South East Asian J. of Mathematics and Mathematical Sciences
Vol. 18, No. 3 (2022), pp. 329-338

DOI: 10.56827/SEAJMMS.2022.1803.27 **ISSN (Online): 2582-0850**

ISSN (Print): 0972-7752

ON METRIC DIMENSION OF BOOLEAN GRAPH $BG_1(G)$

Sameerali C. P. and Sameena K.

PG and Research Department of Mathematics,
MES Mampad College,
Mampad - 676542, Kerala, INDIA

E-mail : sameercup@yahoo.com, sameenakalathodi@gmail.com

(**Received:** *Jul. 20, 2021* **Accepted:** *Nov. 03, 2022* **Published:** *Dec. 30, 2022)*

Abstract: Let G be a simple graph with vertex set V and edge set E. $B_{G,NINC,\overline{K_q}}$ (G), known as boolean graph of G-first kind,simply denoted by $BG_1(G)$ is defined as the graph with vertex set $V \cup E$ and two vertices are adjacent if and only if they correspond to adjacent vertices in G or to a vertex and an edge in G such that the edge is not incident with the vertex. In this paper we give a bound for metric dimension of $BG_1(G)$ and also find expression for metric dimension of boolean graphs of Complete graphs and Star graphs. Finally, an algorithm for finding the metric dimension of $BG_1(G)$ is established.

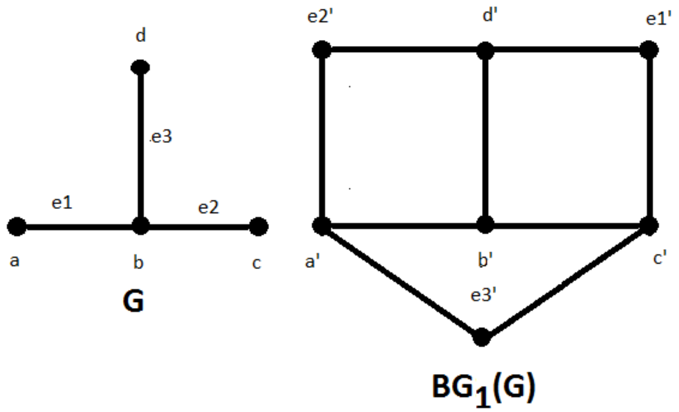
Keywords and Phrases: Boolean graph $BG_1(G)$, metric dimension, resoliving set.

2020 Mathematics Subject Classification: 05C48, 05C85.

1. Introduction

T. N. Janakiraman, M. Bhanumathi, S. Muthammai have contributed much in the study of boolean graph operation and boolean graph [6]. Considering the numerous adjacency relations 32 different kinds of graphs can be generated from a single graph. Globally, like Facebook and Whatsapp, many other networks are also flourishing. Hence, the study of such graphs are trending. The concept of location set and location number have contemporary significance as these concepts have applications in GPS, mobile phone technology and many other networks. The concepts of location set and location number introduced by P J Slater are

renamed as resoliving set and metric dimension respectively. These concepts also have substantial functions in artificial intelligence which is the technology of this era.
A graph G and its boolean graph $BG_1(G)$ is given below.



The graphs that are considered throughout this paper are simple, finite and undirected. Harary [4] and Buckley and Harary [1] are referred for terminologies of graph theory. Let G be a graph with vertex set V and edge set E. The distance from a vertex u to another vertex v, denoted by $d(u, v)$ is the length of any shortest path from u to v. If there is no path from u to v, then $d(u, v) = \infty$. In a simple connected graph G the eccentricity of a vertex v, radius of the graph G and the diameter of the graph G respectively defined as follows.

$$e(v) = \max_{u \in V}\{d(u, v)\}, r(G) = \min_{v \in V}\{e(v)\}, diam(G) = \max_{v \in V}\{e(v)\}$$

If G is a graph with p vertices and q edges, then $BG_1(G)$ is a graph of order $p+q$ and size $q(p-1)$.

Definition 1.1. [3] *Let u,v and w be vertices of a simple connected graph G with vertex set V. If $d(u, w) \neq d(v, w)$, then w is said to resolve the vertices u and v. The set $S \subseteq V$ is called a resoliving set if for every pair of vertices of G there is a resolving vertex in S.*
If $S = \{s_1, s_2, \cdots, s_k\}$ is a resoliving set, then every vertex u of G can be uniquely identified by a k-vector $C_S(u) = (d(u, s_1), d(u, s_2), \cdots, d(u, s_k))$. The vector $C_S(u)$ is called The metric code or location code of u in V. Resolving set of a graph is not unique, since every super set of a resoliving set is a resoliving set.

