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Modular Relations Involving Eisenstein Series

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Abstract : In this paper, we establish several P—Q modular relations for the ratios involving Eisenstein series analogous to those recorded by Ramanujan in his Notebooks. We also introduce a parametrization j_s,t for the series involving Eisenstein series and establish several properties of j_s,t- As an application, we evaluate several explicit values for the Eisenstein series.

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1. Introduction

The Eisenstein series in terms of $L(q)$, $M(q)$ and $N(q)$ are defined by

$$L(q) := 1 - 24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n} \tag{1.1}$$

$$M(q) := 1 + 240\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n} \tag{1.2}$$

and

$$N(q) := 1 - 504\sum_{n=1}^{\infty}\frac{n^5q^n}{1-q^n}. \tag{1.3}$$

where $|q| < 1$. On pages 44, 50, 51 and 53 in his Lost Notebook [16], Ramanujan recorded twelve formulas for Eisenstein series. Most of them are representations for certain Eisenstein series in terms of coefficients of Dedekind eta functions, or more precisely, Hauptmodulus. S. Raghavan and S. S. Rangachari [15] have obtained the proofs for all of Ramanujan's identities of Eisenstein series by employing the theory of modular forms, B. C. Berndt et al. [5] have proved Eisenstein identities using the theorems found in Ramanujan Notebooks.

We define Ramanujan theta-function $f(a,b)$ [1] as follows:

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2}b^{n(n-1)/2}, \quad \text{where } |ab| < 1. \tag{1.4}$$

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Note that, if we set $a = qe^{2iz}$, $b = qe^{-2iz}$ and $q = e^{\pi i\tau}$, where z is complex and $Im(\tau) > 0$, then $f(a,b) = \vartheta_3(z,\tau)$, where $\vartheta_3(z,\tau)$ denotes one of the classical theta-function in its standard notation [19].

Following are the special cases of Ramanujan's general theta function,

$$\varphi(q) = f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q;-q)_\infty}{(q;-q)_\infty}, \tag{1.5}$$

$$\psi(q) = f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{1.6}$$

where

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n), \mid q \mid < 1.$$

Ramanujan recorded 23 beautiful $P-Q$ eta-function identities involving quotients of eta-functions in Chapter 25 of his notebooks [16]. Three of these $P-Q$ identities have been proved by Berndt and Zhang [6] by classical means on employing various modular equations of Ramanujan [2]. To establish the remaining $P-Q$ identities Berndt and Zhang [3, pp.204-236] have used the theory of modular forms. M. S. Mahadeva Naika [8] and S. Bhargava, C. Adiga and Mahadeva Naika [7] have obtained a new class of $P-Q$ identities by employing modular equations belonging to alternative theory of signature 4. These $P-Q$ identities are extremely useful in the computation of class invariants and the values of ratios of theta-functions. For more details on P–Qeta-function identities one can refer [9], [10], [11] and [12].
We introduce the gamma function $\Gamma(x)$ for all complex numbers $x \neq 0, -1, -2, ...$, the gamma function $\Gamma(x)$ is defined by

$$\Gamma(x) = \lim_{k\to\infty} \frac{k!k^{x-1}}{(x)_k}.$$

Recently, M. S. Mahadeva Naika et al., [13] and [14], have established modular relations involving Eisenstein series $M(q)$ with $M(q^n)$ and $N(q)$ with $N(q^n)$ and also introduced the parametrization for $M(q)$ and $N(q)$ and they have studied several properties of them. Motivated by these works, in Section 3, we establish modular relations for the ratios of Eisenstein series involving $L(q)$. In Section 4, we find the transformation formulas and also introduce a parametrization $j_{s,t}$ for the ratios of Eisenstein series $L(q)$ which is analogous to parametrization $h_{k,n}$ introduced by Jinhee Yi[17] [18] for ratios of theta functions. We establish several properties of $j_{s,t}$ for particular positive real numbers s and t and several general formulas for determining explicit values for quotients of Eisenstein series $L(q)$. In Section 5, we establish modular relations between $j_{s,t}$ and $j_{k^2s,t}$ for particular values of k, s, and t. These modular equations

helps us to find values of $j_{s,t}$ for particular values of s and t. Finally, we end this section by defining modular equation. The ordinary hypergeometric series ${}_2F_1(a,b;c;x)$ is defined by

$$ {}_2F_1(a,b;c:x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n x^n}{(c)_n n!},$$

where $(a)_0 = 1$, $(a)_n = a(a+1)(a+2)...(a+n-1)$, for $n \geq 1$, $\mid x \mid < 1$.

Let $z(r) := z(r;x) := {}_2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; x\right)$ and

$$q_r := q_r(x) := exp\left(-\pi csc\left(\frac{\pi}{r}\right) \frac{{}_2F_1(\frac{1}{r}, \frac{r-1}{r}; 1; 1-x)}{{}_2F_1(\frac{1}{r}, \frac{r-1}{r}; 1; x)}\right),$$

where $r = 2, 3, 4, 6$ and $0 < x < 1$.

