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REMARKS ON WEAK FORM OF NANO DERIVED SETS

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Abstract: This paper aims to introduce the concept of nano α- derived set and study the characteristics of nano α- derived set. Further, we investigate the different forms of nano α- derived set using lower and upper approximation.

Keywords and Phrases: Nano topology, Nano Interior, Nano Closure, Nano Derived sets, Nano α-open sets, Nano α-derived sets.

2020 Mathematics Subject Classification: 54A05.

1. Introduction

In 1872, Cantor has introduced the notion of the derived set of a set. He also defined closed subset of the real line as subset containing their derived set. The notion of weak form of open set [4, 5], namely α-open set in topological spaces was introduced by Njastad [7] and since then, these sets have been widely explored. Miguel Caldas [6] introduced and studied topological properties of α- derived set using the concept of α- open set. Recall that, "A subset A of a topological space (X, τ) is defined as α-open if $A \subseteq int(cl(int(A)))$. The complement of a α-open is defined as α-closed set and a point $x \in X$ is said to be a α-limit point of A if for each α-open set G containing x, $G \cap (A - \{x\}) \neq \phi$. The set of all α-limit points of A is known as α-derived set of A". Lellis Thivagar et al. [1] introduced the concept, called nano topological space with respect to a subset X of a universe $\mathcal{U}$ which is derived in terms of approximations of X. Also, he introduced and studied the concept of nano α-open set in nano topological spaces in [1]. In this paper, we introduce the notion of weak form of nano derived set, namely nano α-derived set and derive some of their basic properties. Furthermore, we discuss the different types of nano α- derived set under various cases of lower and upper approximation.

2. Preliminaries

Definition 2.1. [8] *Let $\mathcal{U}$ be a non-empty finite set of objects called the universe and R be an equivalence relation on $\mathcal{U}$ named as the indiscernibility relation. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair $(\mathcal{U}, R)$ is said to be the approximation space. Let $X \subseteq \mathcal{U}$.*

(i) The lower approximation of X with respect to R is the set of all objects, which can be for certain classified as X with respect to R and its is denoted by $L_R(X)$. That is, $L_R(X) = \bigcup_{x\in\mathcal{U}} \{R(x) : R(x) \subseteq X\}$, where R(x) denotes the equivalence class determined by x.

(ii) The upper approximation of X with respect to R is the set of all objects, which can be possibly classified as X with respect to R and it is denoted by $U_R(X)$. That is, $U_R(X) = \bigcup_{x\in\mathcal{U}} \{R(x) : R(x) \cap X \neq \phi\}$

(iii) The boundary region of X with respect to R is the set of all objects, which can be classified neither as X nor as not-X with respect to R and it is denoted by $B_R(X)$. That is, $B_R(X) = U_R(X) - L_R(X)$.

Definition 2.2. [3] *Let $\mathcal{U}$ be the universe, R be an equivalence relation on $\mathcal{U}$ and $\tau_R(X) = \{\mathcal{U}, \phi, L_R(X), U_R(X), B_R(X)\}$ where $X \subseteq \mathcal{U}$. $\tau_R(X)$ satisfies the following axioms:*

(i) $\mathcal{U}$ and $\phi \in \tau_R(X)$.

(ii) The union of the elements of any sub collection of $\tau_R(X)$ is in $\tau_R(X)$.

(iii) The intersection of the elements of any finite sub collection of $\tau_R(X)$ is in $\tau_R(X)$.

That is, $\tau_R(X)$ forms a topology on $\mathcal{U}$ called as the nano topology on $\mathcal{U}$ with respect to X. We call $(\mathcal{U}, \tau_R(X))$ as the nano topological space. The elements of $\tau_R(X)$ are called as nano-open sets. A set A is said to be nano closed if A^C is nano-open.

Definition 2.3. [2] *If $(\mathcal{U}, \tau_R(X))$ is a nano topological space with respect to X where $X \subseteq \mathcal{U}$ and if $A \subseteq \mathcal{U}$, then the nano interior of A is defined as the union of all nano-open subsets of A and it is denoted by $\mathcal{N}Int(A)$. That is, $\mathcal{N}Int(A)$ is the largest nano-open subset of A. The nano closure of A is defined as the intersection of all nano closed sets containing A and it is denoted by $\mathcal{N}Cl(A)$. That is, $\mathcal{N}Cl(A)$ is the smallest nano closed set containing A.*

Proposition 2.4. [3] *Let $\mathcal{U}$ be a non-empty finite universe and $X \subseteq \mathcal{U}$.*

(i) If $L_R(X) = \emptyset$ and $U_R(X) = \mathcal{U}$, then $\tau_R(X) = \{\mathcal{U}, \emptyset\}$, the indiscrete nano topology on $\mathcal{U}$.

(ii) If $L_R(X) = U_R(X) = X$, then the nano topology $\tau_R(X) = \{\mathcal{U}, \emptyset, L_R(X)\}$.

(iii) If $L_R(X) = \emptyset$ and $U_R(X) \neq \mathcal{U}$, then $\tau_R(X) = \{\mathcal{U}, \emptyset, U_R(X)\}$.

(iv) If $L_R(X) \neq \emptyset$ and $U_R(X) = \mathcal{U}$, then $\tau_R(X) = \{\mathcal{U}, \emptyset, L_R(X), B_R(X)\}$.

