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ANTIPODAL DOMINATION NUMBER OF GRAPHS

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Abstract: A dominating set $S \subseteq V$ is said to be an Antipodal Dominating Set(ADS) of a connected graph G if there exist vertices $x, y \in S$ such that $d(x, y) = diam(G)$. The minimum cardinality of an ADS is called the Antipodal Domination Number(ADN), and is denoted by $\gamma_{ap}(G)$. In this paper, we determined the antipodal domination number for various graph products, bound for antipodal domination and characterize the graphs with $\gamma_{ap}(G) = 2$.

Keywords and Phrases: Antipodal Domination, Diameter.

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1. Introduction

Let $G = (V, E)$ be a graph with vertex set V and edge set E. A set $D \subseteq V$ is a **dominating set** of G if every vertex not in D is adjacent to a vertex in D. The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set.

A thorough study of domination, with its many variations, appears in [1, 2]. We introduced a new domination parameter called Antipodal domination by imposing the antipodal condition on the dominating set [3].

Let G be a connected graph. A dominating set S of G is said to be an **Antipodal Dominating Set (ADS)** if there exist vertices $x, y \in S$ such that $d(x, y) = diam(G)$. The minimum cardinality of an ADS is called the **Antipodal Domination Number (ADN)**, and is denoted by $\gamma_{ap}(G)$.

It is easy to note that ADS is superhereditary and $\gamma \leq \gamma_{ap} \leq \gamma + 2$. We have determined γ_{ap} for paths, complete bipartite graphs, generalized wheel, double star, wounded spider and Jahangir graphs in [3]. Moreover, we derived a bound for antipodal domination in graphs and characterize the graphs with $\gamma_{ap}(G) = n, n-1, n-2$. Also we derived a Nordhaus-Gaddum type bound for γ_{ap} [4].

In this paper, we determined the antipodal domination number for various graph products, bound for antipodal domination and characterize the graphs with $\gamma_{ap}(G) = 2$.

2. Ore's Type Theorem

Theorem 2.1. *A dominating set S is a minimal ADS iff for every $u \in S$ one of the following holds:*
(i) u is an isolate in S
(ii) there exists a vertex v in $V - S$ for which $N(v) \cap S = \{u\}$
(iii) For every $x, y \in S - \{u\}$, $d(x, y) \neq diam(G)$.
Proof. Let S be a minimal ADS.
Then for every $u \in S$, $S - \{u\}$ is not an ADS of G.
Then one of the following holds:
(a) For every $x, y \in S - \{u\}$, $d(x, y) \neq diam(G)$.
(b) $S - \{u\}$ is not a dominating set.
Now (a) implies (iii) and (b) implies that S is a minimal dominating set; and so (i) or (ii) holds.
Conversely, suppose that S is not a minimal ADS. Then there exists a vertex $u \in S$ such that $S - \{u\}$ is an ADS.
Hence every vertex in $V - (S - \{u\})$ is adjacent to at least one vertex in $S - \{u\}$; and so condition(i) and (ii) does not hold.
Moreover there exists $x, y \in S - \{u\}$ such that $d(x, y) = diam(G)$; and so condition(iii) does not hold.

3. $\gamma_{ap}-$for Graph Products

Theorem 3.1. *$\gamma_{ap}(P_n \times K_m) = n$, $n, m \geq 3, n \leq m$.*
Proof. Let $u_1, u_2, ..., u_n$ be the vertices of P_n and $v_1, v_2, ..., v_m$ be vertices of K_m.
Let $x_{ij} = (u_i, v_j), 1 \leq i \leq n, 1 \leq j \leq m$ be the vertices of $P_n \times K_m$.
First note that $diam(P_n \times K_m) = n$.
Without loss of generality, let x_{11} and x_{nm} be the vertices with
$d(x_{11}, x_{nm}) = diam(P_n \times K_m)$.
To dominate the vertices in $K_m^{(i)}$ either we need one vertex of $K_m^{(i)}$ or we need m vertices not in $K_m^{(i)}$.