Let n denote a fixed rational number, and assume that

$$n\frac{{}_2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; 1-\alpha\right)}{{}_2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; \alpha\right)} = \frac{{}_2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; 1-\beta\right)}{{}_2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; \beta\right)}, \tag{1.7}$$

where $r = 2, 3, 4$ or 6. Then a modular equation of degree n in the theory of elliptic functions of signature r is a relation between α and β induced by (1.7).

2. Preliminary Results

In this section, we list some identities which are useful in proving our main results.

Lemma 2.1. [2, Ch.17, Entry 13 (viii) - (ix), p.127] *If $L(q)$ is defined as in the equation* (1.1)*, then*

$$\text{(i)}\ \ 2L(e^{-2y}) - L(e^{-y}) = 1 + 24\sum_{k=1}^{\infty} \frac{k}{e^{ky}+1} = z^2(1+x), \tag{2.1}$$

$$\text{(ii)}\ \ 2L(e^{-4y}) - L(e^{-2y}) = 1 + 24\sum_{k=1}^{\infty} \frac{k}{c^{2ky}+1} = z^2(1-x/2). \tag{2.2}$$

where $z := {}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x)$ *and* $y := \pi\frac{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; 1-x)}{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x)}$.

Lemma 2.2. [2, Ch.18, p.215] *If β is of degree 2 over α, then*

$$\beta = \left(\frac{1-\sqrt{1-\alpha}}{1+\sqrt{1-\alpha}}\right)^2. \tag{2.3}$$

Lemma 2.3. [2, Ch.19, Entry 5 (ii), p.230] *If β is of degree 3 over α, then*

$$(\alpha\beta)^{\frac{1}{4}} + \{(1-\alpha)(1-\beta)\}^{\frac{1}{4}} = 1. \tag{2.4}$$

Lemma 2.4. [2, Ch.18, p.215] *If β is of degree 4 over α, then*

$$\beta = \left(\frac{1-\sqrt[4]{1-\alpha}}{1+\sqrt[4]{1-\alpha}}\right)^4. \tag{2.5}$$

Lemma 2.5. [2, Ch.19, Entry 13 (i), p.280] *If β is of degree 5 over α, then*

$$(\alpha\beta)^{\frac{1}{2}} + \{(1-\alpha)(1-\beta)\}^{\frac{1}{2}} + 2\{16\alpha\beta(1-\alpha)(1-\beta)\}^{\frac{1}{6}} = 1. \tag{2.6}$$

Lemma 2.6. [2, Ch.19, Entry 19 (i), p.314] *If β is of degree 7 over α, then*

$$(\alpha\beta)^{\frac{1}{8}} + \{(1-\alpha)(1-\beta)\}^{\frac{1}{8}} = 1. \tag{2.7}$$

Lemma 2.7. [2, Ch.17, Entry 10, p.122] *We have*

$$\text{(i)}\ \ \phi(e^{-y}) = \sqrt{z}, \tag{2.8}$$

$$\text{(ii)}\ \ \phi(-e^{-y}) = \sqrt{z}(1-x)^{\frac{1}{4}}, \tag{2.9}$$

$$\text{(iii)}\ \ \psi(e^{-2y}) = \frac{1}{2}\sqrt{z}(xe^{y})^{\frac{1}{4}}. \tag{2.10}$$

Lemma 2.8. [4, Ch.35, Entry 1, p.325] *Let $h = \dfrac{\pi^{1/4}}{\Gamma(3/4)}$, then*

$$\text{(i)}\ \ \phi(e^{-\pi}) = h,$$

$$\text{(ii)}\ \ \phi(e^{-2\pi}) = h2^{-1}(2+\sqrt{2})^{1/2},$$

$$\text{(iii)}\ \ \phi(e^{-4\pi}) = h2^{-1}(1+2^{-1/4}),$$

$$\text{(iv)}\ \ \phi(e^{-\pi/2}) = h2^{-1/4}(\sqrt{2}+1)^{1/2},$$

$$\text{(v)}\ \ \phi(e^{-\pi/4}) = h(1+2^{-1/4}),$$

$$\text{(vi)}\ \ \psi(e^{-\pi}) = h2^{-5/8}e^{\pi/8},$$

$$\text{(vii)}\ \ \psi(e^{-2\pi}) = h2^{-5/4}e^{\pi/4},$$

$$\text{(viii)}\ \ \psi(e^{-4\pi}) = h2^{-2}e^{\pi/2}(2-\sqrt{2})^{1/2},$$

$$\text{(xi)}\ \ \psi(e^{-8\pi}) = h2^{-2}e^{\pi}(1-2^{-1/4}),$$

$$\text{(x)}\ \ \psi(e^{-\pi/2}) = h2^{-7/16}e^{\pi/16}(\sqrt{2}+1)^{1/4}.$$

Lemma 2.9. [2, Ch.16, Entry 27, p.43] *If $a, b > 0$ such that $ab = \pi^2$, then*

$$\text{(i) } \sqrt[4]{a}\phi(e^{-a}) = \sqrt[4]{b}\phi(e^{-b}), \tag{2.11}$$

$$\text{(ii) } 2\sqrt[4]{a}e^{-a/4}\psi(e^{-2a}) = \sqrt[4]{b}\phi(-e^{-b}). \tag{2.12}$$

3. Main Results

In this section, we establish some new modular relations for the ratios involving Eisenstein series. Throughout this section we set $u := PQ$, $v := \dfrac{P}{Q}$, $\mathbb{P}_\kappa := \sqrt{u^\kappa} + \dfrac{2^{2\kappa}}{\sqrt{u^\kappa}}$, and $\mathbb{Q}_\kappa := \sqrt{v^\kappa} + \dfrac{1}{\sqrt{v^\kappa}}$.