(v) If $L_R(X) \neq U_R(X)$ where $L_R(X) \neq \emptyset$ and $U_R(X) \neq \mathcal{U}$, then $\tau_R(X) = \{\mathcal{U}, \emptyset, L_R(X), U_R(X), B_R(X)\}$ is the discrete nano topology on $\mathcal{U}$.

Definition 2.5. [1] *Let $(\mathcal{U}, \tau_R(X))$ be a nano topological space and $A \subseteq \mathcal{U}$. Then A is said to be nano α-open if $A \subseteq \mathcal{N}int(NCl(NInt(A))$ $\tau_R^{\alpha}(X)$ or $\mathcal{N}\alpha O(\mathcal{U}, X)$ denotes the family of all nano α-open subsets of $\mathcal{U}$.*

Definition 2.6. [1] *Let $(\mathcal{U}, \tau_R(X))$ be a nano topological space and $A \subseteq \mathcal{U}$. A is said to be nano α-closed, if its complement is nano α-open.*

Definition 2.7. [1] *If $(\mathcal{U}, \tau_R(X))$ is a nano topological space and $A \subseteq U$, then intersection of all nano α-closed sets containing A is called the nano α-closure of A, denoted by $\mathcal{N}cl_{\alpha}(A)$.*

3. Nano α-derived set

In this section, we introduce the notion of a weak form of nano derived sets called nano α-derived sets and study its properties.

Definition 3.1. *Let $A \subseteq \mathcal{U}$. A point $x \in \mathcal{U}$ is said to be a nano α-limit point of A if for each nano α-open set G containing x, $G \cap (A - \{x\}) \neq \phi$. The set of all nano α-limit point of A is denoted by $\mathcal{N}D_{\alpha}(A)$ and is called nano α-derived set of A.*

Example 3.2. Let $\mathcal{U} = \{a, b, c, d\}$ with $\mathcal{U}/R = \{\{a\}, \{b, d\}, \{c\}\}$ and $X = \{b, d\}$. Then $\tau_R(X) = \{\mathcal{U}, \phi, \{b, d\}\}$. Then $\{b, d\}, \{a, b, d\}, \{b, c, d\}$, $\mathcal{U}$ and ϕ are nano α-open set in U. If $A = \{a, d\}$, then $\mathcal{N}D_{\alpha}(A) = \{a, b, c\}$.

Theorem 3.3. *If $(\mathcal{U}, \tau_R(X))$ is a nano topological space, and $A, B \subseteq \mathcal{U}$ then*

(i) $\mathcal{N}D_\alpha(A) \subseteq \mathcal{N}D(A)$

(ii) If $A \subseteq B$, $\mathcal{N}D_\alpha(A) \subseteq \mathcal{N}D_\alpha(B)$

(iii) $\mathcal{N}D_\alpha(A) \cup \mathcal{N}D_\alpha(B) = \mathcal{N}D_\alpha(A \cup B)$ *and* $\mathcal{N}D_\alpha(A \cap B) \subseteq \mathcal{N}D_\alpha(A) \cap \mathcal{N}D_\alpha(B)$

(iv) $\mathcal{N}D_\alpha(\mathcal{N}D_\alpha(A)) - A \subseteq \mathcal{N}D_\alpha(A)$

(v) $\mathcal{N}D_\alpha(A \cup \mathcal{N}D_\alpha(A)) \subseteq A \cup \mathcal{N}D_\alpha(A)$.

Proof.

(i) Let $x \in \mathcal{N}D_\alpha(A)$.Then for every nano α-open set G containing x, $G \cap (A - \{x\}) \neq \phi$. If G is a nano-open set containing x, then G is nano α-open and hence $G \cap (A - \{x\}) \neq \phi$. Therefore, $x \in \mathcal{N}D(A)$

(ii) Let $A \subseteq B$. Let $x \in \mathcal{N}D_\alpha(A)$. Then for every nano α-open set G containing x, $G \cap (A - \{x\}) \neq \phi$. Since $A \subseteq B$, $G \cap (A - \{x\}) \subseteq G \cap (B - \{x\})$ and hence $G \cap (B - \{x\}) \neq \phi$. Therefore, $x \in \mathcal{N}D_\alpha(B)$. That is, $\mathcal{N}D_\alpha(A) \subseteq \mathcal{N}D_\alpha(B)$.

(iii) Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, $\mathcal{N}D_\alpha(A) \subseteq \mathcal{N}D_\alpha(A \cup B)$ and $\mathcal{N}D_\alpha(B) \subseteq \mathcal{N}D_\alpha(A \cup B)$ and hence $\mathcal{N}D_\alpha(A) \cup \mathcal{N}D_\alpha(B) \subseteq \mathcal{N}D_\alpha(A \cup B)$. Now, let $x \notin \mathcal{N}D_\alpha(A) \cup \mathcal{N}D_\alpha(B)$. $\Rightarrow x \in \mathcal{N}D_\alpha(A)$ and $x \notin \mathcal{N}D_\alpha(B)$. Therefore there exist nano α-open sets G and H containing x such that $G \cap (A - \{x\}) = \phi$ and $H \cap (B - \{x\}) = \phi$. Since $G \cap H \subseteq G$ and H and G has no point of A other than x and H has no point of B other than x, $G \cap H$ has no point of A other than x and has no point of B other than x. Also $G \cap H$ is a nano α-open set containing x.Thus $G \cap H$ is a nano α-open set containing no pt. of $A \cup B$ other than x.That is,$(G \cap H) \cap (A \cup B - \{x\}) = \phi \Rightarrow x \notin \mathcal{N}D_\alpha(A \cup B)$. Therefore, $\mathcal{N}D_\alpha(A \cup B) \subseteq \mathcal{N}D_\alpha(A) \cup \mathcal{N}D_\alpha(B)$. Thus, $\mathcal{N}D_\alpha(A \cup B) = \mathcal{N}D_\alpha(A) \cup \mathcal{N}D_\alpha(B)$. Since $A \cap B \subseteq A$ and B, using (ii), $\mathcal{N}D_\alpha(A \cap B) \subseteq \mathcal{N}D_\alpha(A) \cap \mathcal{N}D_\alpha(B)$.

