South East Asian J. of Mathematics and Mathematical Sciences
Vol. 18, No. 3 (2022), pp. 229-246

DOI: 10.56827/SEAJMMS.2022.1803.19 **ISSN (Online): 2582-0850**

ISSN (Print): 0972-7752

COMPLETENESS IN MULTI METRIC SPACES

Sujoy Das

Department of Mathematics,
Suri Vidyasagar College,
Suri - 731101, West Bengal, INDIA

E-mail : sujoy_math@yahoo.co.in

(**Received:** *Aug. 07, 2021* **Accepted:** *Oct. 24, 2022* **Published:** *Dec. 30, 2022)*

Abstract: In the present paper a notion of convergence in multi metric space is presented. Complete multi metric space is introduced and some properties are studied. Cantor's intersection theorem and Banach's fixed point theorem are established in multi set settings.

Keywords and Phrases: Multi metric, iterative sequence, Cantor's intersection theorem, Banach's fixed point theorem.

2020 Mathematics Subject Classification: 54E35, 54E50.

1. Introduction

Multiset (bag) is a well established notion both in mathematics and in computer science ([8], [9], [22]). In mathematics, a multiset is considered to be the generalization of a set. In classical set theory, a set is a well-defined collection of distinct objects. If repeated occurrences of any object is allowed in a set, then a mathematical structure, that is known as multiset (mset, for short), is obtained ([21], [23], [24]). In various counting arguments it is convenient to distinguish between a set like $\{a, b, c\}$ and a collection like $\{a, a, a, b, c, c\}$. The latter, if viewed as a set, will be identical to the former. However, it has some of its elements purposely listed several times. We formalize it by defining a multiset as a collection of elements, each considered with certain multiplicity. For the sake of convenience a multiset is written as $\{k_1/x_1, k_2/x_2, ..., k_n/x_n\}$ in which the element x_i occurs k_i times. We observe that each multiplicity k_i is a positive integer.

From 1989 to 1991, Wayne D. Blizard made a through study of multiset theory, real valued multisets and negative membership of the elements of multisets ([1], [2], [3], [4]). K. P. Girish and S. J. John introduced and studied the concepts of multiset topologies, multiset relations, multiset functions, chains and antichains of partially ordered multisets ([12], [13], [14], [15], [16]). Concepts of multigroups and soft multigroups are found in the studies of Sk. Nazmul and S. K. Samanta ([18], [19]). Many other authors like Chakrabarty et al. ([5], [6], [7]), S. P. Jena et al. ([17]), J. L. Peterson ([20]) also studied various properties and applications of multisets.

Classical set theory states that a given element can appear only once in a set; it assumes that all mathematical objects occur without repetition. Thus there is only one number four, one field of complex numbers, etc. So, the only possible relation between two mathematical objects is either they are equal or they are different. However in the physical world it is observed that there is enormous repetition. For instance, there are many hydrogen atoms, many water molecules, many strands of DNA, etc. Coins of the same denomination and year, electrons or grains of sand appear similar, despite being obviously separate.

Functional analysis is an important branch of Mathematics and it has many applications in Mathematics and Sciences. Metric space is the beginning of functional analysis and it has several applications in many branch of functional analysis. An extension of metric spaces is done by using multi set and multi number instead of crisp real set and crisp real number in ([10]). Some topological properties of multi metric spaces are studied in ([11]). In the present paper, a notion of convergence in multi metric space is presented for the first time and complete multi metric space is studied. Multi set version of Cantor's intersection theorem and Banach's fixed point theorem are also established.

The organization of the paper is as follows:

In Section 2, some preliminary results on multi sets, multi real points, multi metric spaces and multi metric topologies are given. Section 3 comprises convergence in multi metric space, complete multi metric space and their properties. Cantor's intersection theorem and Banach's fixed point theorem are established in multi set settings. Section 4 concludes the paper.

2. Preliminaries

Definition 2.1. [12] *A* ***multi set*** *(or mset in short) M drawn from the set X is represented by a function CountM or C_M defined as $C_M : X \to \mathbb{N}$ where $\mathbb{N}$ represents the set of non negative integers.*

Here $C_M(x)$ is the number of occurrences of the element x in the mset M. We repre-

sent the mset M drawn from the set $X = \{x_1, x_2, ..., x_n\}$ as $M = \{m_1/x_1, m_2/x_2, ..., m_n/x_n\}$ where m_i is the number of occurrences of the element x_i in the mset M denoted by $x_i \in^{m_i} M, i = 1, 2, ..., n$. However those elements which are not included in the mset M have zero count.

Example 2.2. [12] Let $X = \{a, b, c, d, e\}$ be any set. Then $M = \{2/a, 4/b, 5/d, 1/e\}$ is an mset drawn from X. Clearly, a set is a special case of a mset.

Definition 2.3. [12] *Let M and N be two msets drawn from a set X. Then, the following are defined:*
(i) $M = N$ if $C_M(x) = C_N(x)$ for all $x \in X$.
(ii) $M \subset N$ if $C_M(x) \leq C_N(x)$ for all $x \in X$.
(iii) $P = M \cup N$ if $C_P(x) = Max\{C_M(x), C_N(x)\}$ for all $x \in X$.
(iv) $P = M \cap N$ if $C_P(x) = Min\{C_M(x), C_N(x)\}$ for al $x \in X$.
(v) $P = M \oplus N$ if $C_P(x) = C_M(x) + C_N(x)$ for all $x \in X$.
(vi) $P = M \ominus N$ if $C_P(x) = Max\{C_M(x) - C_N(x), 0\}$ for all $x \in X$, where $\oplus$ and $\ominus$ represents mset addition and mset subtraction respectively.

Let M be a mset drawn from a set X. The ***support set*** *of M, denoted by M^*, is a subset of X and $M^* = \{x \in X : C_M(x) > 0\}$, i.e., M^* is an ordinary set. M^* is also called root set.*

An mset M is said to be an ***empty mset*** *if for all $x \in X, C_M(x) = 0$. The cardinality of an mset M drawn from a set X is denoted by $Card(M)$ or $|M|$ and is given by $CardM = \sum_{x \in X} C_M(x)$.*

Definition 2.4. [12] *A* ***domain*** *X, is defined as a set of elements from which msets are constructed. The* ***mset space*** *$[X]^w$ is the set of all msets whose elements are in X such that no element in the mset occurs more than w times. The set $[X]^{+\infty}$ is the set of all msets over a domain X such that there is no limit on the number of occurrences of an element in an mset. If $X = \{x_1, x_2, ..., x_k\}$ then $[X]^w = \{\{m_1/x_1, m_2/x_2, ..., m_k/x_k\} : for\ i = 1, 2, ...k;\ m_i \in \{0, 1, 2, ...w\}\}$.*

Definition 2.5. [12] *Let X be a support set and $[X]^w$ be the mset space defined over X. Then for any mset $M \in [X]^w$, the* ***complement*** *M^c of M in $[X]^w$ is an element of $[X]^w$ such that $C^c_M(x) = w - C_M(x)$, for all $x \in X$.*

Definition 2.6. [12] *The* ***maximum*** *mset is defined as Z where*
$C_Z(x) = Max\{C_M(x) : x \in^k M, M \in [X]^m$ and $k \leq m\}$.
Thus $C_Z(x) = m\ \forall x \in X$.

Definition 2.7. [12] *Let $[X]^w$ be an mset space and $\{M_1, M_2, ...\}$ be a collection of msets drawn from $[X]^w$. Then the following operations are possible under an arbitrary collection of msets.*

(i) The ***union*** $\bigcup_{i\in I} M_i = \{C_{\cup M_i}(x)/x : C_{\cup M_i}(x) = max\{C_{M_i}(x) : x \in X\}$.
(ii) The ***intersection*** $\bigcap_{i\in I} M_i = \{C_{\cap M_i}(x)/x : C_{\cap M_i}(x) = min\{C_{M_i}(x) : x \in X\}$.
(iii) The ***mset addition*** $\bigoplus_{i\in I} M_i = \{C_{\bigoplus M_i}(x)/x : C_{\bigoplus M_i}(x) = min\{w, \sum_{i\in I}\{C_{M_i}(x) :\ x \in X\}\}$.
(iv) The ***mset complement*** $M^c = Z \ominus M = \{C_{M}c(x)/x : C_{M}c(x) = C_Z(x) - C_M(x), x \in X\}$.