Theorem 3.1. *If $P := \dfrac{2L(q^2) - L(q)}{2L(q^4) - L(q^2)}$ and $Q := \dfrac{2L(q^4) - L(q^2)}{2L(q^8) - L(q^4)}$, then*

$$7Q^2 + 8Q + 2Q^2P = Q^2P^2 + 16. \tag{3.1}$$

Proof. From the equations (2.1) and (2.2), we find that

$$\alpha := 2\left(\frac{P-1}{P+2}\right) \text{ and } \beta := 2\left(\frac{Q-1}{Q+2}\right). \tag{3.2}$$

Now equation (2.3) can be rewriting as

$$(PQ + 2Q)S = 4 - Q, \tag{3.3}$$

where $S = \sqrt{1-\alpha}$. Squaring both sides of the equation (3.3) and then employing (3.2) in the resultant equation, we get (3.1).

Theorem 3.2. *If $P := \dfrac{2L(q^2) - L(q)}{2L(q^4) - L(q^2)}$ and $Q := \dfrac{2L(q^6) - L(q^3)}{2L(q^{12}) - L(q^6)}$, then*

$$\begin{aligned} &2^6\mathbb{Q}_1\left(3\mathbb{P}_1 - 4\mathbb{P}_3\right) - \mathbb{Q}_2\left(4628 + 192\mathbb{P}_2\right) + 896\mathbb{Q}_3\mathbb{P}_1 - 539\mathbb{Q}_4 \\ &+ 1216\mathbb{P}_2 + 2^6\mathbb{P}_4 + 2142 = 0. \end{aligned} \tag{3.4}$$

Proof. Using the equation (3.2) in the equation (2.4), we find that

$$\begin{aligned} &6S^2PQ - 4SPQ + 2Q + 4PQ + 12S^2Q - 8S^3Q - 4S^3PQ - 8SQ \\ &+ 2P - 16S^3 - 8S^3P - 8SP - 16S + 24S^2 - 8 + 12S^2P = 0, \end{aligned} \tag{3.5}$$

where $S = (\alpha\beta)^{1/4}$. Now by eliminating S from the above equation (3.5) and setting $u := PQ$ and $v := \dfrac{P}{Q}$ along with $\mathbb{P}_k$ and $\mathbb{Q}_k$ in the resulting equation, we arrive at (3.4).

Theorem 3.3. *If* $P := \dfrac{2L(q^2) - L(q)}{2L(q^4) - L(q^2)}$ *and* $Q := \dfrac{2L(q^8) - L(q^4)}{2L(q^{16}) - L(q^8)}$, *then*

$$\begin{aligned}
&\left(2^8v^2 - \frac{847}{v^2}\right) + 49\left(u^2 + \frac{2^8}{u^2}\right) + 2^6\left(2^4v + \frac{37}{v}\right) - \left(u + \frac{2^4}{u}\right) \\
&\times \left(448 + 10\left(2^4v + \frac{3^2}{v}\right)\right) + 28\left(\sqrt{u^3} + \frac{2^6}{\sqrt{u^3}}\right)\left(2^2\sqrt{v} - \frac{7}{\sqrt{v}}\right) \\
&- 4\left(\sqrt{u} + \frac{2^2}{\sqrt{u}}\right)\left[\left(2^6\sqrt{v^3} - \frac{143}{\sqrt{v^3}}\right) - \left(160\sqrt{v} - \frac{184}{\sqrt{v}}\right)\right] + 832 = 0.
\end{aligned} \tag{3.6}$$

Proof of the equation (3.6) is similar to the proof of the equation (3.1) except that in place of the equation (2.3), we use the equation (2.5).

Theorem 3.4. *If* $P := \dfrac{2L(q^2) - L(q)}{2L(q^4) - L(q^2)}$ *and* $Q := \dfrac{2L(q^{10}) - L(q^5)}{2L(q^{20}) - L(q^{10})}$, *then*

$$\begin{aligned}
&10241\mathbb{Q}_6 - 1536\mathbb{P}_4 - 46080\mathbb{P}_2 - \mathbb{Q}_2\left(32000\mathbb{P}_2 + 1280\mathbb{P}_4 - 184335\right) \\
&- 1280\mathbb{Q}_1\left(11\mathbb{P}_1 - 12\mathbb{P}_3\right) - 1280\mathbb{Q}_3\left(39\mathbb{P}_1 - 4\mathbb{P}_3\right) - 17920\mathbb{Q}_5\mathbb{P}_1 \\
&+ 30\mathbb{Q}_4\left(2^7\mathbb{P}_2 + 4659\right) + 281580 = 0.
\end{aligned} \tag{3.7}$$

Proof of the equation (3.7) is similar to the proof of the equation (3.4) except that in place of the equation (2.4), we use the equation (2.6).

