-1} - 2^{n^2-1} = (t-1)2^{n^2-1} = f_{t-1}^*(P_n \Box C_n).$$

Hence by induction we can move $(t-1)$ additional pebbles to the target. Thus the any target vertex receives $1 + (t-1) = t$ pebbles.

Theorem 3.8. *The detour t - pebbling number for $P_n \Box K_n$ is*

$$f_t^*(P_n \Box K_n) = t2^{n^2-1}.$$

Proof. Placing $t2^{n^2-1}-1$ pebbles on (u_1, v_1) we cannot move t pebbles to (u_n, v_n). Thus $f^*(P_n \Box K_n) \geq t2^{n^2-1}$. Let us consider any distribution of $t2^{n^2-1}$ pebbles on the Cartesian product graph. We prove this part by induction on t. For $t = 1$, the result follows from Theorem 3.5. We assume that the result is true for $2 \leq k < t$.

Let (u_i, v_j) be the target vertex, where $u_i \in P_n$ and $v_j \in K_n$. As $t \geq 2$, the Cartesian product graph $P_n \Box K_n$ contains at least 2^{n^2} pebbles on its vertices. Since the distance between the target vertex and any vertex is at most $n^2 - 1$, we can move a pebble to the target vertex from any vertex at a cost of at most 2^{n^2-1} pebbles. Thus the number of remaining pebbles is at least

$$t2^{n^2-1} - 2^{n^2-1} = (t-1)2^{n^2-1} = f^*_{t-1}(P_n \Box K_n).$$

By induction we can move $(t-1)$ additional pebbles to (u_i, v_j). Hence we can move t pebbles to the target. By symmetry we are done for any target vertex.

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