Definition 2.8. [12] *The* ***power set*** *of an mset is denoted by* $P^*(M)$ *and it is an ordinary set whose members are sub msets of* M.

Definition 2.9. [12] *Let* $M \in [X]^w$ *and* $\tau \subseteq P^*(M)$*. Then* τ *is called a* ***multiset topology*** *of* M *if* τ *satisfies the following properties.*
(i) *The mset* M *and the empty mset* $\emptyset$ *are in* τ.
(ii) *The mset union of the elements of any sub collection of* τ *is in* τ.
(iii) *The mset intersection of the elements of any finite sub collection of* τ *is in* τ.

Mathematically a multiset topological space is an ordered pair (M, τ) *consisting of a mset* $M \in [X]^w$ *and a multiset topology* $\tau \subseteq P^*(M)$ *on* M*. Note that* τ *is an ordinary set whose elements are msets. Multiset topology is abbreviated as an M-topology.*

Definition 2.10. [10] ***Multi point:*** *Let* M *be a multi set over a universal set* X*. Then a multi point of* M *is defined by a mapping* $P_x^k : X \longrightarrow \mathbb{N}$ *such that* $P_x^k(x) = k$ *where* $k \leq C_M(x)$*.* x *and* k *will be referred to as the* ***base*** *and the* ***multiplicity*** *of the multi point* P_x^k *respectively.*

Collection of all multi points of an mset M *is denoted by* M_{pt}.

Definition 2.11. [10] *The* ***mset generated by a collection*** B *of multi points is denoted by* $MS(B)$ *and is defined by* $C_{MS(B)}(x) = Sup\{k : P_x^k \in B\}$.

A mset can be generated from the collection of its multi points. If M_{pt} denotes the collection of all multi points of M, then obviously
$C_M(x) = Sup\{k : P_x^k \in M_{pt}\}$ and hence $M = MS(M_{pt})$.

Definition 2.12. [10] (i) *The* ***elementary union*** *between two collections of multi points* C *and* D *is denoted by* $C \sqcup D$ *and is defined as*
$C \sqcup D = \{P_x^k : P_x^l \in C, P_x^m \in D \text{ and } k = max\{l, m\}\}$.

(ii) *The* ***elementary intersection*** *between two collections of multi points* C *and* D *is denoted by* $C \sqcap D$ *and is defined as*
$C \sqcap D = \{P_x^k : P_x^l \in C, P_x^m \in D \text{ and } k = min\{l, m\}\}$.

(iii) *For two collections of multi points* C *and* D*,* C *is said to be an* ***elementary subset*** *of* D*, denoted by* $C \sqsubset D$*, iff* $P_x^l \in C \Rightarrow \exists m \geq l$ *such that* $P_x^m \in D$.

The following results can be easily proved:

Theorem 2.13. [10] (i) *For two collections of multi points C and D, $C \subset D \Rightarrow C \sqsubset D$, but the converse is not true.*

(ii) *For two collections of multi points C and D, $C \cup D \supset C \sqcup D$ and the equality does not hold in general.*

(iii) *For two collections of multi points C and D, $C \cap D \subset C \sqcap D$ and the equality does not hold in general.*

(iv) *For an mset $M, MS(M_{pt}) = M$.*

(v) *For a collection B of multi points, $[MS(B)]_{pt} \supset B$.*

(vi) *For two msets F and G, $F \subset G \Leftrightarrow F_{pt} \subset G_{pt}$.*

(vii) *For two collections of multi points C and D, $C \subset D \Rightarrow MS(C) \subset MS(D)$.*

$(viii)$ *For two collections of multi points C and D, $C \sqsubset D \Leftrightarrow MS(C) \subset MS(D)$.*

(ix) *For two collections of multi points C and D, $MS(C \sqcap D) = MS(C) \cap MS(D)$.*

(x) *For an arbitrary collection $\{B_i : i \in \Delta\}$ of multi points, $MS(\sqcup_{i\in\Delta} B_i) = \cup_{i\in\Delta} MS(B_i)$.*

(xi) *For an arbitrary collection $\{B_i : i \in \Delta\}$ of multi points, $MS(\cup_{i\in\Delta} B_i) = \cup_{i\in\Delta} MS(B_i)$.*

Definition 2.14. [10] *Let $m\mathbb{R}^+$ denotes the multi set over $\mathbb{R}^+$ (set of non-negative real numbers) having multiplicity of each element equal to w, $w \in \mathbb{N}$. The members of $(m\mathbb{R}^+)_{pt}$ will be called **non-negative multi real points**.*

Definition 2.15. [10] *Let P_a^i and P_b^j be two multi real points of $m\mathbb{R}^+$. We define $P_a^i > P_b^j$ if $a > b$ or $P_a^i > P_b^j$ if $i > j$ when $a = b$.*

Definition 2.16. [10] ***(Addition of multi real points)*** *We define $P_a^i + P_b^j = P_{a+b}^k$ where $k = Max\{i, j\}, P_a^i, P_b^j \in (m\mathbb{R}^+)_{pt}$.*

Definition 2.17. [10] ***(Multiplication of multi real points)*** *We define multiplication of two multi real points in $m\mathbb{R}^+$ as follows:*

$P_a^i \times P_b^j = P_0^1$, *if either P_a^i or P_b^j equal to P_0^1;*
$\quad\quad\quad\quad = P_{ab}^k$, *otherwise; where $k =$Max $\{i, j\}$.*

Proposition 2.18. [10] ***(Properties of multiplication)*** *Multiplication of multi real points satisfies the following properties:*
(i) *Multiplication is commutative.*
(ii) *Multiplication is associative.*
(iii) *Multiplication is distributive over addition.*

Definition 2.19. [10] ***Multi Metric:***

Let $d : M_{pt} \times M_{pt} \longrightarrow (m\mathbb{R}^+)_{pt}$ *(*M *being a multi set over a Universal set* X *having multiplicity of any element atmost equal to* w*) be a mapping which satisfies the following:*

(M1) $d(P_x^l, P_y^m) \geq P_0^1, \forall P_x^l, P_y^m, \in M_{pt}$.

(M2) $d(P_x^l, P_y^m) = P_0^1$ *iff* $P_x^l = P_y^m, \forall P_x^l, P_y^m \in M_{pt}$.

(M3) $d(P_x^l, P_y^m) = d(P_y^m, P_x^l), \forall P_x^l, P_y^m \in M_{pt}$.

(M4) $d(P_x^l, P_y^m) + d(P_y^m, P_z^n) \geq d(P_x^l, P_z^n), \forall P_x^l, P_y^m, P_z^n \in M_{pt}$.

(M5) For $l \neq m$, $d(P_x^l, P_y^m) = P_0^k, \Leftrightarrow x = y$ *and* $k = Max\{l, m\}$.

Then d *is said to be a multi metric on* M *and* (M, d) *is called a multi metric (or a M-metric) space.*

Example 2.20. [10] Let M be a multi set over X having multiplicity of any element atmost equal to w. We define
$d : M_{pt} \times M_{pt} \longrightarrow (m\mathbb{R}^+)_{pt}$ such that for all P_x^l, P_y^m of M_{pt},
$d(P_x^l, P_y^m) = P_0^1$ if $P_x^l = P_y^m$ i.e., $x = y$ and $l = m$;
$= P_0^{Max\{l,m\}}$ if $x = y$ and $l \neq m$;
$= P_1^j$ if $x \neq y$. [$1 \leq j \leq w$ is some fixed positive integer]
Then d is a M-metric on M.

Theorem 2.21. [10] *If* $d(P_a^i, P_b^j) = P_r^l$ *and* $d(P_a^p, P_b^q) = P_s^m$*, then* $r = s$*,* $P_a^i, P_b^j, P_a^p, P_b^q$ *are elements of* M_{pt} *and* P_r^l, P_s^m *are elements of* $(m\mathbb{R}^+)_{pt}$.

Definition 2.22. [10] *Let* (M, d) *be an M-metric space and* L *be a non-null sub mset of* M*. Then the mapping* $d_L : L_{pt} \times L_{pt} \longrightarrow (m\mathbb{R}^+)_{pt}$ *given by* $d_L(P_x^a, P_y^b) = d(P_x^a, P_y^b), \forall P_x^a, P_y^b \in L_{pt}$ *is a M-metric on* L*. The metric is known as the* ***relative M-metric*** *induced by* d *on* L*. The M-metric space* (L, d_L) *is called an* ***M-metric subspace*** *or simply an* ***M-subspace*** *of the M-metric space* (M, d).