Theorem 3.5. *If* $P := \dfrac{2L(q^2) - L(q)}{2L(q^4) - L(q^2)}$ *and* $Q := \dfrac{2L(q^{14}) - L(q^7)}{2L(q^{28}) - L(q^{14})}$, *then*

$$\begin{aligned}
&23937893793\mathbb{Q}_8 - 1123634550528\mathbb{P}_2 - 170516295936\mathbb{P}_4 - 14888116224\mathbb{P}_6 \\
&- 188116992\mathbb{P}_8 + 23514624\mathbb{Q}_1\left(40133\mathbb{P}_1 + 4218\mathbb{P}_3 + 1488\mathbb{P}_5 + 2^6\mathbb{P}_7\right) \\
&+ 367416\mathbb{Q}_2\left(2429904\mathbb{P}_2 + 164672\mathbb{P}_4 - 5120\mathbb{P}_6 + 9796063\right) - 5878656\mathbb{Q}_3 \\
&\times \left(210335\mathbb{P}_1 + 43164\mathbb{P}_3 + 1664\mathbb{P}_5\right) + 5878656\mathbb{Q}_5\left(58209\mathbb{P}_1 + 3700\mathbb{P}_3\right) + \\
&183708\mathbb{Q}_4\left(97376\mathbb{P}_4 - 5216\mathbb{P}_2 - 2549119\right) - 367416\mathbb{Q}_6\left(88208\mathbb{P}_2 + 234903\right) \\
&- 20822199552\mathbb{Q}_7\mathbb{P}_1 - 7999815713274 = 0.
\end{aligned} \tag{3.8}$$

The proof of the equation (3.8) is similar to the equation (3.7) except that in place of the equation (2.6), we use the equation (2.7).

4. Transformation Farmula and the Parametrization $j_{s,t}$ for the Eisenstein's Series $L(q)$

In this section, we find transformation formula for Eisenstein's series $L(q)$ and introduce a parametrization $j_{s,t}$ for all positive real numbers s and t, we then establish their properties.

Theorem 4.1. *If $a, b > 0$ such that $ab = \pi^2$, then*

$$a(2L(e^{-2a}) - L(e^{-a})) = 2b(2L(e^{-4b}) - L(e^{-2b})). \tag{4.1}$$

Proof. Employing the equations (2.8) and (2.9) in the equation (2.1), we obtain

$$2L(e^{-2y}) - L(e^{-y}) = \phi^4(e^{-y}) + 2^4 e^{-y}\psi^4(e^{-2y}). \tag{4.2}$$

Using the equations (2.11) and (2.12) in (4.2), we arrive at

$$2L(e^{-2a}) - L(e^{-a}) = \frac{b}{a}\phi^4(e^{-b}) + \frac{b}{a}\phi^4(-e^{-b}). \tag{4.3}$$

Employing the equations (2.8) and (2.9), we get (4.1).

Definition 4.1. For any positive real numbers s and t, we define $j_{s,t}$ as

$$j_{s,t} = \frac{1}{t}\left(\frac{2L(q^2) - L(q)}{2L(q^{2t}) - L(q^t)}\right), \tag{4.4}$$

where $q = e^{-\pi\sqrt{s/t}}$.

Theorem 4.2. *If s and t are any positive real numbers, then*

$$\text{(i)}\ \ j_{s,t}j_{\frac{4}{s},t} = 1, \tag{4.5}$$

$$\text{(ii)}\ \ j_{2,t} = 1, \tag{4.6}$$

$$\text{(iii)}\ \ j_{\frac{4}{s},1} = 1, \tag{4.7}$$

$$\text{(iv)}\ \ j_{s,t} = j_{\frac{4}{s},\frac{1}{t}}. \tag{4.8}$$

proof of (i). Let $a = \pi\sqrt{\frac{s}{t}}$ and $b = \pi\sqrt{\frac{t}{s}}$ in the equation (4.1), we find that

$$\pi\sqrt{\frac{s}{t}}\left(2L(e^{-2\pi\sqrt{s/t}}) - L(e^{-\pi\sqrt{s/t}})\right) = \pi\sqrt{\frac{t}{s}}\left(4L(e^{-4\pi\sqrt{s/t}}) - 2L(e^{-2\pi\sqrt{s/t}})\right). \tag{4.9}$$

Again let $a = \pi\sqrt{st}$ and $\pi\sqrt{1/st}$ in the equation(4.1), we obtain

$$\pi\sqrt{st}\left(2L(e^{-2\pi\sqrt{st}}) - L(e^{-\pi\sqrt{st}})\right) = \pi\sqrt{\frac{1}{st}}\left(4L(e^{-4\pi\sqrt{1/st}}) - 2L(e^{-2\pi\sqrt{1/st}})\right). \tag{4.10}$$

Using the equations (4.9), (4.10) and (4.4), we get (4.5).

proof of (ii). Using (4.4) and replacing a, b by $\pi\sqrt{2/t}$, $\pi\sqrt{t/2}$ respectively in (4.1), we obtain

$$j_{2,t} = \frac{2L(e^{-2\pi\sqrt{2/t}}) - L(e^{-\pi\sqrt{2/t}})}{2L(e^{-2\pi\sqrt{2t}}) - L(e^{-\pi\sqrt{2t}})} = 1. \tag{4.11}$$

Since proofs of (*iii*) and (*iv*) are similar to the proof of (*ii*), we omit the details.