(iv) Let $x \in \mathcal{N}D_\alpha(\mathcal{N}D_\alpha(A)) - A$. Let G be a nano α-open set containing x. Since x is a nano α-limit point of $\mathcal{N}D_\alpha(A)$, $G \cap (\mathcal{N}D_\alpha(A) - \{x\}) \neq \phi$. Let $y \in G \cap (\mathcal{N}D_\alpha(A) - \{x\})$. Then $y \in G$ and $y \in \mathcal{N}D_\alpha(A)$ and $y \neq x$.Since y is a nano α-limit point of A and G is a nano α-open set containing y, $G \cap (A - \{y\}) \neq \phi$. Let $z \in G \cap (A - \{y\})$. Then $z \neq x$, since $z \in A$ but $x \notin A$. Therefore, $z \in G \cap (A - \{x\})$ and hence $G \cap (A - \{x\}) \neq \phi$. Therefore, $x \in \mathcal{N}D_\alpha(A)$. Thus, $\mathcal{N}D_\alpha(\mathcal{N}D_\alpha(A)) - A \subseteq \mathcal{N}D_\alpha(A)$.

(v) Let $x \in \mathcal{N}D_\alpha(A \cup \mathcal{N}D_\alpha(A))$. If $x \in A$, then the result is obvious. If $x \notin A$, then $x \in \mathcal{N}D_\alpha(A \cup \mathcal{N}D_\alpha(A)) - A$. Then for any nano α-open set G containing x, $G \cap [(A \cup \mathcal{N}D_\alpha(A)) - \{x\}] \neq \phi$. That is, $G \cap (A - \{x\}) \neq \phi$ (or) $G \cap (\mathcal{N}D_\alpha(A) - \{x\}) \neq \phi$.Then as in (iv), we get $G \cap (A - \{x\}) \neq \phi$. That is, $x \in \mathcal{N}D_\alpha(A)$. Thus, in both cases, $x \in A \cup \mathcal{N}D_\alpha(A)$. Therefore $\mathcal{N}D_\alpha(A \cup \mathcal{N}D_\alpha(A)) \subseteq A \cup \mathcal{N}D_\alpha(A)$.

Remark 3.4. *Let $\mathcal{U} = \{a, b, c, d\}$ with $\mathcal{U}/R = \{\{a\}, \{b, d\}, \{c\}\}$ and $X = \{b, d\}$. Then $\tau_R(X) = \{\mathcal{U}, \phi, \{b, d\}\}$. Let $A = \{a\}$. Then $\mathcal{N}D(A) = \{c\}$ but $\mathcal{N}D_\alpha(A) = \phi$ and hence $\mathcal{N}D(A) \neq \mathcal{N}D_\alpha(A)$.That is, equality does not hold good in (i) of previous theorem.*

Remark 3.5. $\mathcal{N}D_\alpha(A \cap B) \neq \mathcal{N}D_\alpha(A) \cap \mathcal{N}D_\alpha(B)$. *Let $\mathcal{U} = \{a, b, c, d\}$ with $\mathcal{U}/R = \{\{a\}, \{b, d\}, \{c\}\}$.Let $X = \{b, d\}$. Then $\tau_R(X) = \{\mathcal{U}, \phi, \{b, d\}\}$.Then $\tau_R^\alpha(X) = \{\mathcal{U}, \phi, \{b, d\}, \{a, b, d\}, \{b, c, d\}\}$.Let $A = \{a, b\}; B = \{a, c, d\}$. Then $A \cap B = \{a\}$, where $\mathcal{N}D_\alpha(A \cap B) = \phi$, $\mathcal{N}D_\alpha(A) = \{a, c, d\}$ and $\mathcal{N}D_\alpha(B) = \{a, b, c\}$. Thus, $\mathcal{N}D_\alpha(A) \cap \mathcal{N}D_\alpha(B) = \{a, c\} \neq \mathcal{N}D_\alpha(A \cap B)$.*

Theorem 3.6. *If $(\mathcal{U}, \tau_R(X))$ is a nano topological space, and $A \subseteq \mathcal{U}$ then $\mathcal{N}Cl_\alpha(A) = A \cup \mathcal{N}D_\alpha(A)$.*