Definition 2.23. [10] *Let* (M, d) *be a M-metric space and* L *be a nonempty submset of* M*. Then the diameter of* L*, denoted by* $\delta(L)$ *is defined by:*

$\delta(L) = P_a^k$ *where* $a = Sup\{b : P_b^j = d(P_x^l, P_y^m), P_x^l, P_y^m \in L_{pt}\}$,
$k = 1$ *if* $a > b \; \forall P_b^j = d(P_x^l, P_y^m), P_x^l, P_y^m \in L_{pt}$ *and*
$= Max\{j : P_b^j = d(P_x^l, P_y^m), P_x^l, P_y^m \in L_{pt}\}$ *otherwise.*

If supremum does not exist finitely, we call L *a set of infinite diameter.*

Theorem 2.24. [10] *For a sub mset* L *of* M *in a M-metric space* (M, d)*,* $\delta(L) = P_0^1$ *iff* $L = \{1/a\}$ *ie.* L *consists of a single element of the universal set* X *with multiplicity* 1.

Theorem 2.25. [10] $P \subset Q \Rightarrow \delta(P) \leq \delta(Q)$.

Definition 2.26. [10] *Let* A *and* B *be two sub msets of* M *in a M-metric space*

*(M, d). Then the **distance between** A and B, denoted by $\delta(A, B)$, is defined by*

$\delta(A, B) = P_a^k$ *where* $a = Inf\{b : P_b^j = d(P_x^l, P_y^m), P_x^l \in A_{pt}, P_y^m \in B_{pt}\}$ *and*

$k = w$ *if* $a < b\ \forall P_b^j = d(P_x^l, P_y^m), P_x^l \in A_{pt}, P_y^m \in B_{pt}$;

$k = Min\{j : P_a^j = d(P_x^l, P_y^m), P_x^l \in A_{pt}, P_y^m \in B_{pt}\}$; *otherwise.*

Definition 2.27. [11] *Let (M, d) be a M-metric space, $r > 0$ and $P_a^k \in M_{pt}$. Then the **open ball** with centre P_a^k and radius P_r^1 $[r > 0]$ is denoted by $B(P_a^k, P_r^1)$ and is defined by $B(P_a^k, P_r^1) = \{P_x^l : d(P_x^l, P_a^k) < P_r^1\}$.*

*$MS[B(P_a^k, P_r^1)]$ will be called a **multi open ball** with centre P_a^k and radius $P_r^1 > P_0^1$.*

Definition 2.28. [11] *$B[P_a^k, P_r^1] = \{P_x^l : d(P_x^l, P_a^k) \leq P_r^1\}$ is called the **closed ball** with centre P_a^k and radius P_r^1 $[r > 0]$. $MS[B[P_a^k, P_r^1]]$ will be called a **multi closed ball** with centre P_a^k and radius P_r^1 $[r > 0]$.*

Theorem 2.29. [11] **(Hausdorff Property)**
Let (M, d) be a M-metric space and $P_a^k, P_b^l \in M_{pt}$ such that $a \neq b$. Then $\exists\, r > 0$ such that $MS[B(P_a^k, P_r^1) \cap B(P_b^l, P_r^1)] = \emptyset$ which is equivalent to $B(P_a^k, P_r^1) \cap B(P_b^l, P_r^1) = \phi$.

Definition 2.30. [11] *Let (M, d) be a M-metric space and $P_a^k \in M_{pt}$. A collection $N(P_a^k)$ of multi points of M is said to be a **nbd** of the multi point P_a^k if $\exists r > 0$ such that $P_a^k \in B(P_a^k, P_r^1) \subset N(P_a^k)$. MS[$N(P_a^k)$] will be called a **multi nbd** of the multi point P_a^k.*

Theorem 2.31. [11] *Let N_1 and N_2 are two nbds of a multi point P_a^i in a M-metric space (M, d). Then $N_1 \cap N_2$ is a nbd of P_a^i and hence $MS(N_1 \cap N_2)$ is a multi nbd of P_a^i.*

Definition 2.32. [11] *Let B be a collection of multi points of M in a M-metric space (M, d). Then a multi point P_a^k is said to be an **interior point** of B if $\exists$ an open ball $B(P_a^k, P_r^1)$ with centre at P_a^k and $r > 0$ such that $B(P_a^k, P_r^1) \subset B$.*

Definition 2.33. [11] *Let N be a sub multiset of a M-metric space (M, d). Then a multi point P_a^k is said to be an **interior point of** N if it is an interior point of N_{pt}, ie. $\exists$ an open ball $B(P_a^k, P_r^1)$ with centre at P_a^k, and $r > 0$ such that $B(P_a^k, P_r^1) \subset N_{pt}$.*

Definition 2.34. [11] *Let N be a sub mset of a M-metric space (M, d). Then the **interior** of N is defined to be the set consisting of all interior points of N.*

The interior of the multi set N is denoted by N^o or Int(N).

*MS[Int(N)] is said to be the **multi interior** of N denoted by Mint(N).*

Theorem 2.35. [11] *Let A and B be two non-null sub msets of a M-metric space (M, d). Then*
(i) $Mint(A) \subset A$.
(ii) $A \subset B \Rightarrow Int(A) \subset Int(B)$ and hence $Mint(A) \subset Mint(B)$.
(iii) $Int(A) \cap Int(B) = Int(A \cap B)$.
(iv) (a)$Int(A \cap B) \subset Int(A) \sqcap Int(B)$ (b)$Int(A \cap B) \sqsubset Int(A) \sqcap Int(B)$ (c) $Int(A \cap B) \sqsubset Int(A) \cap Int(B)$.
(v) $Mint(A \cap B) \subset Mint(A) \cap Mint(B)$.
(vi) $Int(A \cup B) \supset Int(A) \cup Int(B)$.

Definition 2.36. [11] *Let (M, d) be a M-metric space. Then a collection B of multi points of M is said to be* ***open*** *if every multi point of B is an interior point of B i.e., for each $P_a^k \in B$, $\exists$ an open ball $B(P_a^k, P_r^1)$ with centre at P_a^k, and $r > 0$ such that $B(P_a^k, P_r^1) \subset B$.*

ϕ is separately considered as an open set.

Definition 2.37. [11] *Let (M, d) be a M-metric space. Then $N \subset M$ is said to be* ***multi open*** *in (M, d) iff $\exists$ a collection B of multi points of N such that B is open and $MS(B) = N$.*

The null multiset Φ separately considered as multi open in (M, d).

Proposition 2.38. [11] *In a M-metric space every open ball is open.*

Theorem 2.39. [11] *In a M-metric space (M, d),*

(i) Union of arbitrary number of open sets of multi points is open.

(ii) Elementary intersection of two open sets of multi points is open.

(iii) Intersection of two open sets of multi points is open.

Theorem 2.40. [11] *In a M-metric space (M, d),*
(i) The null sub mset $\emptyset$ is multi open.
(ii) M is multi open.
(iii) Arbitrary union of multi open sets is multi open.
(iv) Intersection of two multi open sets is multi open.

Example 2.41. [11] Arbitrary intersection of multi open sets may not be multi open.
For example consider $\mathbb{R}$ to be a multi set with multiplicity of each element 1.
Define $d : \mathbb{R}_{pt} \times \mathbb{R}_{pt} \longrightarrow (m\mathbb{R}^+)_{pt}$ by $d(P_x^1, P_y^1) = P_{|x-y|}^1$, $\forall P_x^1, P_y^1 \in \mathbb{R}_{pt}$.
Consider the collection $\{P_n : n \in \mathbb{N}\}$ of multi sets such that
$P_n = \{1/x : -\frac{1}{n} < x < \frac{1}{n}\}$. Then $P_n, n \in \mathbb{N}$ are multi open sets as $(P_n)_{pt} = \{P_x^1 : -\frac{1}{n} < x < \frac{1}{n}\}, n \in \mathbb{N}$ are open sets of multi points in $(\mathbb{R}, d)$ and $P_n = \mathrm{MS}((P_n)_{pt})$.
But, $\bigcap_{n\in\mathbb{N}} P_n = \{1/0\}$ which is not multi open in $(\mathbb{R}, d)$.