Theorem 4.3. *For all positive real numbers s, r and t, we have*

$$j_{sr,t}\, j_{tr,s}^{-1} = j_{\frac{s}{r},\frac{s}{t}}. \tag{4.12}$$

Proof. Replace respectively a, b by $\pi\sqrt{sr/t}$, $\pi\sqrt{t/sr}$ and $\pi\sqrt{tr/s}$, $\pi\sqrt{s/rt}$ in the transformation formula (4.1) and using (4.4), we get (4.3).

Corollary 4.1. *If a, b and c are any positive real numbers, then*

$$\text{(i)}\ \ j_{\frac{1}{b},\frac{1}{b}} j_{\frac{a}{b^2},a} = j_{\frac{a}{b},\frac{a}{b}}, \tag{4.13}$$

$$\text{(ii)}\ \ j_{ab,a} = j_{a,\frac{a}{b}} j_{a^2,b}. \tag{4.14}$$

proof of (*i*). By putting $r = \frac{1}{b}, s = a$ and $t = \frac{a}{b}$ in the equation (4.12), we find that

$$j_{4b,b} j_{\frac{a}{b^2},a} = j_{\frac{a}{b},\frac{a}{b}}. \tag{4.15}$$

Using (4.8) in the equation (4.15), we arrive at (4.13).

proof of (*ii*). Letting $r = a, t = b$ and $s = a$ in the equation (4.12), we find that

$$j_{a^2,b} = j_{\frac{4}{a},\frac{a}{b}} j_{ab,a}. \tag{4.16}$$

Using (4.5) in the equation (4.16), we arrive at (4.14).

Theorem 4.4. *If a, b and c are any positive real numbers, then*

$$j_{b,\frac{c}{a}} = j_{a,\frac{c}{b}}\, j_{c,\frac{b}{a}}. \tag{4.17}$$

Proof. From (4.4), we find that, for all positive real numbers a, b and c,

$$j_{b,\frac{c}{a}}\, j_{a,\frac{c}{b}}^{-1} = \frac{a}{b}\left(\frac{2L(e^{-2\pi\sqrt{\frac{ac}{b}}}) - L(e^{-\pi\sqrt{\frac{ac}{b}}})}{2L(e^{-2\pi\sqrt{\frac{bc}{a}}}) - L(e^{-\pi\sqrt{\frac{bc}{a}}})}\right) = j_{c,\frac{b}{a}}. \tag{4.18}$$

Hence the result.

Theorem 4.5. *If for all positive real numbers x, a, b, c and d, with $ab = cd$, then*

$$j_{a,b}\, j_{xc,xd} = j_{c,d}\, j_{xa,xb}. \tag{4.19}$$

Proof. From (4.4), we find that, for all positive real numbers x, a, b, c and d,

$$j_{xa,xb}\, j_{a,b}^{-1} = \frac{1}{x}\frac{2L(e^{-2\pi\sqrt{ab}}) - L(e^{-\pi\sqrt{ab}})}{2L(e^{-2x\pi\sqrt{ab}}) - L(e^{-x\pi\sqrt{ab}})} \tag{4.20}$$

and

$$j_{xc,xd}\, j_{c,d}^{-1} = \frac{1}{x}\frac{2L(e^{-2\pi\sqrt{cd}}) - L(e^{-\pi\sqrt{cd}})}{2L(e^{-2x\pi\sqrt{cd}}) - L(e^{-x\pi\sqrt{cd}})}. \tag{4.21}$$

The result follows from (4.20), (4.21) and the hypothesis $ab = cd$.

Corollary 4.2 *For all positive real numbers s and t, we have*

$$j_{st,st} = j_{t,t}\, j_{2s,st^2}. \tag{4.22}$$

Proof. The result (4.22) follows from (4.6) and the equation (4.19) with $a = 2$, $b = t^2$, $c = d = t$ and $x = s$.

Theorem 4.6 *For all positive real numbers a, b, c and d, we have*

$$j_{\frac{a}{b},\frac{c}{d}} = \frac{j_{ad,bc}}{j_{ac,bd}}. \tag{4.23}$$

Proof. From (4.4), we find that, for all positive real numbers a, b, c and d,

$$j_{ad,bc}\, j_{ac,bd}^{-1} = \frac{d}{c}\left(\frac{2L(e^{-2\pi\sqrt{\frac{ad}{bc}}}) - L(e^{-\pi\sqrt{\frac{ad}{bc}}})}{2L(e^{-2\pi\sqrt{\frac{ac}{bd}}}) - L(e^{-\pi\sqrt{\frac{ac}{bd}}})}\right) = j_{\frac{a}{b},\frac{c}{d}}. \tag{4.24}$$

Hence the theorem.

