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ON THE S_3-MAGIC GRAPHS

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Abstract: Let $G = (V(G), E(G))$ be a finite (p, q) graph and let $(A, *)$ be a finite non-abelain group with identity element 1. Let $f : E(G) \to N_q = \{1, 2, \dots, q\}$ and let $g : E(G) \to A \setminus \{1\}$ be two edge labelings of G such that f is bijective. Using these two labelings f and g we can define another edge labeling $\ell : E(G) \to N_q \times A \setminus \{1\}$ by

$$\ell(e) := (f(e), g(e)) \text{ for all } e \in E(G).$$

Define a relation $\leq$ on the range of ℓ by:

$$(f(e), g(e)) \leq (f(e'), g(e')) \text{ if and only if } \quad f(e) \leq f(e').$$

This relation $\leq$ is a partial order on the range of ℓ. Let

$$\{(f(e_1), g(e_1)), (f(e_2), g(e_2)), \dots, (f(e_k), g(e_k))\}$$

be a chain in the range of ℓ. We define a product of the elements of this chain as follows:

$$\prod_{i=1}^{k} (f(e_i), g(e_i)) := ((((g(e_1) * g(e_2)) * g(e_3)) * \cdots) * g(e_k).$$

Let $u \in V$ and let $N^*(u)$ be the set of all edges incident with u. Note that the restriction of ℓ on $N^*(u)$ is a chain, say $(f(e_1), g(e_1)) \leq (f(e_2), g(e_2)) \leq \cdots \leq (f(e_n), g(e_n))$. We define

$$\ell^*(u) := \prod_{i=1}^{n} (f(e_i), g(e_i)).$$

If $\ell^*(u)$ is a constant, say a for all $u \in V(G)$, we say that the graph G is A - magic. The map ℓ^* is called an A -magic labeling of G and the corresponding constant a is called the magic constant. In this paper, we consider the permutation group S_3 and investigate graphs that are S_3-magic.

Keywords and Phrases: A-magic labeling, non-abelian group, symmetric group S_3, S_3-magic labeling.

2020 Mathematics Subject Classification: 05C25, 05C78.

1. Introduction

A graph G is an ordered pair $(V(G), E(G))$, where $V(G)$ is a finite nonempty set whose elements are vertices and $E(G)$ is a binary irreflexive and symmetric relation on $V(G)$ whose elements are called edges. For any abelian group A, written additively, any mapping $\ell : E(G) \to A \setminus \{0\}$ is called a labeling. Given a labeling on the edge set of G, one can introduce a vertex set labeling $\ell^+ : V(G) \to A$ as follows:

$$\ell^+(v) = \sum_{uv \in E(G)} \ell(uv)$$

A graph G is said to be A-magic if there is a labeling $\ell : E(G) \to A \setminus \{0\}$ such that for each vertex v, the sum of the labels of the edges incident with v are all equal to the same constant;that is, $\ell^+(v) = a$ for some fixed $a \in A$. The original concept of A-magic graph was introduced by Sedláček [5]. According to him, a graph G is A magic if there exists an edge labeling on G such that

(i) distinct edges have distinct nonnegative labels; and

(ii) the sum of the labels of the edges incident to a particular vertex is the same for all vertices

A natural question arises: given a nonabelian group A. Does there exist graphs that admit A magic labelings? In this paper, we discuss this question.

2. Main Results

Definition 1. [2] *Let $G = (V(G), E(G))$ be a finite (p, q) graph and A be a finite non-abelian group with identity element 1. Let $f : E(G) \to N_q = \{1, 2, \ldots, q\}$ and let $g : E(G) \to A \setminus \{1\}$ be two edge labelings of G such that f is bijective. Define an edge labeling $\ell : E(G) \longrightarrow N_q \times A \setminus \{1\}$ by*

$$l(e) := (f(e), g(e)),\ e \in E(G).$$

Define a relation $\leq$ on the range of ℓ by:

$$(f(e), g(e)) \leq (f(e^{'}), g(e')) \quad \text{if and only if} \quad f(e) \leq f(e').$$