Definition 2.42. [11] *A multi set N in a M-metric space (M,d) is said to be* ***multi closed*** *if its complement N^c is multi open in (M,d).*

Proposition 2.43. [11] *Let $\{N_i : i \in \triangle\}$ be an arbitrary collection of multisets in (M,d). Then $\bigcup_{i\in\triangle}(N_i)^c = (\bigcap_{i\in\triangle} N_i)^c$ and $\bigcap_{i\in\triangle}(N_i)^c = (\bigcup_{i\in\triangle} N_i)^c$.*

Theorem 2.44. [11] *In a M-metric space,*
(i) The null multi set $\emptyset$ is multi closed.
(ii) The absolute multiset M is multi closed.
(iii) Arbitrary intersection of multi closed sets is multi closed.
(iv) Finite union of multi closed sets is multi closed.

Definition 2.45. [11] *Let (M,d) be a M-metric space and B be a collection of multi points of M. Then a multi point P^l_x of M is said to be a* ***limit point*** *of B if every open ball $B(P^l_x, P^1_r)$ $(r > 0)$ containing P^l_x in (M,d) contains at least one point of B other than P^l_x.*

The set of all limit points of B is said to be the ***derived set*** *of B and is denoted by B^d.*

Definition 2.46. [11] *Let (M,d) be a M-metric space and $N \subset M$. Then $P^l_x \in M_{pt}$ is said to be a* ***multi limit point of*** *N if it is a limit point of N_{pt} ie. if every open ball $B(P^l_x, P^1_r)$ $(r > 0)$ containing P^l_x in (M,d) contains at least one point of N_{pt} other than P^l_x.*

A multi limit point of a multi set N may or may not belong to the set N. The multiset generated by the multi limit points of N is called the **multi derived set** of N and is denoted by N^d. Thus $N^d = MS[(N_{pt})^d]$.

Theorem 2.47. [11] *Let A and B be collections of multi points in (M,d). Then (i) $A^d \cup B^d = (A \cup B)^d$ $\quad (ii)$ $(A^d)^d \nsubseteq A^d$ in general.*

Theorem 2.48. [11] *For two sub multi sets P and Q of M, $(P \cup Q)^d = P^d \cup Q^d$.*

Definition 2.49. [11] *Let (M,d) be a M-metric space and $B \subset M_{pt}$. Then the collection of all points of B together with all limit points of B is said to be the* ***closure*** *of B in (M,d) and is denoted by $\overline{B}$. Thus $\overline{B} = B \cup B^d$.*

Theorem 2.50. [11] *If $B \subset M_{pt}$ in (M,d), then $\overline{\overline{B}} = \overline{B}$.*

Definition 2.51. [11] *Let (M,d) be a M-metric space and $N \subset M$. Then the multi set generated by all multi points and all multi limit points of N is said to be the* ***multi closure*** *of N and is denoted by $\overline{N}$.*

Thus the multi set generated by all the multi points of $\overline{N_{pt}}$ is the multi closure of N and we have $\overline{N} = MS[\overline{N_{pt}}] = MS[N_{pt} \cup (N_{pt})^d] = MS[N_{pt}] \cup MS[(N_{pt})^d] =$

$N \cup MS[(N_{pt})^d] = N \cup N^d$.

Theorem 2.52. [11] *Let (M, d) be a M-metric space and $P \subset M$. Then $\overline{P_{pt}} = (\overline{P})_{pt}$.*

Theorem 2.53. [11] *Let (M, d) be a M-metric space and $P, Q \subset M$. Then (i) $\overline{\emptyset} = \emptyset$ and $\overline{M} = M$ (ii) $P \subset \overline{P}$ (iii) $\overline{P} = \overline{\overline{P}}$ (iv) $P \subset Q \Rightarrow \overline{P} \subset \overline{Q}$ (v) $\overline{P} \cup \overline{Q} = \overline{P \cup Q}$ (vi) $\overline{P \cap Q} \subset \overline{P} \cap \overline{Q}$ (vii) $P^l_x \in M_{pt}$ and $\delta(P^l_x, Q) = P^1_0 \Rightarrow P^l_x \in \overline{Q}_{pt}$, but the converse is not true in general.*

3. Completeness of M-metric Spaces

Definition 3.1. *A sequence $\{P^{l_n}_{x_n}\}$ of multi points in $m\mathbb{R}^+$ is said to **converge** to P^1_0 if for any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $P^{l_n}_{x_n} < P^1_\epsilon$ for all $n \geq n_0$.*

Since $l_n \geq 1, \forall n \in \mathbb{N}; P^{l_n}_{x_n} < P^1_\epsilon \iff x_n < \epsilon$.

$\therefore P^{l_n}_{x_n} \to P^1_0 \iff x_n \to 0$ as $n \to +\infty$ in $\mathbb{R}^+$.

Definition 3.2. *Let $\{P^{l_n}_{x_n}\}$ be a sequence of multi points in a M-metric space (M, d). The sequence $\{P^{l_n}_{x_n}\}$ is said to **converge** in (M, d) if there exists P^l_x belongs to M_{pt} such that $d(P^{l_n}_{x_n}, P^l_x) \to P^1_0$ as $n \to +\infty$.* This means that for any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d(P^{l_n}_{x_n}, P^l_x) < P^1_\epsilon \ \forall \ n \geq n_0$.

We denote this by $P^{l_n}_{x_n} \to P^l_x$ as $n \to +\infty$ or by $\lim_{n \to +\infty} P^{l_n}_{x_n} = P^l_x$. P^l_x is said to be the multi limit of the sequence $\{P^{l_n}_{x_n}\}$ as $n \to +\infty$.

Note 3.3. *All convergent sequences of multi points having same bases will converge to multi points having same base ie. if for a sequence $\{P^{l_n}_{x_n}\}$ of multi points $P^{l_n}_{x_n} \to P^l_x$, then $P^{k_n}_{x_n} \to P^k_x$ for any sequence $\{k_n\}$ of natural numbers with $1 \leq k_n \leq C_M(x_n)$ and for any natural number k with $1 \leq k \leq C_M(x)$.*

To prove this, let $\epsilon > 0$ be arbitrary.
Then as $P^{l_n}_{x_n} \to P^l_x$, there exists $m_0 \in \mathbb{N}$ such that, for all $n \geq m_0, d(P^{l_n}_{x_n}, P^l_x) < P^1_{\frac{\epsilon}{2}}$
$\Rightarrow$ For any sequence $\{k_n\}$ of natural numbers with $1 \leq k_n \leq C_M(x_n)$, for any natural number k with $1 \leq k \leq C_M(x)$ and for $n \geq m_0$,
$d(P^{k_n}_{x_n}, P^k_x) \leq d(P^{k_n}_{x_n}, P^{l_n}_{x_n}) + d(P^{l_n}_{x_n}, P^l_x) + d(P^l_x, P^k_x) < P^{u_n}_0 + P^1_{\frac{\epsilon}{2}} + P^u_0$ *[where $u_n = Max\{k_n, l_n\}$ and $u = Max\{k, l\}$]* $= P^{Max\{u_n,u\}}_{\frac{\epsilon}{2}} < P^1_\epsilon \Rightarrow P^{k_n}_{x_n} \to P^k_x$.

Theorem 3.4. Uniqueness of multi limit :
A convergent sequence of multi points converge to multi limit having the same base.

Proof. Let $\{P^{l_n}_{x_n}\}$ be a sequence of multi points in a M-metric space (M, d), $\lim P^{l_n}_{x_n} = P^l_x$ and $\lim P^{l_n}_{x_n} = P^m_y$ where $x \neq y \Rightarrow d(P^l_x, P^m_y) = P^i_a$ where $a > 0$.
Let $0 < \epsilon < \frac{a}{2}$. Since $\lim P^{l_n}_{x_n} = P^l_x$ and $\lim P^{l_n}_{x_n} = P^m_y$, there exist $n_1, n_2 \in \mathbb{N}$ such

that $d(P^{l_n}_{x_n}, P^l_x) < P^1_\epsilon$ for all $n \geq n_1$ and $d(P^{l_n}_{x_n}, P^m_y) < P^1_\epsilon$ for all $n \geq n_2$.
Let $n_0 = n_1 \vee n_2$. Then for all $n \geq n_0$, $d(P^{l_n}_{x_n}, P^l_x) < P^1_\epsilon$ and $d(P^{l_n}_{x_n}, P^m_y) < P^1_\epsilon$.
$\therefore$ for all $n \geq n_0$, $d(P^l_x, P^m_y) \leq d(P^{l_n}_{x_n}, P^l_x) + d(P^{l_n}_{x_n}, P^m_y) < P^1_\epsilon + P^1_\epsilon$
$= P^1_{2\epsilon} < P^i_a$ $[\because 2\epsilon < a]$, which is not possible. $\therefore x = y$.

