

VERTEX-EDGE NEIGHBORHOOD PRIME LABELING IN THE CONTEXT OF CORONA PRODUCT

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Abstract: Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For $u \in V(G)$, $N_V(u) = \{w \in V(G) | uw \in E(G)\}$ and $N_E(u) = \{e \in E(G) | e = uv, \text{ for some } v \in V(G)\}$. A bijective function $f : V(G) \cup E(G) \rightarrow \{1, 2, 3, \dots, |V(G) \cup E(G)|\}$ is said to be a vertex-edge neighborhood prime labeling, if for $u \in V(G)$ with $\deg(u) = 1$, $\gcd\{f(w), f(uw) | w \in N_V(u)\} = 1$; for $u \in V(G)$ with $\deg(u) > 1$, $\gcd\{f(w) | w \in N_V(u)\} = 1$ and $\gcd\{f(e) | e \in N_E(u)\} = 1$. A graph which admits a vertex-edge neighborhood prime labeling is called a vertex-edge neighborhood prime graph. In this paper we prove $K_{m,n} \odot K_1$, $W_n \odot K_1$, $H_n \odot K_1$, $F_n \odot K_1$ and $S(K_{1,n}) \odot K_1$ are vertex-edge neighborhood prime graphs.

Keywords and Phrases: Neighborhood-prime labeling, vertex-edge neighborhood prime labeling, corona product.

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1. Introduction and Definitions

All the graphs considered here are simple, finite, connected and undirected. $V(G)$ and $E(G)$ denote vertex set and edge set of G respectively. For various notations and terminology of graph theory, we follow Gross and Yellen [3] and for number theoretical results, we follow Burton [1].

Let G be a graph with n vertices. A bijective function $f : V(G) \rightarrow \{1, 2, 3, \dots, n\}$ is said to be a **neighborhood-prime labeling** if for every vertex u in $V(G)$ with $\deg(u) > 1$, $\gcd\{f(p) | p \in N(u)\} = 1$, where $N(u) = \{w \in V(G) | uw \in E(G)\}$.

A graph which admits a neighborhood-prime labeling is called a neighborhood-prime graph.

The notion of neighborhood-prime labeling was introduced by Patel and Shrimali [6]. In [7] they proved that union of some graphs are neighborhood-prime graphs. In [8] they proved that product of some graphs are neighborhood-prime graphs. For further list of results regarding neighborhood-prime labeling reader may refer to [2].

For a graph G , a bijective function $f : V(G) \cup E(G) \rightarrow \{1, 2, 3, \dots, |V(G) \cup E(G)|\}$ is said to be a **total neighborhood prime labeling**, if for each vertex in G having degree greater than 1, the gcd of the labels of its neighborhood vertices is 1 and the gcd of the labels of its incident edges is 1. A graph which admits a total neighborhood prime labeling is called a total neighborhood prime graph.

Motivated by neighborhood-prime labeling, Rajesh and Methew [4] introduced the total neighborhood prime labeling. In the total neighborhood prime labeling conditions are applied on neighborhood vertices as well as incident edges of each vertex of degree greater than 1. Shrimali and Pandya [5] extended the condition on the vertices of degree 1 and they defined vertex-edge neighborhood prime labeling.

Let G be a graph. For an arbitrary vertex u in $V(G)$, $N_V(u) = \{w \in V(G) | uw \in E(G)\}$ and $N_E(u) = \{e \in E(G) | e = uv, \text{ for some } v \in V(G)\}$. A bijective function $f : V(G) \cup E(G) \rightarrow \{1, 2, 3, \dots, |V(G) \cup E(G)|\}$ is said to be a **vertex-edge neighborhood prime labeling**, if for $u \in V(G)$ with $deg(u) = 1$, $gcd\{f(w), f(uw) | w \in N_V(u)\} = 1$; for $u \in V(G)$ with $deg(u) > 1$, $gcd\{f(w) | w \in N_V(u)\} = 1$ and $gcd\{f(e) | e \in N_E(u)\} = 1$. A graph which admits a vertex-edge neighborhood prime labeling is called a vertex-edge neighborhood prime graph.