If no vertex of $K_m^{(i-1)}$ lies in S, then to dominate all the vertices in $K_m^{(i)}$, we need $m-$vertices in $K_m^{(i-1)} \cup K_m^{(i+1)}$; and these $m-$vertices dominate at most $3m$ vertices in $K_m^{(i-2)} \cup K_m^{(i-1)} \cup K_m^{(i)} \cup K_m^{(i+1)} \cup K_m^{(i+2)}$.
Hence to dominate $(n-2)m-2$ vertices in $\bigcup_{i=2}^{n-1} K_m^{(i)}$, we need at least $\frac{m}{3m} \times m(n-2)-2 = \left\lceil \frac{m(n-2)-2}{3} \right\rceil$ vertices.
But a single vertex in $K_m^{(i)}$ dominates all the vertices of $K_m^{(i)}$.
Hence to dominate $(n-2)m-2$ vertices in $\bigcup_{i=2}^{n-1} K_m^{(i)}$, it is enough to choose $n-2$ vertices.(one vertex in each $K_m^{(i)}$, $2 \leq i \leq n-1$.
But $\left\lceil \frac{m(n-2)-2}{3} \right\rceil \geq n-2$.
Hence $\gamma_{ap}(P_n \times K_m) \geq n-2+2 = n$.
Now $S = \{x_{11}, x_{12}, ..., x_{(n-1)1}, x_{n2}\}$ is an ADS; and so $\gamma_{ap}(P_n \times K_m) = n$.

Theorem 3.2. $\gamma_{ap}(P_n[K_m]) = \lceil \frac{n-1}{3} \rceil + 1$, $n \geq 2$.
Proof. Let $u_1, u_2, ..., u_n$ be the vertices on P_n and $v_1, v_2, ..., v_m$ be the vertices of K_m.
Let $x_{ij} = (u_i, u_j)$, $1 \leq i \leq n$, $1 \leq j \leq m$ be the vertices of $P_n[K_m]$.
First note that $diam(P_n[K_m]) = n-1$.
Without loss of generality, let x_{11} and x_{n1} be vertices with
$d(x_{11}, x_{n1}) = diam(P_n[K_m]) = n-1$.
x_{11} and x_{m1} dominates four copies of K_m.
Let S_1 be an ADS of the remaining $n-4$ copies of K_m.
Every internal vertex x_{ij}, for some $2 \leq i \leq n$, $1 \leq j \leq m$ is adjacent with at most three copy of K_n; and so $|S_1| \geq \lceil \frac{n-4}{3} \rceil$, and

$$\gamma_{ap}(P_n[K_m]) \geq |S_1| + 2 \geq \left\lceil \frac{n-4}{3} \right\rceil + 2 \geq \left\lceil \frac{n-1}{3} \right\rceil + 1. \tag{3.1}$$

Case(i): $n \equiv 0 (mod\ 3)$
$S = \{x_{11}, x_{n1}\} \cup \{x_{41}, x_{71}, x_{(10)\ 1}, ..., x_{(n-2)}1\}$ is an ADS; and so $|S| = \frac{n}{3} + 1$.
Case(ii): $n \equiv 1 (mod\ 3)$
$S = \{x_{11}, x_{n1}\} \cup \{x_{41}, x_{71}, x_{(10)1}, ..., x_{(n-3)1}\}$ is an ADS; and so $|S| = \frac{n-1}{3} + 1$.
Case(iii): $n \equiv 2 (mod\ 3)$
$S = \{x_{11}, x_{n1}\} \cup \{x_{41}, x_{71}, x_{(10)1}, ..., x_{(n-1)1}\}$ is an ADS ; and so $|S| = \frac{n-2}{3} + 2$.
In all the cases, we have

$$|S| = \left\lceil \frac{n-1}{3} \right\rceil + 1. \tag{3.2}$$

Hence $\gamma_{ap}(P_n[K_m]) = \lceil \frac{n-1}{3} \rceil + 1$.

Theorem 3.3. $\gamma_{ap}(P_n \boxtimes K_m) = \lceil \frac{n-1}{3} \rceil + 1$, $n \geq 2$.
Proof. The proof is similar to proof of Theorem 3.2.

Theorem 3.4. $\gamma_{ap}(P_n \otimes K_m) = n - 2$.
Proof. Let $u_1, u_2, ..., u_n$ be the vertices of P_n and $v_1, v_2, ..., v_m$ be the vertices of K_n.
Let $x_{ij} = (u_i, v_j)$, $1 \leq i \leq n$, $1 \leq j \leq m$.
Let S be any RDS of G.
First note that $diam(P_n \otimes K_m) = n - 1$.
Without loss of generality, let x_{11} and x_{n1} be vertices with
$d(x_{11}, x_{n1}) = diam(P_n \otimes K_m) = n - 1$.
Then x_{11} and $x_{n1} \in S$.
Now x_{11} dominates x_{2j} and x_{n1} dominates $x_{(n-1)j}$, $2 \leq j \leq m$.
Also to dominate x_{1j} and x_{nj}, $2 \leq j \leq m$, we must need x_{21} and x_{3j},
$2 \leq j \leq m$.
Also note that every vertex in $P_n \otimes K_m$ dominates at least m vertices; and so to dominate the remaining $nm - 6m$ vertices we need at least $\frac{nm-6m}{m}$ vertices.
Hence $|S| \geq 4 + n - 6 \geq n - 2$.
Now $S = \{x_{11}, x_{n1}\} \cup \{x_{21}, x_{(n-1)1}\} \cup \{x_{42}, x_{52}, x_{62}, ..., x_{(n-3)2}\}$.
Hence $|S| \leq n - 2$.