Definition 1.2. [3] *A resoliving set with minimum cardinality is known as a* *metric base and this minimum cardinality is called the metric dimension of a graph G, denoted by $\beta(G)$.*

Definition 1.3. *Number of edges incident with a vertex is called degree of the vertex. $\Delta(G)$ and $\delta(G)$ respectively denote the maximum and minimum degrees among the vertices of a graph G.*
Let $G = (V, E)$ and $v \epsilon V$, then the corresponding vertex in $BG_1(G)$is denoted by v' and is called a point vertex . If $e \epsilon E$ is an edge of G, then the corresponding vertex in $BG_1(G)$ is denoted by e' and is called a line vertex.

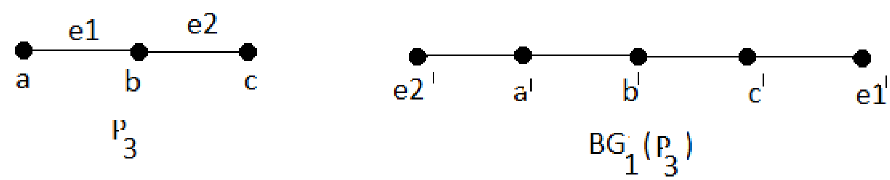
Theorem 1.1. [7] *$\beta(G) = 1$ iff G is a path.*

Theorem 1.2. [5] *$\beta(G) = n - 1$ iff $G = K_n$.*

Theorem 1.3. [2] *$log_3(\Delta + 1) \leq \beta(G) \leq n - diam(G)$.*

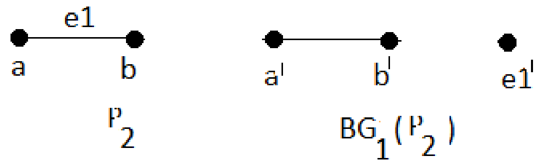
2. Metric Dimension and Boolean Graph $BG_1(G)$

Theorem 2.1. *For any connected undirected non trivial graph $\beta(BG_1(G))$ =1 iff $G = P_3$.*
Proof. Let $\beta(BG_1(G))$ =1. Assume that G is not a path. Then by definition of $BG_1(G)$, it is also not a path. Hence by theorem 1.1, $\beta(BG_1(G)) > 1$. This is a contradiction and so G is a path. Let $G = P_n$ and consider the following cases for n.
Case 1 : $n > 3$.
Let e be an edge meeting a pendant vertex. Then there will be at least two vertices v_1 and v_2 in G which are not meeting the edge e. Since G is connected there is atleast one path $v_1 u_1 u_2 \cdots u_k v_2$ from v_1 to v_2. Then $v_1' u_1' u_2' \cdots u_k' v_2' e' v_1'$ is a cycle in $BG_1(G)$. Hence by theorem 1.1, $\beta(BG_1(G)) > 1$. This is a contradiction.
Case 2: n=3.
From the following figure $BG_1(P_3)$ is P_5. Therefore, $\beta(BG_1(G)) = \beta(P_5) = 1$



Case 3: n=2.
$BG_1(P_2)$ is disconnected . Therefore, $\beta(BG_1(P_2)) \neq 1$.
Thus if $\beta(BG_1(G)) = 1$, then $G = P_3$.

Conversely assume that $G = P_3$, then from Figure $BG_1(P_3)$ is P_5 and hence, $\beta(BG_1(P_3)) = 1$.



Theorem 2.2. *The set of all point vertices of a boolean graph is a resoliving set.*
Proof. Let G=(V,E), where $V = \{v_1, v_2 \cdots v_p\}$ and $E = \{e_1, e_2 \cdots e_q\}$ .Let V' and E' are the sets of point vertices and line vertices. Let $V' = \{v'_1, v'_2 \ldots v'_p\}$ and $E' = \{e'_1, e'_2 \cdots e'_q\}$. $C_{V'}(v'_i) \neq C_{V'}(v'_j)$ since the i^{th} component of $C_{V'}(v'_i)$ is zero and that of $C_{V'}(v'_j)$ is the distance from v_i to v_j which is at least one. Thus, codes of point vertices from V' are different. Let e' be a line vertex of $BG_1(G)$ and assume e joins v_m and v_n in G, then in $BG_1(G)$ e' is adjacent to neither v'_m nor v'_n but it is adjacent to all the remaining point vertices. Then p-2 of components of $C_{V'}(e')$ are 1 and the remaining two components are more than 1. i.e., $C_{V'}(e') = (1, 1, \cdots, a, 1, 1 \cdots, b, 1 \cdots 1)$, a and b occur respectively in m^{th} and n^{th} positions. Now assume e'_i and e'_j are two line vertices of $BG_1(G)$ with $C_{V'}(e'_i) = C_{V'}(e'_j)$. This means components which are not 1 occur in same positions. That is e_i and e_j join same vertices in G and this contradicts the fact that G is simple. Therefore, $C_{V'}(e'_i) \neq C_{V'}(e'_j)$. Moreover, $C_{V'}(v'_i) \neq C_{V'}(e'_j)$ since the i^{th} component of $C_{V'}(v'_i) = 0$ and none of the components of $C_{V'}(e'_j)$ is zero. Hence all the vertices of $BG_1(G)$have different codes. So V' is a resoliving set.