Theorem 3.5. *Let (M, d) be a M-metric space and $P \subset M$. Then a multi point P^l_x is a multi limit point of P iff $\exists$ a sequence $\{P^{l_n}_{x_n}\}$ of multi points of P other than P^l_x, converging to P^l_x.*

Proof. Let P^l_x be a multi point of P. Then for each $n \in \mathbb{N}$, $B(P^l_x, P^1_{\frac{1}{n}}) \cap [P_{pt} \backslash \{P^l_x\}] \neq \phi$.
Let for all $n \in \mathbb{N}$, $P^{l_n}_{x_n} \in B(P^l_x, P^1_{\frac{1}{n}}) \cap [P_{pt} \backslash \{P^l_x\}]$. Then $\{P^{l_n}_{x_n}\}$ is a sequence of multi points of P other than P^l_x such that $d(P^{l_n}_{x_n}, P^l_x) < P^1_{\frac{1}{n}}$, for all $n \in \mathbb{N}$. Since $\frac{1}{n} \to 0$ as $n \to +\infty$, for any $\epsilon > 0$, $\exists$ $n_0 \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$ for all $n \geq n_0$ $\Rightarrow P^1_{\frac{1}{n}} < P^1_\epsilon$ $\forall$ $n \geq n_0$ $\Rightarrow d(P^{l_n}_{x_n}, P^l_x) < P^1_\epsilon$ $\forall n \geq n_0$ $\Rightarrow \lim P^{l_n}_{x_n} = P^l_x$.
Conversely let $\{P^{l_n}_{x_n}\}$ be a sequence of multi points in P other than P^l_x such that $\lim P^{l_n}_{x_n} = P^l_x$ $\Rightarrow$ for any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $P^{l_n}_{x_n} \in B(P^l_x, P^1_\epsilon)$ for all $n \geq n_0$. Also $P^{l_n}_{x_n} \in P_{pt} \backslash \{P^l_x\}$ for all $n \geq n_0$. $\therefore P^{l_n}_{x_n} \in B(P^l_x, P^1_\epsilon) \cap [P_{pt} \backslash \{P^l_x\}]$ for all $n \geq n_0$. $\therefore$ For any $\epsilon > 0, B(P^l_x, P^1_\epsilon) \cap [P_{pt} \backslash \{P^l_x\}] \neq \phi$. $\therefore P^l_x$ is a multi limit point of P.

Theorem 3.6. *Let (M, d) be a M-metric space and $P \subset M$. Then a multi point $P^l_x \in \overline{P_{pt}}$ iff $\exists$ a sequence $\{P^{l_n}_{x_n}\}$ of multi points of P , converging to P^l_x.*

Proof. Let $P^l_x \in \overline{P_{pt}} = P_{pt} \cup (P_{pt})^d$.
If $P^l_x \in P_{pt}$, the sequence $\{P^{l_n}_{x_n}\}$ where $P^{l_n}_{x_n} = P^l_x$ for all $n \in \mathbb{N}$ will serve the purpose.
If $P^l_x \in (P_{pt})^d$, there exists a sequence $\{P^{l_n}_{x_n}\}$ in P_{pt} other than P^l_x, converging to P^l_x.

Conversely let $\exists$ a sequence $\{P^{l_n}_{x_n}\}$ of multi points of P, converging to P^l_x. If $P^l_x \notin P_{pt}$, $\{P^{l_n}_{x_n}\}$ is a sequence other than P^l_x in P_{pt} converging to P^l_x and hence $P^l_x \in (P_{pt})^d$.

Note 3.7. *Since $P^{l_n}_{x_n} \to P^l_x$ for some $1 \leq l \leq C_M(x)$ $\Rightarrow P^{l_n}_{x_n} \to P^k_x$ $\forall$ $1 \leq k \leq C_M(x)$, $\therefore P^l_x \in \overline{P_{pt}}$ for some $1 \leq l \leq C_M(x) \Rightarrow P^k_x \in \overline{P_{pt}}$ $\forall$ $1 \leq k \leq C_M(x)$.*

Definition 3.8. *A sequence $\{P^{i_n}_{x_n}\}$ of multi points in a M-metric space (M, d) is said to be* ***bounded*** *if the set $\{d(P^{i_n}_{x_n}, P^{i_m}_{x_m}) : m, n \in \mathbb{N}\}$ is bounded ie. $\exists$ a nonnegative multi real point P^i_a such that $d(P^{i_n}_{x_n}, P^{i_m}_{x_m}) \leq P^1_a, \forall m, n \in \mathbb{N}$.*

Definition 3.9. *A sequence $\{P^{i_n}_{x_n}\}$ of multi points in a M-metric space (M, d) is said to be a* ***multi Cauchy sequence*** *if for any $\epsilon > 0$, $\exists$ $n_0 \in \mathbb{N}$ such that $d(P^{i_n}_{x_n}, P^{i_m}_{x_m}) < P^1_\epsilon$ $\forall$ $m, n \geq n_0$ ie. $d(P^{i_n}_{x_n}, P^{i_m}_{x_m}) \to P^1_0$ as $m, n \to +\infty$.*

Theorem 3.10. *Every convergent sequence in a M-metric space is Cauchy and every Cauchy sequence is bounded. A Cauchy sequence is convergent iff it has a convergent subsequence.*

Proof. Let$\{P_{x_n}^{l_n}\}$ be a convergent sequence in a M-metric space (M,d) converging to $P_x^l \in M_{pt}$. Then for any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d(P_{x_n}^{l_n}, P_x^l) < P_{\frac{\epsilon}{2}}^1$ for all $n \geq n_0$

Now we have $d(P_{x_n}^{l_n}, P_{x_m}^{l_m}) \leq d(P_{x_n}^{l_n}, P_x^l) + d(P_x^l, P_{x_m}^{l_m}) < P_{\frac{\epsilon}{2}}^1 + P_{\frac{\epsilon}{2}}^1 = P_\epsilon^1$for all $m, n \geq n_0$

$\Rightarrow \{P_{x_n}^{l_n}\}$ is a Cauchy sequence.

Next let $\{P_{x_n}^{l_n}\}$ be a Cauchy sequence in (M,d).

Then for any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d(P_{x_n}^{l_n}, P_{x_m}^{l_m}) < P_\epsilon^1$, for all $m, n \geq n_0$.

Since $\{d(P_{x_n}^{l_n}, P_{x_m}^{l_m}) : m, n \leq n_0\}$ is a finite collection of nonnegative multi real points, we must have $\text{Max}\{d(P_{x_n}^{l_n}, P_{x_m}^{l_m}) : m, n \leq n_0\} = P_r^l \in (m\mathbb{R}^+)_{pt}$

If $m < n_0$ and $n \geq n_0$, $d(P_{x_n}^{l_n}, P_{x_m}^{l_m}) \leq d(P_{x_n}^{l_n}, P_{x_{n_0}}^{l_{n_0}}) + d(P_{x_{n_0}}^{l_{n_0}}, P_{x_m}^{l_m}) < P_r^l + P_\epsilon^1 = P_{r+\epsilon}^l$.

Thus $d(P_{x_n}^{l_n}, P_{x_m}^{l_m}) \leq P_{r+\epsilon}^l$, $\forall m, n \in \mathbb{N}$ and consequently $\{P_{x_n}^{l_n}\}$ is bounded.

Next, if a Cauchy sequence is convergent, each of its subsequence is convergent.

Conversely let us assume that $\{P_{x_n}^{l_n}\}$ is a Cauchy sequence having a convergent subsequence $\{P_{x_{k_n}}^{l_{k_n}}\}$ converging to $P_x^l \in M_{pt}$. Then for any $\epsilon > 0$ we can find $n_0 \in \mathbb{N}$ such that $d(P_{x_{k_n}}^{l_{k_n}}, P_x^l) < P_{\frac{\epsilon}{2}}^1$, $\forall n \geq n_0$ and

$d(P_{x_n}^{l_n}, P_{x_m}^{l_m}) < P_{\frac{\epsilon}{2}}^1$, $\forall\ m, n \geq n_0$

Now $n \geq n_0 \Rightarrow k_n \geq n_0 \Rightarrow\ d(P_{x_n}^{l_n}, P_{x_{k_n}}^{l_{k_n}}) < P_{\frac{\epsilon}{2}}^1$.