Theorem 3.5. *$\gamma_{ap}(P_n \times G) = n + 1$, where G is a graph with $\Delta(G) = n_1 - 1$, where n_1 is the number of vertices in G, $n \leq m$, $n \geq 3$.*
Proof. Let $u_1, u_2, ..., u_n$ be the vertices of P_n and $v_1, v_2, ..., v_m$ be vertices of G.
Let $x_{ij} = (u_i, v_j), 1 \leq i \leq n, 1 \leq j \leq m$ be the vertices of $P_n \times G$.
First note that $diam(P_n \times G) = n$.
Without loss of generality, let x_{11} and x_{n2} be the vertices with
$d(x_{11}, x_{n2}) = diam(P_n \times G) = n$.
To dominate the vertices in $G^{(i)}$ either we need one vertex of G or we need m vertices not in $G^{(i)}$.
If no vertex of $G^{(i-1)}$ lies in S, then to dominate all the vertices in $G^{(i)}$, we need $m-$vertices in $G^{(i-1)} \cup G^{(i+1)}$; and these $m-$vertices dominate at most $3m$ vertices in $G^{(i-2)} \cup G^{(i-1)} \cup G^{(i)} \cup G^{(i+1)} \cup G^{(i+2)}$.
Hence to dominate $nm - (m+3)$ vertices in $\bigcup_{i=2}^{n-1} G^{(i)}$, we need at least $\frac{m}{3m} \times (nm - (m+3)) = \left\lceil \frac{m(n-1)-3}{3} \right\rceil$ vertices.
But a single vertex in $G^{(i)}$ dominates all the vertices of $G^{(i)}$.

Hence to dominate $(n-1)m-3$ vertices in $\bigcup_{i=2}^{n-1} G^{(i)}$, it is enough to choose $n-1$ vertices.(one vertex in each $G^{(i)}$, $2 \le i \le n-1$).
But $\left\lceil \frac{m(n-1)-3}{3} \right\rceil \ge n-1$.
Hence $\gamma_{ap}(P_n \times G) \ge n-1+2 = n+1$.
Now $S = \{x_{11}, x_{n2}\} \cup \{x_{21}, x_{31}, ..., x_{n1}\}$ is an ADS; and so
$\gamma_{ap}(P_n \times G) = n+1$.

Theorem 3.6. $\gamma_{ap}(P_n \boxtimes G) = \left\lceil \frac{n-1}{3} \right\rceil + 1, n \ge 2$.
Proof. The proof is similar to the proof of Theorem 3.2.

Theorem 3.7. $\gamma_{ap}(P_n[G]) = n+1$.
Proof. The proof is similar to proof of Theorem 3.5.

Theorem 3.8. $\gamma_{ap}(P_n \otimes G) = n-2$, *wher* G *is a graph with full degree vertex.*
Proof. The proof is similar to the proof of Theorem 3.5.

4. Bound For Antipodal Domination

Theorem 4.1. *For any connected graph* G *with* $\Delta(G) \le n-r$, $\gamma_{ap}(G) \le r+1, r \ge 2$.
Moreover equality holds iff one of the following conditions holds

1. $|W| = 2$, *where* $W = V - N[v]$, v *is a vertex of maximum degree,* $Z \ne \phi$, Z *does not have a vertex of degree* $|Z|-1$, *where* $Z = V - (N[x] \cup N[y])$ *where* x *and* y *be two vertices with* $d(x,y) = diam(G)$ *and there is no* $x \in V - Z$ *such that* $zx \in E(G)$ *for all* $z \in Z$.

2. $|W| = 1$ *and there exists* $z \in V - (N[u_1] \cup N[u_2])$, *where* u_1 *and* u_2 *be two vertices with* $d(u_1, u_2) = diam(G)$.