Then obviously the relation $\leq$ is a partial order on the range of ℓ. Let $\{(f(e_1), g(e_1)), (f(e_2), g(e_2)), \ldots, (f(e_k), g(e_k))\}$ be a chain in the range of ℓ. We define the product of the elements of this chain as follows:

$$\prod_{i=1}^{k}(f(e_i), g(e_i)) := ((((g(e_1) * g(e_2)) * g(e_3)) * g(e_4)) * \ldots) * g(e_k).$$

Let $u \in V$ and let $N^*(u)$ be the set of all edges incident with u. Note that the range of $\ell|_{N^*(u)}$ is a chain, say $(f(e_1), g(e_1)) \leq (f(e_2), g(e_2)) \leq \cdots \leq (f(e_n), g(e_n))$. We define

$$\ell^*(u) = \prod_{i=1}^{n}(f(e_i), g(e_i)). \tag{2.1}$$

If $\ell^*(u)$ is a constant, say a for all $u \in V(G)$, we say that the graph G is A - magic. The map ℓ^* is called an A -magic labeling of G and the corresponding constant a is called the magic constant.

Example 2. Consider the cycle graph $C_3 = (uv, vw, wu)$ and the permutation group S_3. Note the group S_3 is a nonabelian group of order 6 and its elements are given by

$$\rho_0 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \quad \mu_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix},$$
$$\rho_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad \mu_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix},$$
$$\rho_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \quad \mu_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}.$$

Define $f : E(G) \longrightarrow N_q = \{1, 2, 3\}$ as $f(uv) = 1, f(vw) = 2, f(uw) = 3$ and $g : E(G) \to S_3 \setminus \{\rho_0\}$ as $g(e) = \rho_1 , \forall e \in E(G)$. Thus

$$\ell^*(u) = (1, \rho_1)(3, \rho_1) = \rho_1\rho_1 = \rho_2.$$

Similarly $\ell^*(v) = \rho_1\rho_1 = \rho_2$ and $\ell^*(w) = \rho_1\rho_1 = \rho_2$. Thus C_3 is S_3 - magic with magic constant ρ_2.

In this paper, we consider the symmetric group S_3 and investigate graphs that are $S_3 - magic$.

Theorem 3. *Any regular graph is S_3-magic.*

Proof. Let $G = (V(G), E(G))$ be a regular graph with $|E(G)| = q$. Let $g : E(G) \to S_3 \setminus \{\rho_0\}$ be any constant map and let $f : E(G) \to N_q = \{1, 2, \ldots, q\}$ be any bijective map. Then obviously, ℓ^* is a constant map. This completes the proof of the theorem.

Corollary 4. *For any $n \geq 3$, the cycle graph C_n is S_3-magic.*

Corollary 5. *For any $n \geq 2$, the complete graph K_n is S_3-magic.*

Theorem 6. *If the degrees of the vertices of graph G are either all even or odd, then it is S_3-magic.*

Proof. Let G be a (p, q) graph. We consider two cases:

Case(i) Assume that all the vertices of G are of even degree. Define a map $g : E(G) \to S_3 \setminus \{\rho_0\}$ by $g(e) = \mu_1$, $\forall e \in E(G)$ and let $f : E(G) \to N_q$ be any bijective map. Then obviously, G is S_3-magic with $\ell^*(u) = \rho_0$, for all $u \in V(G)$.

Case(ii) Assume that all vertices are of odd degree. The proof is exactly similar to case (i) and the magic constant is μ_1.

Corollary 7. *All Eulerian graphs are S_3-magic.*

The graph obtained by joining a single pendant edge to each vertex of a cycle is called a *crown graph*.

Corollary 8. *A crown graph is S_3-magic.*

Proof. Since all the vertices of crown graph has odd degree (1 or 3), the proof follows from Theorem 6.