In [5], Shrimali and Pandya proved that path, helm, sunlet, bistar, central edge subdivision of bistar, subdivision of edges of bistar admit a vertex-edge neighborhood prime labeling.

Shrimali and Rathod proved that generalized web graph, generalized web graph without central vertex, splitting graph of path, splitting graph of star, graph obtained by switching of a vertex in path, graph obtained by switching of a vertex in cycle and middle graph of path are vertex-edge neighborhood prime graphs [9].

In [10] Simaringa and Vijayalakshmi proved that Petersen graph $P(n, 2)$ where $n > 4$, prism graph, triangular snake, barycentric cycle, convex polytop, alternate triangular snake $A(T_n)$ for $n = 4, 6, 8, 10, \dots$, triangular book, rectangular book, pentagonal book, quadrilateral snake, alternate quadrilateral snake, double triangular snake, double alternate triangular snake $DA(T_n)$ for $n = 4, 6, 8, 10, \dots$, $P(n, 2) * K_1$, $(C_n \times K_2) * K_1$, $T_n * K_1$, $R_n * K_1$, barycentric cycle attached by pendant edge and m fold petal graphs are vertex-edge neighborhood prime graphs.

Let G_1 be a graph with n vertices. **Corona product** of G_1 with another graph G_2 is a graph obtained by taking n copies of G_2 and joining the i^{th} vertex of G_1 with an edge to every vertex in the i^{th} copy of G_2 .

The **wheel** W_n is defined as the join $C_n + K_1$.

The **helm** H_n is a graph obtained from a wheel W_n by attaching pendent edge to each rim vertex.

The **fan** F_n is defined as the join $P_n + K_1$. The vertex corresponding to K_1 is said to be the apex vertex.

Let G be a graph. If every edge of graph G is subdivided, then the resulting graph is called **barycentric subdivision** of a graph G and it is denoted by $S(G)$.

2. Main Results

Theorem 2.1. *The corona product $K_{m,n} \odot K_1$ is a vertex-edge neighborhood prime graph.*

Proof. Let $W = U \cup V$ be the bipartition of complete bipartite graph $K_{m,n}$, where $U = \{u_1, u_2, \dots, u_m\}$ and $V = \{v_1, v_2, \dots, v_n\}$. Without loss of generality, we assume $m \geq n$. We add new vertices $u'_1, u'_2, \dots, u'_m, v'_1, v'_2, \dots, v'_n$ to obtain the graph $K_{m,n} \odot K_1$ and we denote it by G . Let $e_{i,j} = v_i u_j$, $d_j = u_j u'_j$ and $d'_i = v_i v'_i$ for $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$.

Now, we define $f : V(G) \cup E(G) \rightarrow \{1, 2, 3, \dots, |V(G) \cup E(G)|\}$ as follows.

$$f(u_j) = 2n + 2j - 1, \quad 1 \leq j \leq m$$

$$f(u'_j) = 2(m + n) + mn + j, \quad 1 \leq j \leq m$$

$$f(d_j) = 2n + 2j, \quad 1 \leq j \leq m$$

$$f(v'_i) = 2(m + n) + mn + m + i, \quad 1 \leq i \leq n$$

$$f(e_{i,j}) = 2(m + n) + i + n(j - 1), \quad 1 \leq i \leq n, 1 \leq j \leq m$$

One can easily observe that the labels of $e_{i,1}, e_{i,2}, \dots, e_{i,n}$ are of the form $nk_j^i + r_i$ where $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$ and r_1, r_2, \dots, r_n are equal in some order to $0, 1, 2, \dots, n - 1$.