Corollary 2.1. *If G=(V,E) and $BG_1(G)$ is connected, then $\beta(BG_1(G)) \leq \mid V \mid$.*
Proof. Theorem 2.2 says V' is a resoliving set. Then by definition, metric dimension will not exceed the cardinality of V.

Theorem 2.3. *If G=(V,E) and $BG_1(G)$ is connected, then $log_3(\mid E \mid +1) \leq \beta(BG_1(G) \leq \mid E \mid + \mid V \mid - diam(BG_1(G))$.*
Proof. The degree of a point vertex in $BG_1(G)$ is $\mid E \mid$ and that of a line vertex is $\mid V \mid -2$ [1] ,implies that $\Delta = \mid E \mid$. Theorem 1.3 gives ,
$log_3(\Delta + 1) \leq \beta(G) \leq n - diam(G)$. $BG_1(G)$ has $\mid E \mid + \mid V \mid$ number of vertices and then
$log_3(\mid E \mid +1) \leq \beta(BG_1(G) \leq \mid E \mid + \mid V \mid - diam(BG_1(G))$.

Corollary 2.2. *if G=(V,E) and $BG_1(G)$ are connected, then*
$log_3(\mid E \mid +1) \leq \beta(BG_1(G)) \leq \mid V \mid$.

Proof. Theorem 2.3 gives $log_3(\mid E \mid +1) \leq \beta(BG_1(G))$.
Corollary 2.1 gives $\beta(BG_1(G)) \leq\mid V \mid$.
Combining these results, $log_3(\mid E \mid +1) \leq \beta(BG_1(G)) \leq\mid V \mid$.

Theorem 2.4. *Let G=(V,E) and* $BG_1(G)$ *are connected, then* $\beta(G) < 3^{\beta(BG_1(G))}$.
Proof. For any connected graph G,

$$\beta(G) \leq\mid V \mid -1 \quad [6] \tag{2.1}$$

Since G is connected,

$$\mid V \mid -1 \leq\mid E \mid \tag{2.2}$$

Now log_3 is an increasing function, Then (2.1) gives
$log_3(\beta(G)) \leq log_3(\mid V \mid -1)$
$\leq log_3(\mid E \mid)$ ..... by (2.2)
$< log_3(\mid E \mid) + 1$
$\leq \beta(BG_1(G)) \cdots$ by corollary 2.2,
i.e., $log_3(\beta(G)) < \beta(BG_1(G))$ or
$\beta(G) < 3^{\beta((BG_1(G)}$.

Theorem 2.5. *If* G *is complete and* $\mid V \mid> 2$*, then any* $\mid V \mid -1$ *point vertices form a resoliving set of* $BG_1(G)$.
Proof. Let $S = \{v'_1, v'_2, \cdots, v'_{i-1}, v'_{i+1} \cdots, v'_n\}$ be a set of $\mid V \mid -1$ point vertices. v'_i and all line vertices are out side S, then $C_S(v'_i) = (1, 1, ..., 1)$ .Let e' be a line vertex, then there are two cases.
Case 1: (e is incident with v_i),
Let e=v_iv_j, In this case the distance from e' to a point vertex other than v'_i and v'_j is 1. The distance from e' to both v'_i and v'_j is 2. Hence $C_S(e') = (1, 1, .., 1, 2, 1, ..., 1)$. The position of 2 varies for each edge, so the codes are different.
Case 2: (e is not incident with v_i),
Let e=v_kv_h,here the distances from e' to both v'_k and v'_h are two and the distances from e' to other point vertices are one. Hence $C_S(e') = (1, 1, ...2, ...2, 1...)$.
The 2's occur in positions corresponding to v'_k and v'_h. The position of 2's will vary from line vertex to line vertex, so metric codes of all line vertices are different.
Thus , Metric codes of all point and line vertices are different in both the cases, i.e., S is a resoliving set.