Proof. Let G be a connected graph with $\Delta(G) = n-r$, $r \ge 2$.
Let v be a vertex of maximum degree.
Let $N(v) = \{u_1, u_2, ..., u_{n-r-1}, u_{n-r}\}$ and $V - N[v] = \{w_1, w_2, ..., w_{r-2}, w_{r-1}\}$.
If the diametrical distance lies between u_i and w_j for some $i, j, 1 \le i \le n-r, 1 \le j \le r-1$, then $S = \{w_1, w_2, ..., w_{r-1}, v, u_i\}$ is an ADS.
Otherwise, the diametrical-distance exists between w_i, w_j or v and w_i, or between u_i and u_j, for some i and j.
In these cases also, the set S is an ADS of G.
Hence $\gamma_{ap}(G) \le r+1$.
Equality:
Assume that $\gamma_{ap}(G) = r+1$.

Case 1: diameter exists between w_i and w_j, for some i and j
Then $V - N(v)$ is an ADS of G; and so $\gamma_{ap}(G) \leq r$, a contradiction.
Case 2: diameter exists between w_i and v, for some i
Then $V - N(v)$ is an ADS of G; and so $\gamma_{ap}(G) \leq r$, a contradiction.
Case 3: diameter exists between u_i's
Then $diam(G) \leq 2$.
If $diam(G) = 1$, then G is complete, a contradiction.
If $diam(G) = 2$, then $V - N(v)$ is an ADS of G; and so $\gamma_{ap}(G) \leq r$, a contradiction.
Case 4: diameter exists between w_j and u_i, for some i and j
Case 4.1: W have an edge
Without loss of generality, $w_k w_l \in E(G)$ for some k and l.
Without loss of generality, let $k \neq j$, now $S = (W - \{w_k\}) \cup \{u_i, v\}$ is an ADS of G; and so $\gamma_{ap}(G) \leq r$, a contradiction.
Case 4.1: W does not have an edge
Then every $w \in W$ is adjacent with some $u \in N(v)$. (since G is connected).
Now $2 \leq diam(G) \leq 3$ (Since diameter exists between w_j and u_i; and so $diam(G) \leq 3$; if $diam(G) = 1$, then G is complete).
Case 4.2.1: $diam(G) = 2$
Since $d(v, w_i) = 2$ for all i, we can apply case 2.
Case 4.2.2: $diam(G) = 3$
If $d(w_i, w_j) = 3 (= diam(G))$ for some $w_i, w_j \in W$, then we can apply case 1.
Otherwise $d(w_i, w_j) = 2$ for every pair $w_i, w_j \in W$.
Hence every pair of vertices in W have a common neighbour in $N(v)$.
Case 4.2.2.1: $|W| \geq 3$
Let $w_1, w_2, w_3 \in W$.
Without loss of generality, let w_1 and u_1 be vertices with $d(w_1, u_1) = diam(G)$; and u_2 be a common neighbour of w_2 and w_3.
Then $(W - \{w_2, w_3\}) \cup \{u_1, u_2, v\}$ is an ADS; and so $\gamma_{ap}(G) \leq r$, a contradiction.
Case 4.2.2.2: $|W| = 2$
Then $r = 3$ and $r + 1 = 4$.
Let $w_1, w_2 \in W$ and w_1, w_2 have a common neighbour in $N(v)$ (say)u_2.
If $w_1 w_2 \in E(G)$, then $\{w_1, u_1, v\}$ is an ADS; and so $\gamma_{ap}(G) \leq r$, a contradiction.
If $w_2 u_1 \in E(G)$, then $\{w_1, u_1, v\}$ is an ADS; and so $\gamma_{ap}(G) \leq r$, a contradiction.
Now we shall prove that for every pair of vertices w_j, u_i with $d(w_j, u_i) = diam(G)$, the following 2 claims hold.
Let $Z = V - (N[w_i] \cup N[u_i])$.
Claim 1 : Z does not have a vertex of degree $|Z| - 1$
Claim 2 : There is no $x \in N(w_j) \cup N(u_i)$ such that $zx \in E(G)$ for all $z \in Z$