We give the labels to v_1, v_2, \dots, v_n according to the value of r_1, r_2, \dots, r_n . Once the value of v_1, v_2, \dots, v_n are assigned, we give labels to d'_1, d'_2, \dots, d'_n in such a way that the pairs of labels of v_1 and d'_1 , v_2 and d'_2 , \dots, v_n and d'_n are consecutive numbers. Thus for $l \in \{1, 2, 3, \dots, n\}$

Case-I. n is odd

$$f(v_l) = \begin{cases} n, & r_l = 0 \\ r_l, & r_l \text{ is odd} \\ n + r_l, & r_l \text{ is even} \end{cases}$$

For $i = 1, 2, 3, \dots, n$ $f(d'_i) = f(v_i) + 1$

Case-II. n is even

$$f(v_l) = \begin{cases} r_l, & r_l \text{ is odd} \\ n + 2 + r_l, & r_l = 0 \text{ or } r_l \text{ is even} \end{cases}$$

For $i = 1, 2, 3, \dots, n$

$$f(d'_i) = \begin{cases} f(v_i) + 1, & f(v_i) \leq n \\ f(v_i) - 1, & f(v_i) > n \end{cases}$$

We claim that f is a vertex-edge neighborhood prime labeling. In this graph $u'_1, u'_2, \dots, u'_m, v'_1, v'_2, \dots, v'_n$ are vertices with degree 1. In view of above labeling pattern for each $i, f(d_i)$ and $f(u_i), f(d'_i)$ and $f(v_i)$ are pairs of consecutive integers. So, the condition is satisfied.

Now $u_1, u_2, \dots, u_m, v_1, v_2, \dots, v_n$ are the vertices with degree greater than 1. For each i and $j, N_V(v_i) = \{u_1, u_2, \dots, u_m\}$ and $N_V(u_j) = \{v_1, v_2, \dots, v_n\}$. Since $f(u_1), f(u_2), \dots, f(u_n)$ are consecutive odd integers, $\gcd \{f(p)/p \in N_V(v_i)\} = 1$. Since $1 \in \{f(v_1), f(v_2), \dots, f(v_m)\}$, $\gcd \{f(p)/p \in N_V(u_j)\} = 1$. Now for each u_j the incident edges have consecutive labels, therefore $\gcd \{f(e)/e \in N_E(u_j)\} = 1$. For each v_i , labels of incident edges are of the form $nk_j^i + r_i$ where $j = 1, 2, \dots, n$ and $f(d'_i) = f(v_i) + 1$ or $f(v_i) - 1$ according to n is even or odd. In any case $\gcd \{f(e)/e \in N_E(v_i)\} = 1$. Hence the conditions are satisfied. Thus, f is a vertex-edge neighborhood prime labeling. Hence $K_{m,n} \odot K_1$ is a vertex-edge neighborhood prime graph.

Illustration 1. Vertex-edge neighborhood prime labeling of $K_{8,5} \odot K_1$ is shown in Figure 1.

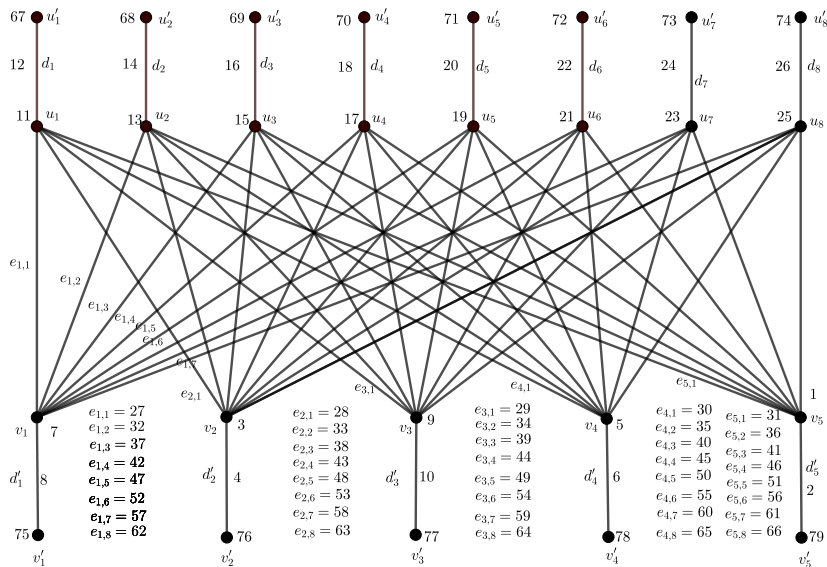


Figure 1: Vertex-edge neighborhood prime labeling of $K_{8,5} \odot K_1$.