Proof. If $x \in A$, then $x \in \mathcal{N}Cl_\alpha(A)$ is obvious. Therefore let $x \notin A$. If $x \in \mathcal{N}D_\alpha(A)$, $G \cap (A - \{x\}) \neq \phi$ for every nano α-open set G containing x. That is, $G \cap A \neq \phi$. If $x \notin \mathcal{N}Cl_\alpha(A)$, then $x \in \mathcal{U} - \mathcal{N}Cl_\alpha(A)$ which is nano α-open and hence $(\mathcal{U} - \mathcal{N}Cl_\alpha(A)) \cap A \neq \phi$. Since $A \subseteq \mathcal{N}Cl_\alpha(A)$, $\mathcal{U} - \mathcal{N}Cl_\alpha(A) \subseteq \mathcal{U} - A$. Therefore, $(\mathcal{U} - \mathcal{N}Cl_\alpha(A)) \cap A \subseteq (\mathcal{U} - A) \cap A = \phi$.That is, $(\mathcal{U} - \mathcal{N}Cl_\alpha(A)) \cap A = \phi$, which is a contradiction.Therefore $x \in \mathcal{N}Cl_\alpha(A)$. Thus $A \cup \mathcal{N}D_\alpha(A) \subseteq \mathcal{N}Cl_\alpha(A)$. If $x \in \mathcal{N}Cl_\alpha(A)$ and $x \in A$, then $x \in A \cup \mathcal{N}D_\alpha(A)$. If $x \in \mathcal{N}Cl_\alpha(A)$ and $x \notin A$,then $G \cap A \neq \phi$ for every nano α-open set G containing x. That is, $G \cap (A - \{x\}) \neq \phi$ for every nano α-open set G containing x . Therefore $x \in \mathcal{N}D_\alpha(A)$ and hence $x \in A \cup \mathcal{N}D_\alpha(A)$. Thus $\mathcal{N}Cl_\alpha(A) \subseteq A \cup \mathcal{N}D_\alpha(A)$. Hence, $\mathcal{N}Cl_\alpha(A) = A \cup \mathcal{N}D_\alpha(A)$.

Theorem 3.7. *If $(\mathcal{U}, \tau_R(X))$ is a nano topological space, and $x \in \mathcal{N}Cl_\alpha(A)$ if and only if $G \cap A \neq \phi$ for every nano α-open set G containing x.*

Proof. Let $x \in \mathcal{N}Cl_\alpha(A)$ and G be a nano α-open set containing x. If $G \cap A = \phi$, then $A \subseteq \mathcal{U} - G$ which is nano α-closed. Since $\mathcal{N}Cl_\alpha(A)$ is the smallest nano α-closed set containing A, $\mathcal{N}Cl_\alpha(A) \subseteq \mathcal{U} - G$, where $x \in \mathcal{N}Cl_\alpha(A)$ but $x \notin \mathcal{U} - G$. $\Rightarrow\Leftarrow$. Therefore $G \cap A \neq \phi$ for every nano α-open set G containing x. Conversely, if $G \cap A \neq \phi$ for every nano α-open G containing x, and if $x \notin \mathcal{N}Cl_\alpha(A)$, then $x \in \mathcal{U} - \mathcal{N}Cl_\alpha(A)$ which is nano α-open and hence $(\mathcal{U} - \mathcal{N}Cl_\alpha(A)) \cap A \neq \phi$. But $\mathcal{U} - \mathcal{N}Cl_\alpha(A) \subseteq \mathcal{U} - A$ since $A \subseteq \mathcal{N}Cl_\alpha(A)$. Therefore $(\mathcal{U} - A) \cap A \neq \phi$. $\Rightarrow\Leftarrow$ Therefore $x \in \mathcal{N}Cl_\alpha(A)$.

Corollary 3.8. *If $(\mathcal{U}, \tau_R(X))$ is a nano topological space, and $A \subseteq \mathcal{U}$. A is nano α-closed if and only if $\mathcal{N}D_\alpha(A) \subseteq A$.*

Proof. A is nano α-closed if and only if $\mathcal{N}Cl_\alpha(A) = A$ if and only if $A \cup \mathcal{N}D_\alpha(A) = A$ if and only if $\mathcal{N}D_\alpha(A) \subseteq A$.

Theorem 3.9. *If A is a singleton subset of $\mathcal{U}$, then $\mathcal{N}D_\alpha(A) = \mathcal{N}Cl_\alpha(A) - A$.*

Proof. If $x \in \mathcal{N}D_\alpha(A)$, then $G \cap (A - \{x\}) \neq \phi$ for every nano α-open set G containing x.Then $x \notin A$ since A is singleton. Since $\mathcal{N}D_\alpha(A) \subseteq \mathcal{N}Cl_\alpha(A)$, $x \in \mathcal{N}Cl_\alpha(A)$. Thus $x \in \mathcal{N}D_\alpha(A) \Rightarrow x \in \mathcal{N}Cl_\alpha(A)$ but $x \notin A \Rightarrow x \in \mathcal{N}Cl_\alpha(A) - A$ Therefore $\mathcal{N}D_\alpha(A) \subseteq \mathcal{N}Cl_\alpha(A) - A$. If $x \in \mathcal{N}Cl_\alpha(A) - A$, $x \in \mathcal{N}Cl_\alpha(A)$ but $x \notin A$. Therefore, $G \cap A \neq \phi$ for every nano α-open G containing x. That is, $G \cap (A - \{x\}) \neq \phi$ for every nano α-open G containing x. Hence, $x \in \mathcal{N}D_\alpha(A)$ .Therefore $\mathcal{N}Cl_\alpha(A) - A \subseteq \mathcal{N}D_\alpha(A)$. Thus, $\mathcal{N}D_\alpha(A) = \mathcal{N}Cl_\alpha(A) - A$ if A is a singleton set.