Theorem 4.7 *If* $h = \frac{\pi^{1/4}}{\Gamma(3/4)}$ *and* $L(e^{-2\pi}) = \frac{3}{\pi}$, *then*

$$\text{(i)} \quad L(e^{-\pi}) = \frac{6}{\pi} - \frac{3}{2}h^4,$$

$$\text{(ii)} \quad L(e^{-4\pi}) = \frac{3}{2\pi} + \frac{3}{8}h^4,$$

$$\text{(iii)} \quad L(e^{-8\pi}) = \frac{3}{4\pi} + \frac{3}{32}h^4(3+2\sqrt{2}),$$

$$\text{(iv)} \quad L(e^{-\pi/2}) = \frac{12}{\pi} - \frac{3}{2}h^4(3+2\sqrt{2}),$$

$$\text{(v)} \quad L(e^{-16\pi}) = \frac{3}{8\pi} + \frac{3}{128}h^4[(7+6\sqrt{2}) + 4(2^{1/4}+2^{3/4})],$$

$$\text{(vi)} \quad L(e^{-\pi/4}) = \frac{24}{\pi} - \frac{3}{2}h^4[(7+6\sqrt{2}) + 4(2^{1/4}+2^{3/4})].$$

Proof of (*i*) to (*vi*) follows from the equation (4.2) and the Lemma (2.8).

Theorem 4.8 *If* $P_n := \dfrac{2L(e^{-2n\pi}) - L(e^{-n\pi})}{2L(e^{-4n\pi}) - L(e^{-2n\pi})}$, *then*

$$\text{(i)} \quad P_1 = 2, \tag{4.25}$$

$$\text{(ii)} \quad P_2 = \frac{-4}{7} + \frac{8\sqrt{2}}{7}, \tag{4.26}$$

$$\text{(iii)} \quad P_3 = \frac{1}{154}\left(52 + 192\sqrt{3} - 77\sqrt{\frac{18432}{5929} + \frac{19968\sqrt{3}}{5929}}\right), \tag{4.27}$$

$$\text{(iv)} \quad P_4 = \frac{143}{360} - \frac{1}{360\sqrt{A+B}} + \frac{1}{2}\left(\sqrt{C-D+E}\right), \tag{4.28}$$

$$\text{(v)} \quad P_5 = \frac{\left(732 + 2944\sqrt{5} - 1463\sqrt{\frac{9625600}{2140369} + \frac{4310016\sqrt{5}}{2140369}}\right)}{2926}. \tag{4.29}$$

where,

$$A := 15488 - 55688955 \times 5^{2/3}\left(\frac{7}{a}\right)^{1/3},$$
$$B := 15 \times 7^{2/3}(5a)^{1/3},$$
$$C := \frac{968}{2025} + \frac{3712597}{864}\left(\frac{7}{5a}\right)^{1/3},$$
$$D := \frac{7^{2/3}a^{1/3}}{864 \times 5^{2/3}},$$
$$E := \frac{101208431}{16200\sqrt{2(A+B)}},$$

and

$$a := 11564095955 + 384\sqrt{1154784010847370}.$$

Proof of (i). Setting $a = b = \pi$ in the transformation formula (4.1), we arrive at (4.25).

Proof of (ii). Setting $q = e^{-\pi}$ in the equation (3.4) and using (4.25), we find that

$$7Q^2 + 8Q - 16 = 0. \tag{4.30}$$

Solving the above equation (4.30), we obtain (4.26).

Proofs of (*iii*), (*iv*) and (*v*) are similar to the proof of (4.26), hence we omit the details.

5. Explicit Evaluations

In this section, we obtain the explicit evaluations of $j_{s,2}$ and $j_{k^2s,2}$ for $k = 2, 3, 4, 5, 7$ and $s = 1, 1/2, 2/3, 2/5, 2/7, 2/9, 2/25$. Throughout this section, we set
$\mathbb{K}_\kappa := \sqrt{K^\kappa} + \dfrac{2^{2\kappa}}{\sqrt{K^\kappa}}$ and $\mathbb{L}_\kappa := \sqrt{L^\kappa} + \dfrac{1}{\sqrt{L^\kappa}}$.

Theorem 5.1 *If* $K = 4j_{s,2}j_{4s,2}$ *and* $L = \frac{j_{4s,2}}{j_{s,2}}$, *then*

$$7L + 2\sqrt{L}\left(\sqrt{K} + \frac{2^2}{\sqrt{K}}\right) = \left(K + \frac{2^4}{K}\right). \tag{5.1}$$

Proof. Using (4.4) with $t = 2$ in (3.1), we arrive at (5.1).

Corollary 5.1. *We have*

$$\text{(i)}\ \ j_{1,2} = \frac{1}{2} + \frac{3\sqrt{2}}{4}, \tag{5.2}$$

$$\text{(ii)}\ \ j_{4,2} = \frac{6\sqrt{2}}{7} - \frac{4}{7}. \tag{5.3}$$

Proof of (5.2) *and* (5.3). Setting $s = 1$ in (5.1) and using the fact that $j_{1,2}j_{4,2} = 1$, we arrive at (5.2) and (5.3).

Theorem 5.2. *If* $K = 4j_{s,2}j_{9s,2}$ *and* $L = \frac{j_{9s,2}}{j_{s,2}}$, *then*

$$\begin{aligned} &2^6\mathbb{L}_1\left(3\mathbb{K}_1 - 4\mathbb{K}_3\right) - \mathbb{L}_2\left(4628 + 192\mathbb{K}_2\right) + 896\mathbb{L}_3\mathbb{K}_1 \\ &- 539\mathbb{L}_4 + 1216\mathbb{K}_2 + 2^6\mathbb{K}_4 + 2142 = 0. \end{aligned} \tag{5.4}$$

Proof. Using (4.4) with $t = 2$ in (3.4), we arrive at (5.4).