Theorem 2.2. *The corona product $W_n \odot K_1$ is a vertex-edge neighborhood prime graph.*

Proof. Let u_0 be the apex vertex and u_1, u_2, \dots, u_n be the rim vertices of wheel graph W_n . We add new vertices $u'_0, u'_1, u'_2, \dots, u'_n$ to obtain the graph $W_n \odot K_1$ and we denote it by G . Let $e_i = u_0u_i$, $e'_i = u_iu_{i+1}$, $d = u_0u'_0$ and $d_i = u_iu'_i$ for $i = 1, 2, \dots, n$ where values of i taken modulo n . In the graph G , vertex set $V(G) = \{u_0, u'_0, u_i, u'_i / i = 1, 2, \dots, n\}$ and edge set $E(G) = \{e_i, e'_i, d_i, d / i = 1, 2, \dots, n\}$. So, $|V(G)| = 2n + 2$ and $|E(G)| = 3n + 1$.

Now we define $f : V(G) \cup E(G) \longrightarrow \{1, 2, 3, \dots, |V(G) \cup E(G)|\}$ as follows.

- $f(u_0) = 1$
- $f(u'_0) = 5n + 2$
- $f(u_i) = 2i + 1, \quad 1 \leq i \leq n$
- $f(u'_i) = 4n + 1 + i, \quad 1 \leq i \leq n$
- $f(e_i) = 2n + 1 + 2i, \quad 1 \leq i \leq n$
- $f(e'_i) = 2n + 2i, \quad 1 \leq i \leq n$
- $f(d_i) = 2i, \quad 1 \leq i \leq n$
- $f(d) = 5n + 3$