Theorem 2.6. *If* G *is complete and* $\mid V \mid> 2$*, then* $\beta(BG_1(G)) =\mid V \mid -1$.
Proof. There are following possibilities to discuss in the proof
Case 1: ($\mid V \mid= 3$),
Theorem 2.5 implies, $\beta(BG_1(G)) \leq 2$. Also, $BG_1(G)$ is not a path. Therefore,

$\beta(BG_1(G))$=2.
Case 2: $(| V |> 3)$,
Using Theorem 2.5, $\beta(BG_1(G))) \leq | V | -1$. Assume that S is a resoliving set of $BG_1(G)$ with less than $| V | -1$ vertices. Now following subcases are there to be discussed.
Sub case 1: (S contains point vertices only),
Here, at least two point vertices lie outside S, say v'_i and v'_j.
$C_S(v_i) = (1, 1, ..., 1) = C_S(v_j)$,which contradicts the fact that S is a resoliving set. Hence, there must be at least one line vertex in S.
Sub case 2: (S contains line vertices only),
Choose two line vertices e'_i and e'_j out side from S. Then,
$C_S(e'_i) = (2, 2, ..., 2) = C_S(e'_j)$, Which is not possible. Hence, there must be at least one point vertex in S.
Sub case 3: (S contains at least one point vertex and one line vertex),
In this case, there must be at least 3 point vertices out from S. Consider the ${}^3C_2$ line vertices associated to these point vertices. If at least two of these line vertices lie outside S, metric codes of those line vertices will be same and is of the form (...,1,...,2,..,1,...) where the 1's correspond to point vertices in S and 2's corresponding to line vertices of S.This is a contradiction ,so two or more line vertices among the ${}^3C_2$ must lie in S. Then at least four point vertices lie out side S. Now consider the ${}^4C_2$ line vertices corresponding to these point vertices. If any two of them lie outside S, their codes will be same and that is not possible, so there must be at least 7 point vertices outside S. Repeating the same arguments for a finite number of times, the number of line vertices in S will be greater than or equal to cardinality of S. Which is a contradiction as S cannot include line vertices alone.
Therefore, there is no resoliving set for $BG_1(G)$ with less than $| V | -1$ elements. i.e., $\beta(BG_1(G)) =| V | -1$.

Corollary 2.3. *If* $| V |> 3$*, then* $\beta(BG_1(G)) =| V | -1 = \beta(G)$.
Proof. The graph G is complete implies $\beta(G) =| V | -1$ [4]. Now the result follows from the Theorem 2.6.

Theorem 2.7. *Let G be a star graph with* $| V |> 3$*, then the set of all point vertices except any of the leaves is a resoliving set of* $BG_1(G)$.
Proof. Let V=$\{v_1, v_2..., v_n\}$ and v_1 be the centre vertex of G.
Let S=$\{v'_1, v'_2, \cdots, v'_{i-1}, v'_{i+1} \cdots, v'_n\}$. Here,$C_S(v'_i) = (1, 2, 2..., 2)$.
Let e=$v_1v_k, k \neq i$. Then $C_S(e') = (2, 1, ..., 2, ..., 1)$, where the 2's occur in positions corresponding to both v'_1 and v'_k . Every line vertex is uniquely determined by the

vertex v_k, So the metric code of each line vertex is also different. Therefore, S is a resoliving set.

Corollary 2.4. *If G is a star graph and $| V |> 3$, then $\beta(BG_1(G)) \leq| V | -1$.*

Theorem 2.8. *If G is a star graph and $| V |> 3$, then $\beta(BG_1(G)) =| V | -1$.*
Proof. From corollary 2.4, $\beta(BG_1(G)) \leq| V | -1$. Assume that $BG_1(G)$ has a resoliving set S with fewer elements than $| V | -1$. Without loss of Generality, assume that S have $| V | -2$ elements. Let e'_i be the line vertex corresponding to the edge $e_i = v_1v_i$. Now consider the following $| V |$ pairs of vertices of $BG_1(G)$. $(v'_1, v'_1), (v'_2, e'_2), (v'_3, e'_3)...(v'_n, e'_n)$. Eliminate the pairs if at least one component is in S, then there remain minimum two pairs and which leads to following cases.
Case 1: (v'_1 is in S),
In this case, there must be two pairs (v'_i, e'_i) and (v'_j, e'_j), whose components are not in S. Also, $C_S(e'_i) = (2, 1, 1...2, ...2) = C_S(e'_j)$ where the 2's corresponding to v'_1 and other line vertices in S. 1's corresponding to point vertices in S. which is not possible.
Case 2: (v'_1 is not in S),
Here v'_1 and the components of at least one pair (v'_i, e'_i) are not in S. Then $C_S(v'_1) = (1, 1, ..., 2, 2...2) = C_S(e'_i)$, where the 1's are corresponding to point vertices in S and 2's are corresponding to line vertices of S, It is a contradiction. Hence, S is not a resoliving set. Therefore, $\beta(BG_1(G)) =| V | -1$.