Corollary 5.2. *We have*

$$\text{(i)}\ \ j_{\frac{2}{3},2} = \frac{13}{11} + \frac{4\sqrt{3}}{11}, \tag{5.5}$$

$$\text{(ii)}\ \ j_{6,2} = \frac{143}{121} - \frac{44\sqrt{3}}{121}, \tag{5.6}$$

$$\text{(iii)}\ \ j_{\frac{2}{9},2} = \frac{13}{77} + \frac{48\sqrt{3}}{77} + \frac{4\sqrt{72+78\sqrt{3}}}{77}, \tag{5.7}$$

$$\text{(iv)}\ \ j_{18,2} = \frac{13}{77} + \frac{48\sqrt{3}}{77} - \frac{4\sqrt{72+78\sqrt{3}}}{77}. \tag{5.8}$$

Proof of (*i*) *and* (*ii*). Setting $s = 2/3$ in (5.4) and using the fact that $j_{\frac{2}{3},2}j_{6,2} = 1$, we find that

$$(11x^2 - 26x + 11)(x+1)^2(7x^2 - 22x + 7)^2 = 0, \tag{5.9}$$

where $x = j_{\frac{2}{3},2}$. We observe that the first factor of (5.9) vanishes for particular value of $q = e^{-\pi\sqrt{1/3}}$, whereas the other factors do not vanish. Hence, we obtain

$$11x^2 - 26x + 11 = 0. \tag{5.10}$$

Solving the equation (5.10) for x, we arrive at (5.5) and (5.6).

Proof of (*iii*) *and* (*iv*). Setting $s = 2$ in (5.4) and using the fact that $j_{2,2} = 1$, we find that

$$8624y^4 - 5824y^3 - 21984y^2 - 5824y + 8624 = 0. \tag{5.11}$$

On simplification we obtain

$$8624Y^2 - 5824Y - 39232 = 0, \tag{5.12}$$

where $Y = y + \frac{1}{y}$. Again solving for y, we arrive at (5.7) and (5.8).

Theorem 5.3 *If* $K = 4j_{s,2}j_{16s,2}$ *and* $L = \frac{j_{16s,2}}{j_{s,2}}$, *then*

$$\begin{aligned}&\left(2^8L^2 - \frac{847}{L^2}\right) + 49\left(K^2 + \frac{2^8}{K^2}\right) + 2^6\left(2^4L + \frac{37}{L}\right) - \left(K + \frac{2^4}{K}\right)\\ &\times\left(448 + 10\left(2^4L + \frac{3^2}{L}\right)\right) + 28\left(\sqrt{K^3} + \frac{2^6}{\sqrt{K^3}}\right)\left(4\sqrt{L} - \frac{7}{\sqrt{L}}\right)\\ &- 4\left(\sqrt{K} + \frac{4}{\sqrt{K}}\right)\left[\left(2^6\sqrt{L^3} - \frac{143}{\sqrt{L^3}}\right) - \left(160\sqrt{L} - \frac{184}{\sqrt{L}}\right)\right] + 832 = 0.\end{aligned} \tag{5.13}$$

Proof. Using (4.4) with $t = 2$ in (3.6), we arrive at (5.13).

Corollary 5.3 *We have*

$$\text{(i)}\ \ j_{\frac{1}{2},2} = \frac{1}{2} + \sqrt{2}, \tag{5.14}$$

$$\text{(ii)}\ \ j_{8,2} = \frac{-2}{7} + \frac{4\sqrt{2}}{7}. \tag{5.15}$$

Proof of (i) and (ii). Setting $s = 1/2$ in (5.13) and using the fact that $j_{\frac{1}{2},2}j_{8,2} = 1$, we find that

$$(4x^2 - 4x - 7)(2x - 1)^2(4x^2 - 4x - 11)^2 = 0, \tag{5.16}$$

where $x = j_{\frac{1}{2},2}$. We observe that the first factor of (5.16) vanishes for particular value of $q = e^{-\pi\sqrt{1/4}}$, whereas the other factors do not vanish. Hence, we obtain

$$4x^2 - 4x - 7 = 0. \tag{5.17}$$

Solving the equation (5.17) for x, we arrive at (5.5) and (5.6).

Theorem 5.4. *If* $K = 4j_{s,2}j_{25s,2}$ *and* $L = \frac{j_{25s,2}}{j_{s,2}}$, *then*

$$\begin{aligned}&10241\mathbb{L}_6 - 1536\mathbb{K}_4 - 46080\mathbb{K}_2 - \mathbb{L}_2\left(32000\mathbb{K}_2 + 1280\mathbb{K}_4 - 184335\right)\\ &- 1280\mathbb{L}_1\left(11\mathbb{K}_1 - 12\mathbb{K}_3\right) - 1280\mathbb{L}_3\left(39\mathbb{K}_1 - 4\mathbb{K}_3\right) - 17920\mathbb{L}_5\mathbb{K}_1\\ &+ 30\mathbb{L}_4\left(2^7\mathbb{K}_2 + 4659\right) + 281580 = 0.\end{aligned} \tag{5.18}$$

Proof. Using (4.4) with $t = 2$ in (3.7), we arrive at (5.18).