Theorem 9. *For any $n \geq 3$, the path of order n is not S_3-magic.*

Proof. Let $P_n = (u_1, u_2, \ldots, u_n)$ be a path of order n. Assume to the contrary that P_n admits a S_3-magic labeling. This implies that, there exist two maps f and g such that $\ell^*(u_1) = \ell^*(u_2) = \cdots = \ell^*(u_n) = a,$ for some $a \in S_3 \setminus \{\rho_0\}$. Since u_1

and u_n are vertices of degree 1, $g(u_1u_2) = g(u_{n-1}u_n) = a$. Let $g(u_2u_3) = b$, $b \in S_3 \setminus \{\rho_0\}$ and let $f(u_1u_2) = m_1, f(u_2u_3) = m_2$ for some $m_1, m_2 \in N_{n-1}$. Now
$\ell^*(u_2) = \begin{cases} (m_1, a)(m_2, b), & \text{if} \quad m_1 < m_2, \\ (m_2, b)(m_1, a), & \text{if} \quad m_2 < m_1. \end{cases}$
This implies that $\ell^*(u_2) = ab$, if $m_1 < m_2$ and $\ell^*(u_2) = ba$, if $m_2 < m_1$. This implies that either $a = \rho_0$ or $b = \rho_0$, which is a contradiction. Hence the path P_n is not S_3-magic.

Comb graph is a graph obtained by joining a single pendant edge to each vertex of a path P_n.

Theorem 10. *Comb graphs are not S_3-magic.*
Proof. Let the vertices of P_n be $u_1, u_2, \ldots, u_n$ and the end vertex of each pendent edge at u_i be u_{n+i}. Suppose to the contrary that comb graph G is S_3 magic. Then by the definition, there exist functions $f : E(G) \to N_{2n-1}$ and $g : E(G) \to S_3 \setminus \{\rho_0\}$ such that $l^*(u_1) = l^*(u_2) = \cdots = l^*(u_{2n}) = a$, for some $a \in S_3 \setminus \{\rho_0\}$. Since each $u_{n+i}, 0 \leq i \leq n$ are of degree 1, it follows that $g(u_iu_{n+i}) = a, 0 \leq i \leq n$. This implies that there exists $b \in S_3 \setminus \{\rho_0\}$ such that $g(u_1u_2) = b$ and $l^*(u_1) = a * b$ or $l^*(u_1) = b * a$ according to the value of $f(u_1u_2)$ and $f(u_1u_{n+1})$. Since $l^*(u_1) = a$, it follows that $ab = a$ or $ba = a$ which implies either $a = \rho_0$ or $b = \rho_0$. This contradiction shows that G is not S_3-magic.

A splitting graph $S(G)$ of a graph G is that graph obtained from G by adding to G a new vertex z' for each vertex z of G and joining z' to the neighbors of z in G.

Theorem 11. *Splitting graph of a path P_n, where $n \geq 3$ is S_3-magic.*
Proof. Let P_n be a path of order n, where $n \geq 3$. Let $u_1, u_2, \ldots, u_n$ be the vertices of P_n. Then $S(P_n)$ has $2n$ vertices and $3n - 3$ edges. Let u_{n+i} be the vertex corresponding to the i^{th} vertex in $S(P_n)$. Observe that there are two pendant edges in $S(P_n)$, one with end points u_2 and u_{n+1} and the other with end points u_{n-1} and u_{2n}. Here we consider 2 cases.

Case (i) $n = 3$.

Define $f : E(S(P_3)) \to N_6$ as $f(u_1u_2) = 1, \quad f(u_3u_5) = 2, \quad f(u_2u_3) = 3, \quad f(u_1u_5) = 4, \quad f(u_2u_4) = 5, \quad f(u_2u_6) = 6$ and $g : E(S(P_3)) \to S_3 \setminus \{\rho_0\}$ as

$$g(u_1u_2) = g(u_3u_5) = \rho_1, \; g(u_2u_4) = g(u_2u_6) = \mu_1,$$
$$g(u_1u_5) = g(u_2u_3) = \mu_2.$$

Note that $\ell^*(u) = \mu_1, \forall u \in V(S(P_3))$. Hence the graph $S(P_3)$ is S_3-magic.