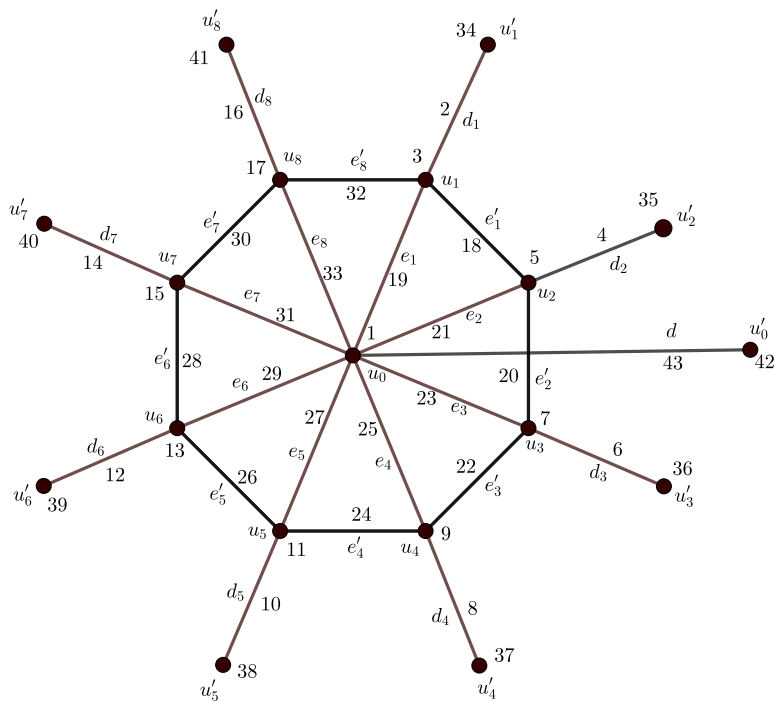


Figure 2: Vertex-edge neighborhood prime labeling of $W_8 \odot K_1$

In G , u'_0, u'_1, \dots, u'_n are the vertices of degree 1. For u'_0 , $\gcd \{f(u_0), f(u_0u'_0)\} = 1$ because $f(u_0) = 1$. For u'_i $1 \leq i \leq n$, $\gcd \{f(u_i), f(d_i)\} = 1$ because $f(u_i)$ and $f(d_i)$ are consecutive numbers. So the condition for vertices of degree 1 is satisfied. Let w be any vertex of degree greater than 1. If $w \neq u_0$, $u_0 \in N_V(w)$. Since $f(u_0) = 1$, $\gcd \{f(p)/p \in N_V(w)\} = 1$. If $w = u_0$ then $N_V(w) = \{u_1, u_2, \dots, u_n\}$. Since $f(u_1), f(u_2), \dots, f(u_n)$ are consecutive odd numbers, $\gcd \{f(p)/p \in N_V(w)\} = 1$. For any vertex w , at least two incident edges have consecutive labels. Therefore, $\gcd \{f(e)/e \in N_E(w)\} = 1$. Thus, f satisfies the conditions of vertex-edge neighborhood prime labeling. So, the graph G is a vertex-edge neighborhood prime graph.

Illustration 2. Vertex-edge neighborhood prime labeling of $W_8 \odot K_1$ is shown in Figure 2.

Theorem 2.3. The corona product $H_n \odot K_1$ is a vertex-edge neighborhood prime graph.

Proof. Let u_0 be the apex vertex, u_1, u_2, \dots, u_n be the rim vertices and v_1, v_2, \dots, v_n be the pendent vertices of helm graph H_n . We add new vertices $u'_0, u'_1, u'_2, \dots, u'_n, v'_1, v'_2, \dots, v'_n$ to obtain the graph $H_n \odot K_1$ and we denote it by G . Let $e = u_0u'_0$, $e_i = u_0u_i$, $e'_i = u_iu_{i+1}$, $d_i = u_iv_i$, $g_i = u_iu'_i$ and $g'_i = v_iv'_i$ for $i = 1, 2, \dots, n$ where value of i taken modulo n . In G , vertex set $V(G) = \{u_0, u'_0, u_i, u'_i, v_i, v'_i/i = 1, 2, \dots, n\}$ and edge set $E(G) = \{e, e_i, e'_i, d_i, g_i, g'_i/i = 1, 2, \dots, n\}$. So, $|V(G)| = 4n + 2$ and $|E(G)| = 5n + 1$.

Now, we define $f : V(G) \cup E(G) \longrightarrow \{1, 2, 3, \dots, |V(G) \cup E(G)|\}$ as follows.

$$\begin{aligned} f(u_0) &= 1 \\ f(u'_0) &= 9n + 2 \\ f(u_i) &= 7i - 4, \quad 1 \leq i \leq n \\ f(u'_i) &= 8n + 1 + i, \quad 1 \leq i \leq n \\ f(v_i) &= 7i - 2, \quad 1 \leq i \leq n \\ f(v'_i) &= 7i - 3, \quad 1 \leq i \leq n \\ f(e_i) &= 7n + 1 + i, \quad 1 \leq i \leq n \\ f(e'_i) &= 7i + 1, \quad 1 \leq i \leq n \\ f(g_i) &= 7i - 5, \quad 1 \leq i \leq n \\ f(g'_i) &= 7i - 1, \quad 1 \leq i \leq n \\ f(d_i) &= 7i, \quad 1 \leq i \leq n \\ f(d) &= 9n + 3 \end{aligned}$$

$u'_0, u'_1, \dots, u'_n, v'_1, v'_2, \dots, v'_n$ are the vertices with degree 1 in the graph G . For v'_i and u'_i , the pairs $f(v_i)$ and $f(g'_i)$, $f(u_i)$ and $f(g_i)$ are of consecutive numbers for each i and for u'_0 , $f(u_0) = 1$. So, the condition for vertices of degree 1 is satisfied. Let w be any vertex with degree greater than 1. If $w = v_i$, $N_V(w) = \{u_i, v'_i\}$ and

$f(u_i)$ and $f(v'_i)$ are consecutive numbers, if $w = u_i$, $\{f(p)/p \in N_V(w)\}$ contains 1 and if $w = u_0$, $\{f(p)/p \in N_V(w)\}$ contains n integers of the form $7k + 3$ where $k = 0, 1, 2, \dots, n - 1$. In each case $\gcd \{f(p)/p \in N_V(w)\} = 1$. For any vertex w , $\{f(e)/e \in N_E(w)\}$ contains at least two consecutive numbers, so $\gcd \{f(e)/e \in N_E(w)\} = 1$. Since f satisfies all the conditions of vertex-edge neighborhood prime labeling, G is a vertex-edge neighborhood prime graph.