$\therefore\ \forall\ n \geq n_0,\ d(P_{x_n}^{l_n}, P_x^l) \leq d(P_{x_n}^{l_n}, P_{x_{k_n}}^{l_{k_n}}) + d(P_{x_{k_n}}^{l_{k_n}}, P_x^l) < P_{\frac{\epsilon}{2}}^1 + P_{\frac{\epsilon}{2}}^1 = P_\epsilon^1$.

$\Rightarrow \{P_{x_n}^{l_n}\}$ is convergent and converges to P_x^l.

Definition 3.11. (Complete M-metric space)

A M-metric space (M,d) is said to be complete if every multi Cauchy sequence in (M,d) converges to a multi point of M. A M-metric space is said to be incomplete if it is not complete.

Example 3.12. Consider the M-metric space $(\mathbb{R}, d)$ as in the example [2.41]. Let $P = \{1/x : 0 < x \leq 1\}$ and for any $n \in \mathbb{N}, P_{a_n}^1 = P_{\frac{1}{n}}^1$. Then $\{P_{a_n}^1\}$ is a sequence of multi points in P.

For any $\epsilon > 0$, we choose $m \in \mathbb{N}$ such that $m > \frac{1}{\epsilon}$.

Then for $i \geq j \geq m$, $|\frac{1}{i} - \frac{1}{j}| = \frac{|i-j|}{ij} \leq \frac{i}{ij} = \frac{1}{j} \leq \frac{1}{m} < \epsilon$

$\Rightarrow d(P_{a_i}^1, P_{a_j}^1) = d(P_{\frac{1}{i}}^1, P_{\frac{1}{j}}^1) = P_{|\frac{1}{i} - \frac{1}{j}|}^1 < P_\epsilon^1\ \forall\ i \geq j \geq m.$

$\Rightarrow \{P_{a_n}^1\}$ is a Cauchy sequence in P.

Also the sequence converges to P_0^1 which is not a multi point of P. So the sequence cannot converge in (P, d_P).
$\therefore (P, d_P)$ is not a complete multi metric space.

Theorem 3.13. *In a M-metric space (M, d), for any sub mset P of M, if $\delta(P) = P_a^i$, then $\delta(\overline{P}) = P_a^j$ where $i \leq j$.*
Proof. Let $\delta(\overline{P}) = P_b^j$.
Since $P \subset \overline{P}$, $\delta(P) \leq \delta(\overline{P})$
$\Rightarrow P_a^i \leq P_b^j$
$\Rightarrow a \leq b$ ———(1)
Let $P_c^k, P_d^l \in (\overline{P})_{pt} = \overline{P_{pt}}$ and $d(P_c^k, P_d^l) = P_e^m$.
Then for any $\epsilon > 0$, $B(P_c^k, P_{\frac{\epsilon}{2}}^1) \cap P_{pt} \neq \phi$ and $B(P_d^l, P_{\frac{\epsilon}{2}}^1) \cap P_{pt} \neq \phi$.
Let $P_f^n \in B(P_c^k, P_{\frac{\epsilon}{2}}^1) \cap P_{pt}$ and $P_g^p \in B(P_d^l, P_{\frac{\epsilon}{2}}^1) \cap P_{pt}$
$\Rightarrow P_f^n, P_g^p \in P_{pt}$, $d(P_c^k, P_f^n) < P_{\frac{\epsilon}{2}}^1$ and $d(P_d^l, P_g^p) < P_{\frac{\epsilon}{2}}^1$
$\Rightarrow P_e^m = d(P_c^k, P_d^l) \leq d(P_c^k, P_f^n) + d(P_f^n, P_g^p) + d(P_d^l, P_g^p)$
$< P_{\frac{\epsilon}{2}}^1 + \delta(P) + P_{\frac{\epsilon}{2}}^1$ $[\because P_f^n, P_g^p \in P_{pt}, d(P_f^n, P_g^p) \leq \delta(P)]$
$\Rightarrow P_e^m < P_\epsilon^1 + P_a^i = P_{\epsilon+a}^i \Rightarrow e \leq \epsilon + a$.
Since this is true for any e such that $P_e^m = d(P_c^k, P_d^l)$, $P_c^k, P_d^l \in (\overline{P})_{pt}$,
so $\mathrm{Sup}\{e : P_e^m = d(P_c^k, P_d^l), P_c^k, P_d^l \in (\overline{P})_{pt}\} \leq \epsilon + a$
$\Rightarrow b \leq \epsilon + a$ $[\because \delta(\overline{P}) = P_b^j]$.
Since this is true for any $\epsilon > 0$, $b \leq a$——— (2)
Thus from (1)& (2), $b = a$.
$\therefore \delta(P) = P_a^i \Rightarrow \delta(\overline{P}) = P_a^j$ where $i \leq j$ as $P \subset \overline{P}$.

Definition 3.14. *If $\{P_n\}$ is a sequence of sub msets of M in (M, d) and $\delta(P_n) \to P_0^1$ as $n \to +\infty \Rightarrow \delta(\overline{P_n}) \to P_0^1$ as $n \to +\infty$ in $m\mathbb{R}^+$.*
Proof. Let $\delta(P_n) = P_{a_n}^{i_n}$for all $n \in \mathbb{N}$.
Then $\delta(\overline{P_n}) = P_{a_n}^{j_n}$ where $i_n \leq j_n$ for all $n \in \mathbb{N}$.
Now $\delta(P_n) = P_{a_n}^{i_n} \longrightarrow P_0^1$ as $n \to +\infty$ in $m\mathbb{R}^+$
$\Rightarrow$ For any $\epsilon > 0$, $\exists\ n_0 \in \mathbb{N}$ such that $P_{a_n}^{i_n} < P_\epsilon^1$ for all $n \geq n_0$ $[\because i_n \geq 1$ for all $n \in \mathbb{N}]$
$\Rightarrow P_{a_n}^{j_n} < P_\epsilon^1$ for all $n \geq n_0$
$\Rightarrow \delta(\overline{P_n}) < P_\epsilon^1$ for all $n \geq n_0$
$\Rightarrow \delta(\overline{P_n}) \to P_0^1$ as $n \to +\infty$.

Proposition 3.15. *If $\{P_i : i \in \Delta\}$ be an arbitrary collection of sub msets in a M-metric space (M, d), then $\bigcap_{i\in\Delta} (P_i)_{pt} = (\bigcap_{i\in\Delta} P_i)_{pt}$.*
Proof. We have $\bigcap_{i\in\Delta} P_i \subset P_i\ \forall\ i \in \Delta$
$\Rightarrow (\bigcap_{i\in\Delta} P_i)_{pt} \subset (P_i)_{pt}\ \forall\ i \in \Delta$

$\Rightarrow (\bigcap_{i\in\Delta} P_i)_{pt} \subset \bigcap_{i\in\Delta}(P_i)_{pt}$.

Next let $P_a^k \in \bigcap_{i\in\Delta}(P_i)_{pt}$
$\Rightarrow P_a^k \in (P_i)_{pt}$ for all $i \in \Delta$.
$\Rightarrow C_{P_i}(a) \geq k$ for all $i \in \Delta \Rightarrow \bigwedge_{i\in\Delta} C_{P_i}(a) \geq k$
$\Rightarrow C_{\bigcap_{i\in\Delta} P_i}(a) \geq k$
$\Rightarrow P_a^k \in (\bigcap_{i\in\Delta} P_i)_{pt}$
$\Rightarrow \bigcap_{i\in\Delta}(P_i)_{pt} \subset (\bigcup_{i\in\Delta} P_i)_{pt}$.

Theorem 3.16. (Cantor's intersection theorem in M-metric space)
A M-metric space (M, d) is complete iff for any sequence $\{P_n\}$ of non null sub mset of M with $P_n = \overline{P_n}$ for all $n \in \mathbb{N}$ and $P_1 \supset P_2 \supset$ .... $\supset P_n \supset$ ... such that $\delta(P_n) \to P_0^1$ as $n \to +\infty$, the intersection $P = \cap_{n\in\mathbb{N}} P_n$ consists of multi points having same base.