Case(ii) $n > 3$.

Define $f : E(S(P_n)) \to N_{3n-3}$ as

$$f(u_1u_2) = 1, f(u_2u_{n+1}) = 2n, f(u_{n-1}u_n) = n,$$
$$f(u_iu_{n+i+1}) = n + i, \quad 1 \le i \le n-2, f(u_iu_{n+(i-1)}) = i-1, \quad 3 \le i \le n,$$
$$f(u_iu_{i+1}) = 2n + (i-1), \quad 2 \le i \le n-2, f(u_{n-1}u_{2n}) = 2n-1.$$

Now define $g : E(S(P_n)) \to S_3 \setminus \{\rho_0\}$ as

$$g(u_1u_2) = \rho_1, \; g(u_2u_{n+1}) = \mu_1 = g(u_{n-1}u_{2n}),$$
$$g(u_iu_{i+1}) = \mu_1, \quad 2 \le i \le n-2, \; g(u_iu_{n+(i+1)}) = \mu_2, 1 \le i \le n-2,$$
$$g(u_{n-1}u_n) = \mu_2, \; g(u_iu_{n+(i-1)}) = \rho_1, \; 3 \le i \le n.$$

Obviously, $S(P_n)$ is S_3-magic with magic constant μ_1.

This completes the proof of the theorem.

Theorem 12. *The star graph $K_{1,n}$ is S_3-magic if and only if either n is odd or $n \equiv 1(mod\ 3)$.*

Proof. Let $G = K_{1,n}$. First, assume that n is odd. Define $g : E(G) \to S_3 \setminus \{\rho_0\}$ by $g(e) = \mu_1, \forall e \in E(G)$. Let $f : E(G) \to N_n = \{1, 2, \dots, n\}$ be any bijection. Obviously $\ell^*(u) = \mu_1$, $\forall u \in V(G)$. Similarly, we can prove that if $n \equiv 1 (\text{mod } 3)$ then $K_{1,n}$ is S_3-magic.

Conversely, assume that $K_{1,n}$ is S_3-magic. Thus, each pendant edge should be labeled by the same element of S_3 under the map g. Hence $g : E(G) \to S_3 \setminus \{\rho_0\}$ must be a constant map. Let $u_1, u_2, \dots, u_n$ be the vertices of $K_{1,n}$ having degree 1 and let v be the vertex of $K_{1,n}$ having degree n. Let $f : E(G) \to \{1, 2, 3, \dots, n\}$ be a bijection which make $K_{1,n}$ S_3-magic. By our assumption $\ell^*(u_i) = a$, for some $a \in S_3 \setminus \{\rho_0\}$, $i = 1, 2, \dots, n$. Thus $\ell^*(v) = \ell^*(u_i) = a$. This implies that $\underbrace{aa\cdots a}_{n \text{ times}} = a$. Since the maximum order of an element in S_3 is 3 this implies that $n \equiv 1 \,(\text{mod } 3)$ or n is odd. Hence the proof.

Theorem 13. *For $m, n \ge 2$, the complete bipartite graph $K_{m,n}$ is S_3-magic.*

Proof. Let G be the graph $K_{m,n}$. Here we consider four cases.

Case (i) Assume that m and n have the same parity. We can define $g : E(G) \to S_3 \setminus \{\rho_0\}$ as $g(e) = \mu_1, \forall e \in E(G)$ and $f : E(G) \to \{1, 2, \dots, m+n\}$ be any bijection. Then obviously $\ell^*(u)$ is either ρ_0 or $\mu_1, \forall u \in V(G)$.