Illustration 3. Vertex-edge neighborhood prime labeling of $H_8 \odot K_1$ is shown in Figure 3.

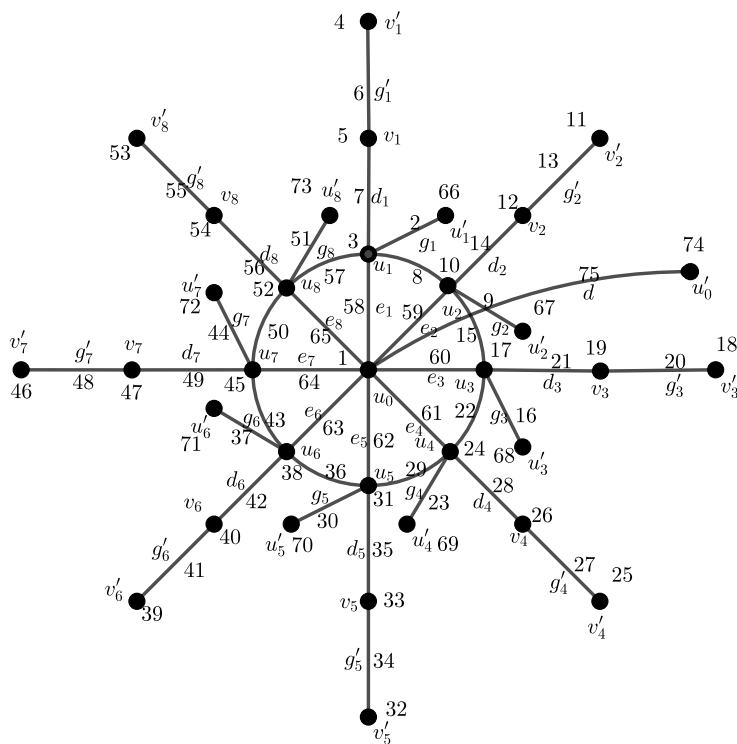


Figure 3: Vertex-edge neighborhood prime labeling of $H_8 \odot K_1$

Theorem 2.4. The corona product $F_n \odot K_1$ is a vertex-edge neighborhood prime graph.

Proof. Let $u_0, u_1, u_2, \dots, u_n$ be the vertices of the fan graph F_n , where u_0 is the apex vertex. We add new vertices $u'_0, u'_1, u'_2, \dots, u'_n$ to obtain the graph $F_n \odot K_1$ and we denote it by G . Let $e_i = u_0u_i$, $d_i = u_iu'_i$ and $d = u_0u'_0$ for $i = 1, 2, \dots, n$. Let $e'_i = u_iu_{i+1}$ for $i = 1, 2, \dots, n - 1$. In the graph G vertex set $V(G) = \{u_0, u'_0, u_i, u'_i/i =$

$1, 2, \dots, n\}$ and edge set $E(G) = \{d, d_i, e_i/i = 1, 2, \dots, n\} \cup \{e'_i/i = 1, 2, \dots, n-1\}$. So, $|V(G)| = 2n + 2$ and $|E(G)| = 3n$.

Now, we define $f : V(G) \cup E(G) \rightarrow \{1, 2, 3, \dots, |V(G) \cup E(G)|\}$ as follows.

- $f(u_0) = 1$
- $f(u'_0) = 4n + 2$
- $f(u_i) = 2i + 1, \quad 1 \leq i \leq n$
- $f(u'_i) = 4n + 2 + i, \quad 1 \leq i \leq n$
- $f(e_i) = 2n + 2i, \quad 1 \leq i \leq n$
- $f(e'_i) = 2n + 2i + 1, \quad 1 \leq i \leq n - 1$
- $f(d_i) = 2i, \quad 1 \leq i \leq n$
- $f(d) = 4n + 1$