4. Different forms of Nano α-derived set

In this section, we study about the characterization of Nano α-derived sets under different types of lower and upper approximations.

Theorem 4.1. *If $L_R(X) = U_R(X)$ in a nano topological space $(\mathcal{U}, \tau_R(X))$ and $A \subseteq \mathcal{U}$ has more than one element, then*

$$\mathcal{N}D_\alpha(A) = \begin{cases} \mathcal{U}, & \text{if } A \subseteq L_R(X) \\ \phi, & \text{if } A \subseteq L_R(X)^C \\ \mathcal{U} - \{x\}, & \text{if } A \text{ has a single element of } L_R(X) \text{ \& } \\ & \text{at least one element of } L_R(X)^C \\ \mathcal{U}, & \text{if } A \text{ has more than one element of } L_R(X) \text{ and} \\ & \text{at least one element of } L_R(X)^C \end{cases}.$$

Proof. The nano α-open sets in $\mathcal{U}$ are $\mathcal{U}$, ϕ and any $B \supseteq L_R(X)$.

(i) Let $A \subseteq L_R(X)$. If $x \in L_R(X)$, $L_R(X) \cap [A - \{x\}] \neq \phi$, since A has more than one element. Also, $B \cap [A - \{x\}] \neq \phi$ for any $B \supseteq L_R(X)$ since $L_R(X) \cap [A - \{x\}] \subseteq B \cap [A - \{x\}]$. Thus for every nano α-open set G containing x, $G \cap (A - \{x\}) \neq \phi$. Therefore, every element of $L_R(X)$ is a nano α-limit point of A. If $x \in \mathcal{U} - L_R(X)$, then $x \notin A$. Also, $L_R(X) \cup \{x\}$ is the only nano α-open set containing x apart from $\mathcal{U}$ and further $[L_R(X) \cup \{x\}] \cap [A - \{x\}] = L_R(X) \cap aA = A \neq \phi$, unless $A = \phi$. Thus, every element of $\mathcal{U} - L_R(X)$ is also a nano α-limit point of A. Therefore, $\mathcal{N}D_\alpha(A) = L_R(X) \cap [\mathcal{U} - L_R(X)] = \mathcal{U}$ if $A \subseteq L_R(X)$.

(ii) Let $A \subseteq L_R(X)^c$. If $x \in L_R(X)$, $L_R(X) \cap [A - \{x\}] = L_R(X) \cap A = \phi$ and hence x is not a nano α-limit point of A. If $x \in \mathcal{U} - L_R(X)$, then nano α-open sets containing x are $\mathcal{U}$ and $L_R(X) \cup \{x\}$ where $[L_R(X) \cup \{x\}] \cap [A - \{x\}] = L_R(X) \cap A = \phi$.Therefore x is not a α-limit point of A.Thus, $\mathcal{N}D_\alpha(A) = \phi$

(iii) Let A have a single element, x, of $L_R(X)$ and atleast one element of $L_R(X)^c$. Since $L_R(X) \cap (A - \{x\}) = \phi$, x is not a nano α-limit point of A. Let $y \in L_R(X)$ but $y \neq x$. Then $L_R(X) \cap [A - \{y\}] \neq \phi$ since $x \in L_R(X) \cap (A - \{y\})$ and also, $B \cap [A - \{x\}] \neq \phi$ for every $B \supseteq L_R(X)$. Therefore, every $y \in L_R(X)$ such that $y \neq x$ is a nano α-limit point of A. If $y \in L_R(X)^c$, then $y \neq x$. Also, $L_R(X) \cup \{y\}$ is the only nano α-open set containing y and $[L_R(x) \cup \{y\}] \cap [A - \{y\}] = L_R((X) \cap A \neq \phi$, since $x \in L_R(X) \cap A$. That is, any element of $L_R(X)^c$ is a nano α-limit point of A. Thus, every element of $L_R(X)$ other than x and every element of $L_R(X)^c$ is a nano α-limit point of A. Hence, $\mathcal{N}D_\alpha(A) = \mathcal{U} - \{x\}$, if A has a single element x of $L_R(X)$ and atleast one element of $L_R(X)^c$.

(iv) Let A have more than one element of $L_R(X)$ and atleast one element of $L_R(X)^c$. If $x \in L_R(X)$, $L_R(X) \cap (A - \{x\}) \neq \phi$ and $B \cap (A - \{x\}) \neq \phi$ for every $B \supseteq L_R(X)$. Therefore, every element of $L_R(X)$ is a nano α-limit point of A. If $x \in L_R(X)^C$, $[L_R(X) \cup \{x\}] \cap [A - \{x\}] = L_R(X) \cap A \neq \phi$, since A has alteast two elements of $L_R(X)$. Therefore, every element of $L_R(X)^C$ is a nano α-limit point of A.

Thus, $\mathcal{N}D(A) = \mathcal{U}$, if A has more than one element of $L_R(X)$ and atleast one element of $L_R(X)^C$. Hence the theorem follows.

Theorem 4.2. *In a nano topological space $(\mathcal{U}, \tau_R(X))$, if $L_R(X) = \phi$, then for any $A \subseteq \mathcal{U}$ with more than one element*

$$\mathcal{N}D_\alpha(A) = \begin{cases} \mathcal{U}, & \text{if } A \subseteq U_R(X) \\ \phi, & \text{if } A \subseteq U_R(X)^C \\ \mathcal{U} - \{x\}, & \text{if } A \text{ has a single element } x \text{ of} \\ & U_R(X) \text{ but more than one element of } \mathcal{U} \end{cases}$$

Proof. Since $L_R(X) = \phi$, $\tau_R(X) = \{\mathcal{U}, \phi, U_R(X)\}$. Then $U_R(X)$ and any set $B \supseteq U_R(X)$ are the nano α-open sets in $\mathcal{U}$.