Proof. Let (M, d) be a complete M-metric space and $\{P_n\}$ be a sequence of non null sub mset of M with $P_n = \overline{P_n}$ for all $n \in \mathbb{N}$ and $P_1 \supset P_2 \supset$ .... $\supset P_n \supset$ ... such that $\delta(P_n) \to P_0^1$ as $n \to +\infty$.
Since $\forall\ n \in \mathbb{N}, P_n \neq \emptyset,\ \exists$ a multi point $P_{a_n}^{k_n} \in (P_n)_{pt},\ \forall\ n \in \mathbb{N}$.
To show that $\{P_{a_n}^{k_n}\}$ is Cauchy, we have for $m, n \in \mathbb{N}, m \geq n$,
$P_{a_n}^{k_n}, P_{a_m}^{k_m} \in (P_n)_{pt},\ [P_m \subset P_n\ \forall\ m \geq n\ \Rightarrow (P_m)_{pt} \subset (P_n)_{pt},\ \forall\ m \geq n]$
and hence $d(P_{a_n}^{k_n}, P_{a_m}^{k_m}) \leq \delta(P_n)\ \to P_0^1$ as $n \to +\infty$ [and hence $m \to +\infty$]
$\therefore d(P_{a_n}^{k_n}, P_{a_m}^{k_m}) \to P_0^1$ as $m, n \to +\infty$
$\therefore \{P_{a_n}^{k_n}\}$ is Cauchy in (M, d)
Since (M, d) is complete, by hypothesis $\exists\ P_a^k \in M_{pt}$ such that $\lim P_{a_n}^{k_n} = P_a^k$.

To show that P_a^k is a multi point of $\cap_{n\in\mathbb{N}} P_n$ we have for any $n \in \mathbb{N}$,
$\{P_{a_m}^{k_m}\}_{m\geq n} \in (P_n)_{pt}$ and hence $lim_m P_{a_m}^{k_m} = P_a^k \in (\overline{P_n})_{pt} = (P_n)_{pt}$
$\Rightarrow P_a^k \in \bigcap_{n\in\mathbb{N}} (P_n)_{pt} = (\bigcap_{n\in\mathbb{N}} P_n)_{pt}$.
$\therefore \bigcap_{n\in\mathbb{N}} P_n$ consists of at least one multi point.

Also since $P_{a_n}^{k_n} \to P_a^k \ \Rightarrow\ P_{a_n}^{k_n} \to P_a^l\ \forall\ 1 \leq l \leq C_M(a)$, it follows that $P_a^l \in (\bigcap_{n\in\mathbb{N}} P_n)_{pt}\ \forall\ 1 \leq l \leq C_M(a)$.

To prove the uniqueness of the base a, let $P_b^j \in (\bigcap_{n\in\mathbb{N}} P_n)_{pt} = \bigcap_{n\in\mathbb{N}} (P_n)_{pt}$
$\Rightarrow P_b^j \in (P_n)_{pt}\ \forall\ n \in \mathbb{N}$. Also since $P_a^k \in (P_n)_{pt}\ \forall\ n \in \mathbb{N}$, it follows that $\forall\ n \in \mathbb{N},\ d(P_a^k, P_b^j) \leq \delta(P_n) \to P_0^1$ as $n \to +\infty$.
$\Rightarrow d(P_a^k, P_b^j) < P_\epsilon^1$ for any $\epsilon > 0$ which gives $a = b$.

Conversely, let the given condition be true. Let $\{P_{a_n}^{k_n}\}$ be a Cauchy sequence of multi points in (M, d) and for each $n \in \mathbb{N}$,

$P_n =$MS$\{P_{a_n}^{k_n}, P_{a_{n+1}}^{k_{n+1}}, P_{a_{n+2}}^{k_{n+2}}, ...\}$.

Then $P_1 \supset P_2 \supset$ .... $\supset P_n \supset$ ... and hence $\overline{P_1} \supset \overline{P_2} \supset$ .... $\supset \overline{P_n} \supset$ ...

Thus $\{\overline{P_n}\}$ is a sequence of non null sub msets of M such that $\overline{P_1} \supset \overline{P_2} \supset .... \supset \overline{P_n} \supset ...$ and $\overline{\overline{P_n}} = \overline{P_n}$, $\forall\ n \in \mathbb{N}$.
Also $\delta(P_n) \to P_0^1$ as $n \to +\infty$ as $\{P_{a_n}^{k_n}\}$ is a Cauchy sequence.
$\therefore \delta(\overline{P_n}) \to P_0^1$ as $n \to +\infty$.
$\therefore$ by the given condition $\bigcap_{n\in\mathbb{N}} \overline{P_n}$ consists of multi points having the same base.
Let P_a^k be one of them.
Then $d(P_{a_n}^{k_n}, P_a^k) \le \delta(\overline{P_n}) \to P_0^1$ as $n \to +\infty$.
$\Rightarrow d(P_{a_n}^{k_n}, P_a^k) \to P_0^1$ as $n \to +\infty$
$\lim P_{a_n}^{k_n} = P_a^k$.
Hence $\{P_{a_n}^{k_n}\}$ converges to P_a^k proving that (M, d) is complete.

Definition 3.17. (Contraction mapping)
Let (M, d) be a M-metric space. Then a mapping $T : M_{pt} \longrightarrow M_{pt}$ is said to be a contraction mapping if $\exists\ 0 < \alpha < 1$ and $\exists\ 1 \le u \le w$ such that $d[T(P_a^i), T(P_b^j)] \le P_\alpha^u \times d(P_a^i, P_b^j)$ for all $P_a^i, P_b^j \in M_{pt}$.

Definition 3.18. (Iterative sequence)
Let (M, d) be a M-metric space, $P_a^i \in M_{pt}$ and $T : M_{pt} \longrightarrow M_{pt}$ be a mapping. Now we construct a sequence as follows:
$P_{a_1}^{i_1} = T(P_a^i), P_{a_2}^{i_2} = T(P_{a_1}^{i_1}) = T^2(P_a^i)$. *Similarly* $P_{a_3}^{i_3} = T(P_{a_2}^{i_2}) = T^3(P_a^i), ...,$
$P_{a_n}^{i_n} = T(P_{a_{n-1}}^{i_{n-1}}) = T^n(P_a^i)$.
Then the sequence $\{P_{a_n}^{i_n}\}$ is called an iterative sequence constructed by the multi point P_a^i.

Theorem 3.19. (Banach's fixed point theorem)
Every contraction mapping defined over a complete M-metric space has fixed points with same base.

Proof. Let (M, d) be a complete M-metric space and $T : M_{pt} \longrightarrow M_{pt}$ be a contraction mapping. Let $P_a^i \in M_{pt}$ and we construct the iterative sequence as follows:

$P_{a_1}^{i_1} = T(P_a^i)$
$P_{a_2}^{i_2} = T(P_{a_1}^{i_1}) = T^2(P_a^i)$
$P_{a_3}^{i_3} = T(P_{a_2}^{i_2}) = T^3(P_a^i)$
$P_{a_n}^{i_n} = T(P_{a_{n-1}}^{i_{n-1}}) = T^n(P_a^i)$
....
....

Now we show that $\{P_{a_n}^{i_n}\}$ is a Cauchy sequence.
We have for $m \in \mathbb{N}, d(P_{a_{m+1}}^{i_{m+1}}, P_{a_m}^{i_m}) = d(T(P_{a_m}^{i_m}), T(P_{a_{m-1}}^{i_{m-1}})))$
$\le P_\alpha^u \times d(P_{a_m}^{i_m}, P_{a_{m-1}}^{i_{m-1}})$ [where $0 < \alpha < 1$ and $1 \le u \le w$]

$= P^u_\alpha \times d(T(P^{i_{m-1}}_{a_{m-1}}), T(P^{i_{m-2}}_{a_{m-2}}))$
$\leq P^u_\alpha \times P^u_\alpha \times d(P^{i_{m-1}}_{a_{m-1}}, P^{i_{m-2}}_{a_{m-2}})$
$= P^u_{\alpha^2} \times d(P^{i_{m-1}}_{a_{m-1}}, P^{i_{m-2}}_{a_{m-2}})$ [From associative property of multiplication]
....
....