Case(ii) Suppose $m \equiv 0 (\text{mod } 2)$ and $n \equiv 0 (\text{mod } 3)$. Then $m = 2k$ for some k and $n = 3l$ for some l. Let $U := \{u_1, u_2, \ldots, u_{2k}\}$ and $V := \{v_1, v_2, \ldots, v_{3l}\}$ be the two partite sets of $K_{m,n}$. Define $g : E(G) \to S_3 \setminus \{\rho_0\}$ by

$$g(u_i v_j) = \begin{cases} \rho_1, & \text{if } i \text{ is odd,} \\ \rho_2, & \text{if } i \text{ is even.} \end{cases}$$

Now define $f : E(G) \to \{1, 2, \ldots, m+n\}$ by
$f(u_i v_j) = (i-1)m + j,\ 1 \le i \le 2k,\ 1 \le j \le 3l$. Obviously, $\ell^*(u) = \rho_0,\ \forall u \in V(G)$.

Case (iii) Assume that $m \equiv 0 (\text{mod } 2)$, $n \equiv 2 (\text{mod } 3)$ and n odd. Note that in this case $n = 5 + (k-1)6,\ k \in \mathbb{N}$. Let $U = \{u_1, u_2, \ldots, u_{2l}\}$ and $V = \{v_1, v_2, \ldots, v_n\}$, where $2l = m$ be the two partite sets of $K_{m,n}$. Define $g : E(G) \to S_3 \setminus \{\rho_0\}$ by:

$$g(u_i v_j) = \begin{cases} \rho_1, & \text{if } i \text{ is odd and } j \le n-2, \\ \rho_2, & \text{if } i \text{ is even and } j \le n-2, \\ \mu_1, & \text{if } j = n-1, n. \end{cases}$$

Now define $f : E(G) \to N_{m+n} = \{1, 2, \ldots, m+n\}$ by
$f(u_i v_j) = (i-1)m + j, i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$.
Then $\ell^*(u) = \rho_0,\ \ \forall u \in V(G)$.

Case(iv) Assume that $m \equiv 0 (\text{mod } 2)$, $n \equiv 1 (\text{mod } 3)$ and n is odd. Here the number n is of the form $7 + (k-1)6$, where $k \in \mathbb{N}$. Define $g : E(G) \to S_3 \setminus \{\rho_0\}$ by

$$g(u_i v_j) = \begin{cases} \rho_1, & \text{if } i \text{ and } j \text{ odd}, j \ne 3, j \le 6, \\ \rho_2, & \text{if } i \text{ is odd and } j = 3, \\ \mu_1, & \text{if } j \text{ is even,} \\ \mu_1, & \text{if } j \ge 6, \\ \rho_2, & \text{if } i \text{ is even } j \text{ is odd}, j \ne 3,\ j \le 6, \\ \rho_1, & \text{if } i \text{ is even and } j = 3. \end{cases}$$

Now define the map $f : E(G) \to N_{m+n} = \{1, 2, \ldots, m+n\}$ by $f(u_i v_j) = (i-1)m + j$, where $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Then $\ell^*(u) = \rho_0$, $\forall u \in V(G)$.

This completes the proof.

A wheel graph W_n of order $n+1$, is a graph that contains a cycle of order n and for which every vertex in the cycle is connected to one other vertex (which is known as the hub). The edges of a wheel which include the hub are called spokes. Equivalently, $W_n = K_1 + C_n$.

Theorem 14. *If $n \geq 3$, the wheel W_n is S_3-magic.*
Proof. Let G be the wheel W_n and let the vertices of C_n be $u_1, u_2, \ldots, u_n$ and the vertex of K_1 be k. Here we consider two cases:

Case(i) Assume that n is odd. Define $g : E(G) \to S_3 \setminus \{\rho_0\}$ as follows:
Label each spokes by μ_1 and all the outer edges by μ_2 and define $f : E(G) \to N_{2n} = \{1, 2, \ldots, 2n\}$ as:

$$f(ku_i) = i,\ i = 1, 2, \ldots, n,\ f(u_i u_{i+1}) = n + i,\ i < n,\ f(u_n u_1) = 2n.$$

Then obviously, $\ell^*(e) = \mu_1$, for all $e \in E(W_n)$.