(i) Let A have more than one element of $U_R(X)$. If $x \in U_R(X)$, then $U_R(X)$ and any $B \supseteq U_R(X)$ are the nano α-open sets containing x and $U_R(X) \cap [A - \{x\}] \neq \phi$ and hence $B \cap [A - \{x\}] \neq \phi$ for any $B \supseteq U_R(X)$, since A has more than one element of $U_R(X)$. Therefore, every element of $U_R(X)$ is a nano α-limit point of A. If $x \in [U_R(X)]^c$, then $U_R(X) \cup B$ for $B \subseteq [U_R(X)]^c$ containing x is a nano α-open set containing x and $[U_R(X) \cup B] \cap [A - \{x\}] \neq \phi$, since A has more than one element of $U_R(X)$. Therefore, every element of $[U_R(X)]^c$ is a nano α-limit point of A. Thus $\mathcal{N}D_\alpha(A) = \mathcal{U}$ if A has more than one element of $U_R(X)$. (A may or may not have elements of $U_R(X)^c$).

(ii) Let $A \subseteq U_R(X)^c$. Let $x \in U_R(X)$. Then $U_R(X) \cap (A - \{x\}) = U_R(X) \cap A = \phi$ and hence x is not a nano α-limit point of A. Let $x \notin U_R(X)$. Then $U_R(X) \cup B$ where $B \subseteq [U_R(X)]^c$ containing x is a nano α-open set containing x and In particular, $U_R(X) \cup \{x\}$ is a nano α-open set containing x and $(U_R(X) \cup \{x\}) \cap (A - \{x\}) = U_R(X) \cap A = \phi$. Therefore, x is not a nano α-limit point of A. Thus, $\mathcal{N}D_\alpha(A) = \phi$ if $A \subseteq U_R(X)^c$

(iii) Let A have a single element say x, of $U_R(X)$, so that A will have alteast one element of $U_R(X)^c$ since A has more than one element of $\mathcal{U}$. Then $U_R(X)$ is a nano α-open set containing x and $U_R(X) \cap (A - \{x\}) = \phi$. Therefore, x is not a nano α-limit point of A. If $y \in U_R(X)$ and $y \neq x$, $U_R(X) \cap (A - \{y\}) \neq \phi$, since $x \in U_R(X) \cap (A - \{y\})$ and $B \cap (A - \{y\}) \neq \phi$ for all $B \supseteq U_R(X)$ and hence y is a nano α-limit point of A. If $y \in U_R(X)^c$, then $y \neq x$ and $[U_R(X) \cup B] \cap [A - \{y\}] \neq \phi$ for every $B \subseteq U_R(X)^C$ containing y. Therefore every $y \in U_R(X)^c$ is a nano α-limit point of A. Thus, $\mathcal{N}D_\alpha(A) = (U_R(x) - \{x\}) \cup [U_R(X)]^c = \mathcal{U} - \{x\}$ if A has a single element, x, of $U_R(X)$.

Hence the theorem holds.

Theorem 4.3. *In a nano topological space $(\mathcal{U}, \tau_R(X))$, if $L_R(X) \neq \phi$ and $U_R(X) = \mathcal{U}$ then for any sbset A of $\mathcal{U}$ with more than one element,*

$$\mathcal{N}D_\alpha(A) = \begin{cases} L_R(X), & \text{if } A \subseteq L_R(X) \\ B_R(X), & \text{if } A \subseteq B_R(X) \\ \mathcal{U} - \{x, y\} & \text{if A has one element, x, of } L_R(X) \text{ and} \\ & \text{one element , y, of } B_R(X) \\ \mathcal{U} - \{x\}, & \text{if A has a single elt. x of } L_R(X) \text{ and} \\ & \text{more than one element of } B_R(X)^C \text{ (or) A has} \\ & \text{one element, x, of } B_R(X) \text{ and more than} \\ & \text{one element of } L_R(X) \\ \mathcal{U}, & \text{otherwise} \end{cases}$$

Proof. $U_R(X) = \mathcal{U}$ and $L_R(X) \neq \phi$, the nano topology with respect to X on $\mathcal{U}$ is given by $\tau_R(X) = \{\mathcal{U}, \phi, L_R(X), B_R(X)\}$ where $B_R(X) = L_R(X)^C$. Then $\mathcal{U}, \phi, L_R(X)$ and $B_R(X)$ are the only nano α-open sets in $\mathcal{U}$.

(i) Let $A \subseteq L_R(X)$. If $x \in L_R(X)$,then $L_R(X)$ and $\mathcal{U}$ are the nano α-open sets containing x. Also, $L_R(X) \cap (A - \{x\}) \neq \phi$ since A has more than one element of $\mathcal{U}$, in particular, of $L_R(X)$. Also $\mathcal{U} \cap (A - \{x\}) \neq \phi$. Therefore, every element of $L_R(X)$ is a nano α-limit point of A.If $x \in B_R(X)$, $B_R(X)$ and $\mathcal{U}$ are the nano α-open sets containing x. Since $B_R(X) \cap (A - \{x\}) = B_R(X) \cap A = \phi$, and hence, x is not a nano α-limit point of A. Thus, no element of $B_R(X)$ can be a nano α-limit point of A. Therefore $\mathcal{N}D_\alpha(A) = L_R(X)$ if $A \subseteq L_R(X)$.