u'_0, u'_1, \dots, u'_n are the vertices with degree 1 in the graph G . For u'_i , $f(d_i)$ and $f(u_i)$ is pair of consecutive integers, where $i = 1, 2, \dots, n$ and for u'_0 , $f(u_0) = 1$. So, the condition of vertices of degree 1 is satisfied. $u_0, u_1, u_2, \dots, u_n$ are the vertices with degree greater than 1 and $f(u_0) \in \{f(p)/p \in N_V(u_i)\}$ for $i = 1, 2, 3, \dots, n$. Since, $f(u_0) = 1$, $\gcd\{f(p)/p \in N_V(u_i)\} = 1$. $\{f(e)/e \in N_E(u_i)\}$ contains at least two consecutive numbers, so $\gcd\{f(e)/e \in N_E(u_i)\} = 1$, for $i = 0, 1, 2, 3, \dots, n$. So, G is a vertex-edge neighborhood prime graph.

Illustration 4. Vertex-edge neighborhood prime labeling of $F_8 \odot K_1$ is shown in Figure 4.

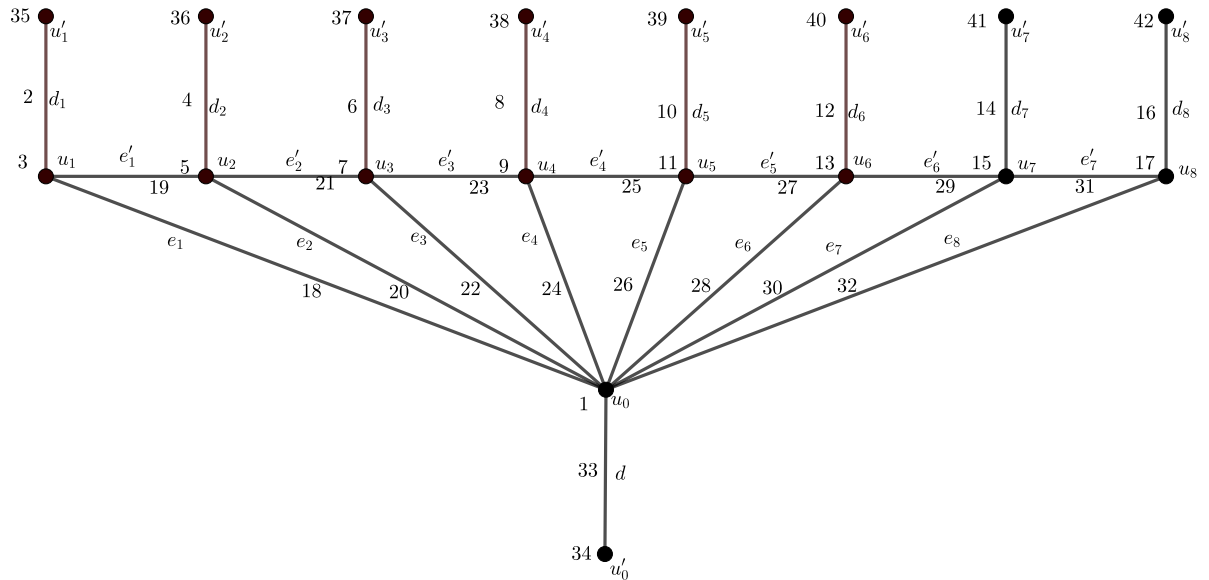


Figure 4: Vertex-edge neighborhood prime labeling of $F_8 \odot K_1$

Theorem 2.5. *The corona product $S(K_{1,n}) \odot K_1$ is a vertex-edge neighborhood prime graph.*

Proof. Let $S(K_{1,n})$ be barycentric subdivision of star graph $K_{1,n}$. Let $u_0, u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n$ be the vertices of $S(K_{1,n})$, where u_0 is apex vertex, u_1, u_2, \dots, u_n are the vertices of degree 2 and v_1, v_2, \dots, v_n are the pendent vertices. We add new vertices $u'_0, u'_1, u'_2, \dots, u'_n, v'_1, v'_2, \dots, v'_n$ to obtain the graph $S(K_{1,n}) \odot K_1$ and we denote it by G . Let $g = u_0u'_0, e_i = u_0u_i, d_i = u_iv_i, g_i = u_iu'_i$ and $g'_i = v_iv'_i$ for $i = 1, 2, \dots, n$. Here, vertex set $V(G) = \{u_0, u'_0, u_i, u'_i, v_i, v'_i / i = 1, 2, \dots, n\}$ and edge set $E(G) = \{g, e_i, d_i, g_i, g'_i / i = 1, 2, \dots, n\}$. So, $|V(G)| = 4n + 2$ and $|E(G)| = 4n + 1$.