$\leq P^u_{\alpha^m} \times d(P^{i_1}_{a_1}, P^i_a)$.
Now for $n > m$,
$d(P^{i_m}_{a_m}, P^{i_n}_{a_n}) \leq d(P^{i_m}_{a_m}, P^{i_{m+1}}_{a_{m+1}}) + d(P^{i_{m+1}}_{a_{m+1}}, P^{i_{m+2}}_{a_{m+2}}) + ... + d(P^{i_{n-1}}_{a_{n-1}}, P^{i_n}_{a_n})$
$\leq P^u_{\alpha^m} \times d(P^{i_1}_{a_1}, P^i_a) + P^u_{\alpha^{m+1}} \times d(P^{i_1}_{a_1}, P^i_a) + ... + P^u_{\alpha^{n-1}} \times d(P^{i_1}_{a_1}, P^i_a)$
$= [P^u_{\alpha^m} + P^u_{\alpha^{m+1}} + P^u_{\alpha^{n-1}}] \times d(P^{i_1}_{a_1}, P^i_a)$
$= P^u_{\alpha^m+\alpha^{m+1}+.......+\alpha^{n-1}} \times d(P^{i_1}_{a_1}, P^i_a)$
$= P^u_{\frac{\alpha^m(1-\alpha^{n-m})}{1-\alpha}} \times P^k_c$ [where $d(P^{i_1}_{a_1}, P^i_a) = P^k_c$]
$= P^l_{\frac{\alpha^m(1-\alpha^{n-m})c}{1-\alpha}}$ [Where $l = Max\{u, k\}$ and assuming without any loss of generality, neither P^k_c nor $P^u_{\frac{\alpha^m(1-\alpha^{n-m})}{1-\alpha}}$ equal to P^1_0]
$< P^l_{\frac{\alpha^m c}{1-\alpha}}$ [$\because 0 < \alpha < 1$ and so $0 < \alpha^{n-m} < 1$]
$< P^1_\epsilon \ \forall\ n > m \geq n_0$ [As $\lim \alpha^m = 0$, for any $\epsilon > 0$, $\exists\ n_0 \in \mathbb{N}$ such that $\alpha^m < \frac{(1-\alpha)\epsilon}{c}$, $\forall\ m \geq n_0$]
$\therefore d(P^{i_m}_{a_m}, P^{i_n}_{a_n}) \to P^1_0$ as $m, n \to +\infty$
$\Rightarrow \{P^{i_n}_{a_n}\}$ is a Cauchy sequence.
Since (M, d) is complete,$\{P^{i_n}_{a_n}\}$ converges to a multi point $P^j_b \in M_{pt}$.
Now we have $d(T(P^j_b), P^j_b) \leq d(T(P^j_b), P^{i_n}_{a_n}) + d(P^{i_n}_{a_n}, P^j_b)$ [For any $n \in \mathbb{N}$]
$= d[T(P^j_b), T(P^{i_{n-1}}_{a_{n-1}})] + d(P^{i_n}_{a_n}, P^j_b)$
$\leq P^u_\alpha \times d(P^j_b, P^{i_{n-1}}_{a_{n-1}}) + d(P^{i_n}_{a_n}, P^j_b)$
$\longrightarrow P^1_0$ as $n \to +\infty$ [$\because \lim d(P^{i_n}_{a_n}, P^j_b) = P^1_0,\ \lim d(P^j_b, P^{i_{n-1}}_{a_{n-1}}) = P^1_0$
and so $\lim P^u_\alpha \times d(P^j_b, P^{i_{n-1}}_{a_{n-1}}) = P^1_0$]
$\therefore d(T(P^j_b), P^j_b) = P^1_0 \Rightarrow T(P^j_b) = P^j_b$.
Thus P^j_b is a fixed point of T.

Since $P^{i_n}_{a_n} \to P^j_b$ for some $1 \leq j \leq C_M(b)$ $\Rightarrow P^{i_n}_{a_n} \to P^k_b\ \forall\ 1 \leq k \leq C_M(b)$, so each P^k_b, $1 \leq k \leq C_M(b)$ i.e., each multi point having base b is a fixed point of T.

Next to show the uniqueness of the base b let P^k_c be a fixed point of T where $b \neq c$. Then we have $d(P^j_b, P^k_c) = P^l_d$ where $d > 0$.
Now $P^l_d = d(P^j_b, P^k_c) = d(T(P^j_b), T(P^k_c)) \leq P^u_\alpha \times d(P^j_b, P^k_c)$ [$0 < \alpha < 1$ *and* $1 \leq u \leq w$] $\Rightarrow P^l_d \leq P^u_\alpha \times P^l_d = P^{Max\{u,l\}}_{\alpha d}$
$\Rightarrow d \leq \alpha d \ \Rightarrow \alpha \geq 1$ [$\because d > 0$], which is a contradiction. So $b = c$.

4. Conclusions

Functional analysis is an important branch of Mathematics and it has many applications in Mathematics and Sciences. Metric space is the beginning of functional analysis and it has several applications in many branch of functional analysis. In this paper convergence in multi metric space and complete multi metric space are studied. Cantor's intersection theorem and Banach's fixed point theorem are established in multi set settings. There is an ample scope for further research on multi metric space. Research on Multi norm and multi inner product can be of special interest.

References

[1] Blizard Wayne D., Multiset theory, Notre Dame Journal of Formal Logic, 30, (1) (1989), 36-66.

[2] Blizard Wayne D., Real-valued multisets and fuzzy sets, Fuzzy Sets and Systems, 33, (1) (1989), 77-97.

[3] Blizard Wayne D., Negative membership, Notre Dame Journal of Formal Logic, 31, (3) (1990), 346-368.

[4] Blizard Wayne D., The development of multiset theory, Modern Logic, 1, (4) (1991), 319-352.

[5] Chakrabarty K., Bags with interval counts, Foundations of Computing and Decision Sciences, 25, (1) (2000), 23-36.

[6] Chakrabarty K. and Despi I., n^k-bags, Int. J. Intell. Syst., 22, (2) (2007), 223-236.

[7] Chakrabarty K., Biswas R. and Nanda S., Fuzzy shadows, Fuzzy Sets and Systems, 101, (3) (1999), 413-421.

[8] Clements G. F., On multiset k-families, Discrete Mathematics, 69, (2) (1988), 153-164.

[9] Conder M., Marshall S. and Slinko Arkadii M., Orders on multisets and discrete cones, A Journal on The Theory of Ordered Sets and Its Applications, 24 (2007), 277-296.

[10] Das S., Roy R., An introduction to multi metric spaces, Adv. Dyn. Syst. Appl., 16, (2) (2021), 605-618.

[11] Das S., Roy R., Some topological properties of multi metric spaces, J. Math. Comput. Sci., 11 (2021), 7253-7268.

[12] Girish K. P., John S. J., Multiset topologies induced by multiset relations, Information Sciences, 188 (2012), 298-313.

[13] Girish K. P., John S. J., On multiset topologies, Theory and applications of Mathematics and Computer Science, 2, (1) (2012), 37-52.

[14] Girish K. P., John S. J., General relations between partially ordered multisets and their chains and antichains, Mathematical Communications, 14, (2) (2009), 193-206.

[15] Girish K. P., John S. J., Relations and functions in multiset context, Inf. Sci., 179, (6) (2009), 758-768.

[16] Girish K. P., John S. J., Rough multisets and information multisystems, Advances in Decision Sciences, (2011), p. 17.

[17] Jena S. P., Ghosh S. K. and Tripathy B. K., On the theory of bags and lists, Information Sciences, 132, (14) (2001), 241-254.

[18] Nazmul Sk., Samanta S. K., On soft multigroups, Annals of Fuzzy Mathematics and Informatics, 10, (2) (2015), 271-285.

[19] Nazmul Sk., Majumdar P., Samanta S. K., On multisets and multigroups, Annals of Fuzzy Mathematics and Informatics, 6, (30) (2013), 643-656.

[20] Peterson J. L., Computation sequence sets, Journal of Computer and System Sciences, 13, (1) (1976), 1-24.

[21] Singh D., A note on the development of multiset theory, Modern Logic, 4, (4) (1994), 405-406.

[22] Singh D., Ibrahim A. M., Yohana T. and Singh J. N., Complementation in multiset theory, International Mathematical Forum, 6, (38) (2011), 1877-1884.

[23] Singh D., Ibrahim A. M., Yohana T. and Singh J. N., An overview of the applications of multisets, Novi Sad J. Math, 37, (2) (2007), 73-92.

[24] Singh D., Ibrahim A. M., Yohana T. and Singh J. N., Some combinatorics of multisets, International Journal of Mathematical Education in Science and Technology, 34, (4) (2003), 489-499.