(ii) Let $A \subseteq B_R(X)$. If $x \in L_R(X)$, then $L_R(X) \cap (A - \{x\}) = L_R(X) \cap A = \phi$ and hence x is not a nano α-limit point of A. If $x \in B_R(X)$, then $B_R(X) \cap (A - \{x\}) \neq \phi$, since A has more than one element of $\mathcal{U}$ and hence atleast two elements of $B_R(X)$. Since $B_R(X) \subseteq \mathcal{U}, \mathcal{U} \cap (A - \{x\}) \neq \phi$. Thus, any element of $B_R(X)$ is a nano α-limit point of A. Therefore, $\mathcal{N}D_\alpha(A) = B_R(X)$, if $A \subseteq B_R(X)$.

(iii) Let A have one element of x, of $L_R(X)$ and one element y of $B_R(X)$. That is $A = \{x, y\}$. Then x and y are not nano α-limit points of A, since $L_R(X) \cap (A - \{x\}) = \phi$ and $B_R(X) \cap (A - \{y\}) = \phi$. Let $z \in L_R(X)$ such that $z \neq x$. Then, $L_R(X) \cap (A - \{z\} \neq \phi$ since $x \in L_R(X) \cap (A - \{z\})$ and $\mathcal{U} \cap (A - \{z\}) \neq \phi$. Therefore, every element of $L_R(X)$ other than x is a nano α-limit point A. Similarly, it can be shown that every element of $B_R(X)$, other than y is a nano α-limit point A. Therefore, $\mathcal{N}D_\alpha(A) = \mathcal{U} - \{x, y\}$.

(iv) If A has one element, x of $L_R(X)$ and more than one element of $B_R(X)$, x is not a nano α-limit point of A, since $L_R(X) \cap (A - \{x\}) = \phi$. If $y \in L_R(X)$ such that $y \neq x$, $x \in L_R(X) \cap (A - \{y\})$ and hence $L_R(X) \cap (A - \{y\}) \neq \phi$. Therefore, every element of $L_R(X)$ other than x is a nano α-limit point of A. If $y \in B_R(X)$, $B_R(X) \cap (A - \{y\}) \neq \phi$ since A has more than one element of $B_R(X)$ and hence every element of $B_R(X)$ is a nano α-limit point of A. Thus, $\mathcal{N}D_\alpha(A) = (L_R(X) - \{x\}) \cup B_R(X) = \mathcal{U} - \{x\}$. Similarly, if A has one element, x of $B_R(X)$ and more than one element of $L_R(X)$,then it can be shown that $\mathcal{N}D_\alpha(A) = (B_R(X) - x) \cup L_R(X) = \mathcal{U} - \{x\}$.

(v) If A has more than one element of $L_R(X)$ and more than one element of $B_R(X)$, then it is obvious that $L_R(X) \cap (A - \{x\}) \neq \phi$ for every $x \in L_R(X)$ and $B_R(X) \cap (A - \{y\}) \neq \phi$ for every $y \in B_R(X)$. Hence $\mathcal{N}D_\alpha(A) = L_R(X) \cup B_R(X) = \mathcal{U}$.

Theorem 4.4. *In a nano topological space $(\mathcal{U}, \tau_R(X))$, if $L_R(X) \neq U_R(X)$ where $U_R(X) \neq \mathcal{U}$ and $L_R(X) \neq \phi$, and $A \subseteq \mathcal{U}$ has more than one element of $\mathcal{U}$, then $\mathcal{N}D_\alpha(A)$ is given by*

$$\mathcal{N}D_\alpha(A) = \begin{cases} (L_R(X) - \{x\}) \cup [U_R(X)]^c, & \textit{if A has one element, x of } L_R(X) \textit{ and no element of } B_R(X) \\ (B_R(X) - \{x\}) \cup [U_R(X)]^c, & \textit{If A has one element, x of } B_R(X) \textit{ and no element of } L_R(X) \\ \mathcal{U} - \{x, y\}, & \textit{if A has one element, x of } L_R(X) \textit{ and one element, y of } B_R(X) \textit{ and may or may not have elements of } [U_R(X)]^c \\ \mathcal{U} - \{x\}, & \textit{if A has one element, x of } L_R(X) \textit{ and more than one element of } B_R(X) \textit{ or A has no element, x of } B_R(X) \textit{ and more than one element of } L_R(X) \\ \mathcal{U}, & \textit{otherwise} \end{cases}$$

Proof. Since $L_R(X) \neq U_R(X)$, the nano topology $\tau_R(X) = \{\mathcal{U}, \phi, L_R(X), U_R(X), B_R(X)\}$. Then the nano α-open sets in $\mathcal{U}$ are $\mathcal{U}, \phi, L_R(X), U_R(X)$ and $B_R(X)$ and any set $\supseteq U_R(X)$.