Corollary 2.5. *If G is a star graph with $| V |> 3$, then $\beta(BG_1(G))$ exceeds $\beta(G)$ by one.*
Proof. If G is a Star graph, then
$\beta(G) =| V | -2$ (set of $| V | -2$ leaves is the smallest resoliving set).
From theorem 2.8 , $\beta(BG_1(G)) =| V | -1$.

3. An Algorithmic Approach to Metric Dimension of $BG_1(G)$.

In this section an algorithm is established to find the metric dimension of the boolean graph of a given graph G.The inputs are
1. number of vertices of G (Must be greater than 4)
2. a_{ij},i¿j (upper triangular part of the adjacency matrix).
The out put part includes
1. adjacency matrix of $BG_1(G)$
2. the distance matrix of $BG_1(G)$
3. metric dimension of $BG_1(G)$
4. a metric base(a smallest resoliving set) of $BG_1(G)$.
Step 1: Start.

Step 2: Declare two dimensional arrays a, b and c.
Declare the integer variables $e = 0, n, r, i, j, k, flag = 0$.
Step 3: Read n,the number of vertices in G.
Step 4: For i and j from 0 to $n-1$, read a_{ij} for $i > j$, the upper triangular part of adjacency matrix of G.
If a_{ij}=1,
$\{e = e + 1$, $d_{ij} = 1$,
$b_{i,n+e-1} = 0, b_{j,n+e-1} = 0$, $d_{i,n+e-1} = 2, d_{j,n+e-1} = 2$.
for $k = 0$ to $k = n - 1$,
if k is different from i and j,
then set $b_{k,n+e-1} = 1, d_{k,n+e-1} = 1$. $\}$
If $a_{ij} = 0$, then set $d_{ij} = 2$, $b_{ij} = 0$.
Step 5: For i and j from n to $n + e - 1$,
If $i \neq j$, then set $b_{ij} = 0$ and $d_{ij} = 2$.
Step 6: For all possible i and j; set $a_{ii} = 0$, $b_{ii} = 0$, $d_{ii} = 0$, $a_{ji} = a_{ij}$, $b_{ji} = b_{ij}$, d_{ji}=d_{ij}.
Set $r = 2$
Step 7: If $flag = 0$, then go to step 8.
If $flag = 1$, then go to step 9.
Step 8: Consider all possible ${}^{n+e}C_r$ combinations of vertices of $BG_1(G)$. Check whether any combination is a resoliving set with the help of d_{ij}'s. If any of the combination is a resoliving set, then set $flag = 1$, store the combination and goto step 7.
Reset $r = r + 1$ and goto step 7.
Step 9: Print b_{ij}-adjacency matrix of $BG_1(G)$.
Print d_{ij} -distace matirix of $BG_1(G)$.
Print r -metric dimension of $BG_1(G)$.
Print the stored combination -metric base of $BG_1(G)$.
Step 10 Stop.

References

[1] Buckley F. and Harary F., Distance in graphs, Addition Wesley publishing company, 1990.

[2] Chartrand, G., Erwin, D., Harary, F., and Zhang, P., Radio Antipodal Colorings of Cycles, Congr. Numer., 144 (2000), 129-141.

[3] Chartrand G., Zhang P., Introduction to Graph Theory, Boston The Mc Graw Hill Companies, Inc., (2006), 341-346.

[4] Harary F., Graph theory, Addition Wesley publishing company Reading, Mass, 1972.

[5] Harary, F., and Melter, R. A., On the metric dimension of a graph, Ars. Combinatoria, 2 (1976), 191-195.

[6] Janakiramam T. N., Bhanumathi M. and Muthamai S., boolean graph $BG_1(G)$ of a graph G, International Journal of Engineering Science, Advanced computing and Bio technology, Vol. 6, No. 2 (2015), 1-16.

[7] Khuller, S., Raghavachari, B., Rosenfeld, A., Landmarks in graphs, Disc. Appl. Math., 70 (1996), 217-229.

[8] Salter P. J., Leaves of tree, Proc. of the sixth S. E. Conf. on combinatorics, graph theory and computing, Utlitas Math (1975), 549-559.

[9] Salter P. J., Dominating and Reference sets in graphs, J. Math. Phy. Sci., 22 (1988), 445-455.