Case(ii) Suppose n is even. Define $g : E(G) \to S_3 \setminus \{\rho_0\}$ by labeling each spokes by μ_1 and all the outer edges by μ_2 and ρ_2 alternatively such that

$$g(u_i u_{i+1}) = \begin{cases} \mu_2, & \text{if } i \text{ is odd,} \\ \rho_2, & \text{if } i \text{ is even.} \end{cases},\ g(u_n u_1) = \rho_2.$$

Now for $i = 1, 2, \ldots, n$, define $f : E(G) \to N_{2n}$ as:

$$f(ku_i) = i,\ f(u_1 u_n) = 2n,\ f(u_i u_{i+1}) = \begin{cases} \frac{(i+1)}{2} + n, & \text{if } i \text{ is odd,} \\ \frac{i}{2} + \frac{3n}{2}, & \text{if } i \text{ is even.} \end{cases}$$

Hence the wheel W_n becomes S_3-magic with magic constant ρ_0.

A shell $S_{n,n-3}$ of width n is a graph obtained by taking $n-3$ concurrent chords in a cycle C_n of n vertices. The vertex at which all chords are concurrent is called *apex*. The two vertices adjacent to the *apex* have degree 2, apex has degree $n-1$ and all other vertices have degree 3.

Theorem 15. *Shell graphs $S_{n,n-3}$ are S_3-magic.*
Proof. Let G be the shell graph $S_{n,n-3}$ and denote the vertices of $S_{n,n-3}$ by $u_1, u_2, \ldots, u_n$. There are n vertices and $2n-3$ edges in $S_{n,n-3}$. Without loss of generality let the *apex* be u_1. Here we consider two cases:

Case(i) n is even.

We define $f : E(G) \to N_{2n-3}$ as follows:

$$f(u_1u_2) = 1, f(u_nu_1) = \frac{n}{2} + 1, f(u_{n-1}u_n) = 2n - 3,$$

$$f(u_iu_{i+1}) = \begin{cases} \frac{i}{2} + 1, & \text{if } i \text{ is even and } 2 \le i \le n-2, \\ \frac{n+i+1}{2}, & \text{if } i \text{ is odd and } 3 \le i \le n-3. \end{cases}$$

$$f(u_1u_{n-1}) = n \text{ and } f(u_1u_i) = n + (i-2) \text{ where } i \ne n-1, 2.$$

and now define $g : E(G) \to S_3 \setminus \{\rho_0\}$ as
$g(u_1u_2) = \rho_1,\ g(u_1u_n) = \rho_2,\ g(u_{n-1}u_n) = \mu_1,\ g(u_1u_{n-1}) = \mu_2,$
$g(u_1u_i) = \mu_1,$ where $i \ne 2, n-1, n.$

$$g(u_iu_{i+1}) = \begin{cases} \mu_2, & \text{if } i \text{ is odd and } 3 \le i \le n-3, \\ \mu_3, & \text{if } i \text{ is even and } 2 \le i \le n-1. \end{cases}$$

Under these maps, shell graphs $S_{n,n-3}$ with even number of vertices are S_3-magic with magic constant μ_2.

Case(ii) n is odd.

Define $f(u_iu_{i+1}) = \begin{cases} \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{n+i}{2}, & \text{if } i \text{ is odd.} \end{cases}$ $,f(u_1u_n) = n$ and $f(u_1u_i) = n+(i-2)$, where $i \ne 2, n$.