Corollary 5.4 *We have*

$$\text{(i)}\ \ j_{\frac{2}{5},2} = \frac{97}{209} + \frac{72\sqrt{5}}{209} + \frac{12\sqrt{97\sqrt{5}-58}}{209}, \tag{5.19}$$

$$\text{(ii)}\ \ j_{10,2} = \frac{97}{209} + \frac{72\sqrt{5}}{209} - \frac{12\sqrt{97\sqrt{5}-58}}{209}, \tag{5.20}$$

$$\text{(iii)}\ \ j_{\frac{2}{25},2} = \frac{183}{1463} + \frac{736\sqrt{5}}{1463} + \frac{8\sqrt{4209\sqrt{5}+9400}}{1463}, \tag{5.21}$$

$$\text{(iv)}\ \ j_{50,2} = \frac{183}{1463} + \frac{736\sqrt{5}}{1463} - \frac{8\sqrt{4209\sqrt{5}+9400}}{1463}. \tag{5.22}$$

Proof of (i) and (ii). Setting $s = 2/5$ in (5.18) and using the fact that $j_{\frac{2}{5},2}j_{10,2} = 1$, we find that

$$(209x^4 - 388x^3 + 102x^2 - 388x + 209)(x-1)^2(x+1)^2(7x^2 - 18x + 7)^2 = 0. \tag{5.23}$$

where $x = j_{\frac{2}{5},2}$. We observe that the first factor of (5.23) vanishes for particular value of $q = e^{-\pi\sqrt{1/5}}$, whereas the other factors do not vanish. On simplification, we obtain

$$209X^2 - 388X - 316 = 0, \tag{5.24}$$

where $X = x + \frac{1}{x}$. Again solving for x , we arrive at (5.19) and (5.20).

Proof of (iii) and (iv). Setting $s = 2$ in (5.18) and using the fact that $j_{2,2} = 1$, we find that

$$10241y^4 - 5124y^3 - 30714y^2 - 5124y + 10241 = 0. \tag{5.25}$$

By setting $Y = y + \frac{1}{y}$ in the above equation (5.25), we deduce that

$$10241Y^2 - 5124Y - 51196 = 0. \tag{5.26}$$

Again solving for y, we arrive at (5.21) and (5.22).

Theorem 5.5 *If* $K = 4j_{s,2}j_{49s,2}$ *and* $L = \frac{j_{49s,2}}{j_{s,2}}$ *then*

$$\begin{aligned}
&23937893793\mathbb{L}_8 - 1123634550528\mathbb{K}_2 - 170516295936\mathbb{K}_4 - 14888116224\mathbb{K}_6 \\
&- 188116992\mathbb{K}_8 + 23514624\mathbb{L}_1\left(40133\mathbb{K}_1 + 4218\mathbb{K}_3 + 1488\mathbb{K}_5 + 2^6\mathbb{K}_7\right) \\
&+ 367416\mathbb{L}_2\left(2429904\mathbb{K}_2 + 164672\mathbb{K}_4 - 5120\mathbb{K}_6 + 9796063\right) - 5878656\mathbb{L}_3 \\
&\times\left(210335\mathbb{K}_1 + 43164\mathbb{K}_3 + 1664\mathbb{K}_5\right) + 5878656\mathbb{L}_5\left(58209\mathbb{K}_1 + 3700\mathbb{K}_3\right) + \\
&183708\mathbb{L}_4\left(97376\mathbb{K}_4 - 5216\mathbb{K}_2 - 2549119\right) - 367416\mathbb{L}_6\left(88208\mathbb{K}_2 + 234903\right) \\
&- 20822199552\mathbb{L}_7\mathbb{K}_1 - 7999815713274 = 0.
\end{aligned} \tag{5.27}$$

Proof. Using (4.4) with $t = 2$ in (3.8), we arrive at (5.27).

Corollary 5.5. *We have*

$$\text{(i)}\ \ j_{\frac{2}{7},2} = \frac{71}{57} + \frac{16\sqrt{7}}{57}, \tag{5.28}$$

$$\text{(ii)}\ \ j_{14,2} = \frac{71}{57} - \frac{16\sqrt{7}}{57}. \tag{5.29}$$

Proof of (i) and (ii). By setting $s = 2/7$ in the equation (5.27) and using the fact that $j_{\frac{2}{7},2}j_{14,2} = 1$, we find that

$$\begin{aligned}&(57x^2 - 142x + 257)(23x^4 + 20x^3 - 150x^2 + 20x + 23)^2\\ &(x+1)^2(11x^2 - 26x + 11)^2 = 0,\end{aligned} \tag{5.30}$$

where $x = j_{\frac{2}{7},2}$. We observe that the first factor of (5.30) vanishes for particular value of $q = e^{-\pi\sqrt{1/7}}$, whereas the other factors do not vanish. Hence, we obtain

$$57x^2 - 142x + 257 = 0. \tag{5.31}$$

Solving the equation (5.31) for x, we arrive at (5.28) and (5.29).

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