Without loss of generality, let w_1, u_1 be vertices with $d(w_1, u_1) = diam(G)$.
Already we have noted that $w_1w_2 \notin E(G)$ and $w_2u_1 \notin E(G)$.
Now to dominate w_2, either we need $w_2 \in S$ or $u_2 \in S$.
Suppose w_2 is adjacent to all u_i's in Z, then $\{w_1, u_1, w_2\}$ is an ADS; and so $\gamma_{ap}(G) \leq 3$, a contradiction.
Then w_2 is not adjacent to at least one u_j in Z.
Hence w_2 is not a vertex of degree $|Z| - 1$.
Suppose u_k, for some k is a vertex of degree $|Z| - 1$ in Z; then $\{w_1, u_1, u_k\}$ is an ADS; and so $\gamma_{ap}(G) \leq 3$, a contradiction.
Hence u_k is not adjacent to at least one vertex in Z.
Hence there is no vertex in Z of degree $|Z| - 1$.
Hence the claim holds.
Suppose if there exists $x \in N(u_1) \cup N(w_1)$ such that $zx \in E(G)$ for all $z \in Z$.
Hence $\{u_1, w_1, x\}$ is an ADS; and so $\gamma_{ap}(G) \leq 3$, a contradiction.
Hence claim 2 holds.
Case 4.2.2.3: $|W| = 1$
Then $r = 2$; and so $r + 1 = 3$.
Without loss of generality, let w_1 and u_1 be vertices with $d(w_1, u_1) = diam(G)$, $w_1, u_1 \in S$.
Claim : There exists $z \in V - (N[w_1] \cup N[u_1])$
Assume the contrary that $N[w_1] \cup N[u_1] = V(G)$, then $W \cup \{u_1\}$ is an ADS; and so $\gamma_{ap}(G) \leq r$, a contradiction.
Then there exists $u_k \in V - (N[w_1] \cup N[u_1])$ adjacent to v or some vertices in $V - \{u_1, w_1\}$.
Now to dominate u_k we need at least one more vertex; and so $\gamma_{ap}(G) \geq 3$.
Now $\{u_1, w_1, v\}$ is an ADS; and so $\gamma_{ap}(G) = 3 = r + 1$.

5. Characterization of graphs with $\gamma_{ap} = 2$

Definition 5.1. *Let G_1 be a graph obtained by adding zero or more leaves to the pendant vertex of the path P_3.*
Let G_2 be a graph obtained by adding zero or more leaves to the pendant vertex of the path P_4

Theorem 5.1. *For any connected graph G of order $n \geq 2$, $\gamma_{ap}(G) = 2$ iff $diam(G) \leq 3$ and one of the following holds*
(i) G is complete
(ii) $diam(G) = 2$ and G_1 is a spanning subgraph of G.
(iii) $diam(G) = 3$ and G_2 is a spanning subgraph of G,
where G_1 and G_2 are graphs defined in Definition 5.1.

Proof. Assume that $\gamma_{ap}(G) = 2$.
Let $S = \{u, v\}$ be a $\gamma_{ap}-$set of G.
Then $N[u] \cup N[v] = V(G)$ (5.1)
Case 1 : $uv \in E(G)$
Then G is complete; and so condition (i) holds.
Case 2 : $uv \notin E(G)$
Case 2.1: $N(u) \cap N(v) \neq \phi$
Then there exists $w \in N(u) \cap N(v)$.
Then G has $\langle P_3 \rangle : uwv$ as an induced path.
Now, by (1), it follows that $diam(G) = 2$ and G_1 is a spanning subgraph of G.
Case 2.2 : $N(u) \cap N(v) = \phi$
Since G is connected, then there exists vertices $x_1 \in N(u)$ and $x_2 \in N(v)$ such that $x_1x_2 \in E(G)$.
Then G has $\langle P_4 \rangle : ux_1x_2v$ as an induced path.
Now, by (5.1), it follows that $diam(G) = 3$ and G_2 is a spanning subgraph of G.
Conversely, assume that one of the following holds
If G is complete, then $\gamma_{ap}(G) = 2$.
If $diam(G) = 2$ and G_1 is a spanning subgraph of G, then G has $\langle P_3 \rangle : w_1w_2w_3$(say).
Also if there exists $x \in V - \{w_1, w_2, w_3\}$ such hat xw_1 or $xw_3 \in E(G)$.
Now $\{w_1, w_2\}$ is an ADS; and so $\gamma_{ap}(G) = 2$.
If $diam(G) = 3$ and G_2 is a spaning subgraph of G, then G has $\langle P_4 \rangle : w_1w_2w_3w_4$(say).
Suppose there exists $x \in V - \{w_1, w_2, w_3, w_4\}$, such that xw_1 or $xw_4 \in E(G)$.
Hence $\{w_1, w_4\}$ is an ADS; and so $\gamma_{ap}(G) = 2$.

Corollary 5.1. *Let G be a disconnected graph. Then $\gamma_{ap}(G) = 2$ iff $G = 2K_1$.*

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