Now define $g(u_1u_2) = g(u_nu_1) = g(u_2u_3) = g(u_{n-1}u_n) = \rho_1,\ g(u_1u_i) = \mu_1,$ where $i \ne 2, n$ and $g(u_iu_{i+1}) = \begin{cases} \mu_3, & \text{if } i \text{ is odd,} \\ \rho_1, & \text{if } i \text{ is even.} \end{cases}$

Thus the shell graph $S_{n,n-3}$ with odd number of vertices becomes S_3-magic with magic constant ρ_2.

Hence the proof.

When k copies of C_n share a common edge it will form the $n - gon\,book$ of k pages and is denoted by $B(n, k)$.

Theorem 16. *For any $n \ge 3$ and $k \ge 1$, the $n - gon\,book$ of k pages are S_3-magic.*

Proof. Here we consider two cases:

Case(i) Suppose k is odd. Then all the vertices of $B(n, k)$ will be even. Define $g(e) = \mu_1,\ \forall e \in E(B(n, k))$ and f as any bijection from $E(G)$ to $\{1,\ 2, \ldots,\ k(n-1) + 1\}$. Then the graph $B(n, k)$ becomes S_3-magic with magic constant ρ_0.

Case(ii) Suppose k is even. We denote the common edge of $B(n,k)$ by c. Now define the labeling $g : E(B(n,k)) \to S_3 \setminus \{\rho_0\}$ as follows:
Let $g(c) = \rho_1$ also label the outer edges of the first page by μ_1 and all other edges by μ_3. Denote the edges in the first page by $c,\ a_1,\ a_2, \ldots,\ a_{n-1}$. Now define $f(c) = 1$ and $f(a_i) = i+1$ and map other edges to the set $\{n+1, \ldots, k(n-1)+1\}$ such that $f(e_i) \neq f(e_j),\ \ e_i, e_j \in E(B(n,k))$. Then obviously, $\ell^*(v) = \rho_0$, $\forall v \in V(B(n,k))$.

This completes the proof of the theorem.

Theorem 17. *The cycle graph C_n with a pendant edge is S_3-magic.*
Proof. Let us denote the vertices of C_n by $u_1, u_2, \ldots, u_n$. Without loss of generality assume that the pendant edge e is on the vertex u_1 and let its other end vertex be u_{n+1}.

Case(i) Suppose n is odd. Define $g : E(G) \to S_3 \setminus \{\rho_0\}$ as

$$g(u_i u_{i+1}) = \begin{cases} \mu_1, & \text{if } i \text{ is odd and } i < n, \\ \mu_3, & \text{if } i \text{ is even and } i < n. \end{cases},$$

$$g(u_n u_1) = \mu_1 \ \text{ and } g(u_1 u_{n+1}) = \rho_2.$$

Now define

$$f(u_i u_{i+1}) = \begin{cases} \frac{i+1}{2}, & \text{if } i \text{ is odd and } i < n, \\ \frac{n+1}{2} + \frac{i}{2}, & \text{if } i \text{ is even and } i < n. \end{cases},$$

$$f(u_n u_1) = \frac{n+1}{2},\ f(u_1 u_{n+1}) = n+1.$$

Hence the graph is S_3-magic with magic constant ρ_2.

Case(ii) Suppose n is even. Here we define

$$g(u_i u_{i+1}) = \begin{cases} \mu_3, & \text{if } i \text{ is odd and } i < n, i \neq 1 \\ \mu_2, & \text{if } i \text{ is even and } i \neq n. \end{cases},$$

$$g(u_1 u_2) = \mu_1 = g(u_n u_1) \text{ and } g(u_1 u_{n+1}) = \rho_1.$$

Moreover, define f as:

$$f(u_1 u_2) = 1, f(u_n u_1) = n, f(u_1 u_{n+1}) = n+1,$$

$$f(u_i u_{i+1}) = \begin{cases} \frac{i}{2} + 1, & \text{if } i \text{ even and } i < n, \\ \frac{n}{2} + \frac{i-1}{2}, & \text{if } i \text{ is odd } i \neq 1 \text{ and } i < n. \end{cases}$$

Hence the magic constant is ρ_1.

This completes the proof of the theorem.

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