(i) Let A have one element x, of $L_R(X)$ and no element of $B_R(X)$. Since $L_R(X) \cap (A - \{x\}) = \phi$, x is not a nano α-limit point of A. Let $y \in L_R(X)$ such that $y \neq x$. Then $L_R(X)$, $U_R(X)$ and any set $B \supseteq U_R(X)$ are the nano α-open sets containing y and, $L_R(X) \cap (A - \{y\}) \neq \phi$, since $x \in L_R(X) \cap (A - \{y\})$ and hence $U_R(X) \cap (A - \{y\}) \neq \phi$ and $B \cap (A - \{y\}) \neq \phi$ for any $B \supseteq U_R(X)$. Thus, every element of $L_R(X)$ other than x is a nano α-limit point of A. If $y \in B_R(X)$, then y is not a nano limit point of A since $B_R(X) \cap (A - \{y\}) = B_R(X) \cap A, = \phi$. Therefore, no element of $B_R(X)$ can be a nano α-limit point of A. If $y \in [U_R(X)]^c$, then $U_R(X) \cup \{y\}$ and $\mathcal{U}$ are the only nano α-open sets containing y and $(U_R(X) \cup \{y\}) \cap (A - \{y\}) \neq \phi$, since $y \notin U_R(X)$, $y \notin L_R(X)$ and hence $y \neq x$. Thus any element of $U_R(X)^c$ is a nano α-limit point of A. Thus, $\mathcal{N}D_\alpha(A) = (L_R(X) - \{x\}) \cup U_R(X)^c$.

(ii) If A has one element, x of $B_R(X)$ and no element of $L_R(X)$, then it will automatically contain atleast one element of $U_R(X)^c$, since A has more than one element. Then as in (i), it can be show that only, element of $B_R(X)$ other than x and elements of $U_R(X)^c$ are the nano α-limit points of A and hence $\mathcal{N}D_\alpha(A) = (B_R(X) - \{x\}) \cup U_R(X)^c$.

(iii) Let A have one element x, of $L_R(X)$ and one element y of $B_R(X)$.

(a) If A has no other element of $\mathcal{U}$, then x and y are not nano α-limit points

A, since $L_R(X) \cap (A - \{x\}) = \phi$ and $B_R(X) \cap (A - \{y\}) = \phi$ and any $z \neq x$ in $L_R(X)$ is a nano α-limit point of A, since $L_R(X) \cap (A - \{z\}) \neq \phi$ and $B \cap (A - \{z\}) \neq \phi$ for any $B \supseteq U_R(X)$. Hence every element of $L_R(X)$, other than x is a nano α-limit point of A. Similarly, every element of $B_R(X)$ other than y is a nano α-limit point of A. If $z \in (U_R(X))^c$, then $U_R(X) \cup \{z\}$ and $\mathcal{U}$ are the nano α- open sets containing z. Therefore, $[U_R(X) \cup \{z\}] \cap [A - \{z\}] \neq \phi$, as both x and y belong to this intersection. Thus, z is a nano α-limit point of A. Therefore, $\mathcal{N}D_\alpha(A) = [L_R(X) - \{x\}] \cup [B_R(X) - \{y\}] \cup [U_R(X)]^c = [U_R(X) - \{x, y\}] \cup [U_R(X)]^c = \mathcal{U} - \{x, y\}$.

(b) If A has at least one element of $U_R(X)^C$, then as in (a), every element of $L_R(X)$ other than x and every element of $B_R(X)$ other than y are nano α-limit point of A. if $z \in U_R(X)^C$, then again $[U_R(X) \cup \{z\}] \cap [A - \{z\}] \neq \phi$ since x and y belong to this intersection. Thus, $\mathcal{N}D_\alpha(A) = \mathcal{U} - \{x, y\}$ as in (a).

(iv) Let A have only one element, x of $L_R(X)$ and more than one element of $B_R(X)$. Then every element of $L_R(X)$ other than x is a nano α-limit point of A. If $y \in B_R(X)$, then $B_R(X) \cap [A - \{y\}] \neq \phi$, since A has more than one element of $B_R(X)$ and hence B $\cap [A - \{y\}] \neq \phi$ for any B $\supseteq U_R(X)$ also. Thus every y in $B_R(X)$ is a nano α- limit point of A. If $y \in U_R(X)^C$, then $U_R(X) \cup \{y\}$ and $\mathcal{U}$ are the nano α- open sets containing y. Also $(U_R(X) \cup \{y\}) \cap (A - \{y\}) \neq \phi$ and hence $\mathcal{U} \cap (A - \{y\}) \neq \phi$. Thus any element of $[U_R(X)]^C$ is a nano α-limit point of A. Thus, $\mathcal{N}D_\alpha(A) = (L_R(X) - \{x\}) \cup B_R(X) \cup [U_R(X)]^c = \mathcal{U} - \{x\}$.

(v) Let A have more than one element of $L_R(X)$ and more than one element of $B_R(X)$. Then every element of $L_R(X)$ and every element of $B_R(X)$ are nano α-limit points of A. Also for any element $x \in [U_R(X)]^C$, $(U_R(X) \cup \{x\}) \cap (A - \{x\}) \neq \phi$, since A has atleast one element of $L_R(X)$ and atleast one element of $B_R(X)$. Thus, any element of $[U_R(X)]^C$ is a nano α-limit point of A. Hence $\mathcal{N}D_\alpha(A) = \mathcal{U}$.

5. Conclusion

In this paper, we developed the concepts known as Nano α-derived sets and also studies their properties as well as characterization under various forms of Nano topology. We believe that these results will enhance the further study of Nano topological spaces in different aspects and it will helps to establish the applications through Nano topology.

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