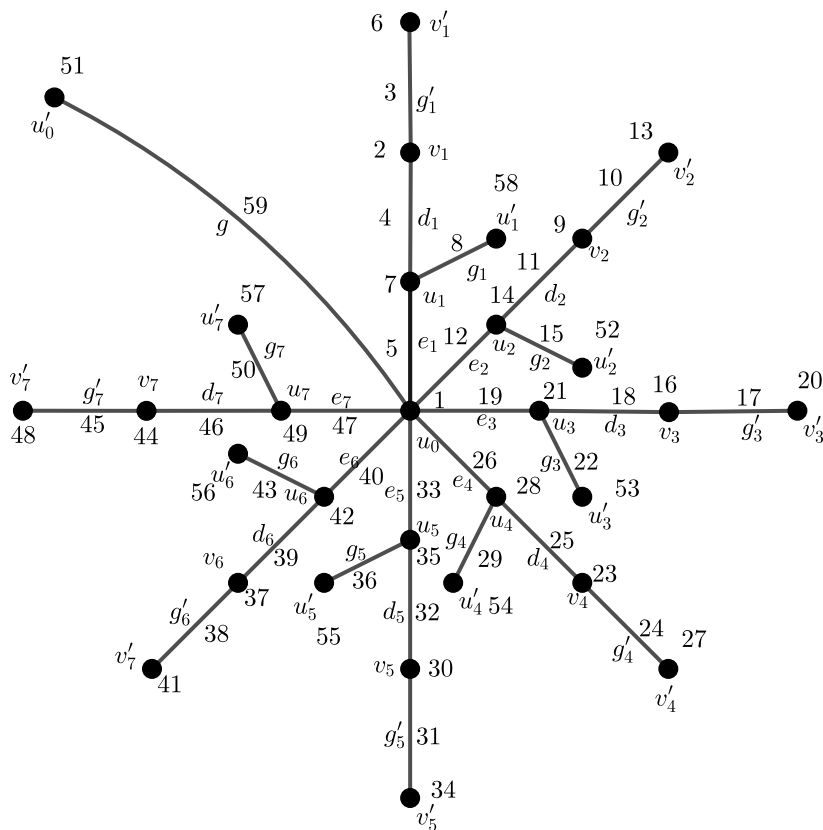


Figure 5: Vertex-edge neighborhood prime labeling of $S(K_{1,7}) \odot K_1$.

Now, we define $f : V(G) \cup E(G) \longrightarrow \{1, 2, 3, \dots, |V(G) \cup E(G)|\}$ as follows.

$$f(u_0) = 1$$

$$f(u'_0) = 7n + 2$$

$$\begin{aligned}
f(u_i) &= 7i, & 1 \leq i \leq n \\
f(u'_i) &= \begin{cases} 8n + 2, & i = 1 \\ 7n + 1 + i, & 2 \leq i \leq n \end{cases} \\
f(v_i) &= 7i - 5, & 1 \leq i \leq n \\
f(v'_i) &= 7i - 1, & 1 \leq i \leq n \\
f(e_i) &= 7i - 2, & 1 \leq i \leq n \\
f(d_i) &= 7i - 3, & 1 \leq i \leq n \\
f(g_i) &= 7i + 1, & 1 \leq i \leq n \\
f(g'_i) &= 7i - 4, & 1 \leq i \leq n \\
f(g) &= 8n + 3
\end{aligned}$$

In view of the above labeling pattern, one can easily see that all the conditions of vertex-edge neighborhood prime graph are satisfied. So, G is a vertex-edge neighborhood prime graph.

Illustration 5. Vertex-edge neighborhood prime labeling of $S(K_{1,7}) \odot K_1$ is shown in Figure 5.

Acknowledgment

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