South East Asian J. of Mathematics and Mathematical Sciences
Vol. 18, No. 3 (2022), pp. 193-206

DOI: 10.56827/SEAJMMS.2022.1803.16 **ISSN (Online): 2582-0850**

ISSN (Print): 0972-7752

EQUIVALENT STRUCTURES ON $N(\kappa)$ MANIFOLD ADMITTING GENERALIZED TANAKA WEBSTER CONNECTION

Shweta Naik and H. G. Nagaraja

Department of Mathematics,
Bangalore University,
Jnana Bharathi Campus, Bengaluru - 560056, Karnataka, INDIA

E-mail : naikshwetamaths@gmail.com, hgnraj@yahoo.com

(**Received:** *Aug. 12, 2021* **Accepted:** *Nov. 13, 2022* **Published:** *Dec. 30, 2022)*

Abstract: The main objective of the present paper is to study the equivalence of semi-symmetric and pseudo-symmetric conditions imposing on different curvature tensors in $N(\kappa)$ manifolds admitting generalized Tanaka Webster ($\tilde{\nabla}$) connection. Classification is done according as expression of Ricci tensor and scalar curvature with respect to $\tilde{\nabla}$. Finally an example is given.

Keywords and Phrases: $N(\kappa)$ manifold, generalized Tanaka Webster connection, pseudo-symmetry, semi-symmetry.

2020 Mathematics Subject Classification: 53D10, 53D35.

1. Introduction

In 1988 Tanno [12] introduced the notion of κ-nullity distribution of a contact metric manifold as a distribution such that characteristic vector field ξ of contact metric manifold belongs to the κ-nullity distribution. The contact metric manifold with ξ belonging to the κ-nullity distribution is called $N(\kappa)$-contact metric manifold. Such manifold have been also studied by several authors such as Blair ([4], [3]), [8], [7] and many others. In 2014, Shaikh and Khundu [10] studied the equivalency of various geometric structures obtained by some restrictions imposing on different curvature tensors. In 2016 same authors studied semi-symmetric type and pseudo-symmetric type curvature restricted geometric structures due to projective curvature tensor and characterized such structures on Riemannian and semi-Riemannian manifolds [9].

In differential geometry, various curvature tensors arise as invariants of different transformations, e.g., projective, conformal, concircular , M-projective , W_2 curvature tensor etc.

Shaikh and Kundu [10] proved that the conditions

i) $R\cdot P=0$, $R\cdot R=0$, $R\cdot C=0$, $R\cdot P^*=0$, $R\cdot\mathcal{M}=0$, $R\cdot\mathcal{W}_i=0$ and $R\cdot\mathcal{W}_i^*=0$ (for all $i=1,2,....9$) are equivalent and we name such a class by C_1

ii) $C\cdot C=LQ(g,C)$, $C\cdot P=LQ(g,P)$, $C\cdot P^*=LQ(g,P^*)$, $C\cdot\mathcal{M}=LQ(g,\mathcal{M})$, $C\cdot R=LQ(g,R)$, $C\cdot\mathcal{W}_i=LQ(g,\mathcal{W}_i)$ and $C\cdot\mathcal{W}_i^*=LQ(g,\mathcal{W}_i^*)$ (for all $i=1,2,....9$) are equivalent and we name such a class by C_2

iii) $C\cdot R=0$, $C\cdot P=0$, $C\cdot C=0$, $C\cdot P^*=0$, $C\cdot\mathcal{M}=0$, $C\cdot\mathcal{W}_i=0$ and $C\cdot\mathcal{W}_i^*=0$ (for all $i=1,2,....9$) are equivalent and we name such a class by C_3

iv) $K\cdot R=0$, $K\cdot P=0$, $K\cdot C=0$, $K\cdot P^*=0$, $K\cdot\mathcal{M}=0$, $K\cdot\mathcal{W}_i=0$ and $K\cdot\mathcal{W}_i^*=0$ (for all $i=1,2,....9$) are equivalent and we name such a class by C_4

v) $R\cdot K=0$ and $R\cdot W=0$ are equivalent and we name such a class by C_5

vi) $R\cdot W=LQ(g,W)$ and $R\cdot K=LQ(g,K)$ are equivalent and we name such a class by C_6

vii) $R\cdot C=LQ(g,C)$, $R\cdot P=LQ(g,P)$, $R\cdot P^*=LQ(g,P^*)$, $R\cdot M=LQ(g,M)$, $R\cdot R=LQ(g,R)$, $R\cdot\mathcal{W}_i=LQ(g,\mathcal{W}_i)$ and $R\cdot\mathcal{W}_i^*=LQ(g,\mathcal{W}_i^*)$ (for all $i=1,2,....9$) are equivalent and we name such a class by C_7

viii) $K\cdot W=0$ and $K\cdot K=0$ are equivalent and we name such a class by C_8,

ix) $R=0, W=0$, $P=0$, $P^*=0$, $\mathcal{M}=0$, $\mathcal{W}_i=0$ and $\mathcal{W}_i^*=0$ (for all $i=1,2,....9$) are equivalent and we name such a class by C_9,

where the symbols R, W, C, P, K, $\mathcal{M}$ and $\mathcal{W}_i$ stand for Riemann curvature tensor, conformal curvature tensor, concircular curvature tensor, projective curvature tensor, conharmonic curvature tensor, M-projective curvature tensor, $\mathcal{W}_i$-curvature tensor and $\mathcal{W}_i^*$-curvature tensor, $i=1,2,...,9$.

An important invariant of concircular transformation is the concircular curvature tensor C, which is defined by [1]

$$C(X,Y)Z=R(X,Y)Z-\frac{r}{2n(2n+1)}[g(Y,Z)X-g(X,Z)Y],\text{ for all}X,Y,Z\in\chi(M). \tag{1.1}$$

The projective curvature tensor P [9], and the conformal curvature tensor W [7] are respectively given by,

$$P(X,Y)Z=R(X,Y)Z-\frac{1}{2n}[S(Y,Z)X-S(X,Z)Y], \tag{1.2}$$

$$W(X,Y)Z = R(X,Y)Z - \frac{1}{2n-1}(S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ -g(X,Z)QY) + \frac{r}{2n(2n-1}(g(Y,Z)X - g(X,Z)Y), \tag{1.3}$$

for all $X, Y, Z \in \chi(M)$.
The conharmonic curvature tensor K [11] is defined by

$$K(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}(S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY), \tag{1.4}$$

for all $X, Y, Z \in \chi(M)$.

2. Preliminaries

Let M be an almost contact metric manifold with the structure tensors (ϕ, ξ, η, g), where ϕ is a tensor field of type $(1,1)$, ξ a vector field, η a 1-form and g is a Riemannian metric on M [4]. Then

$$\phi^2 X = -X + \eta(X)\xi, g(X,\xi) = \eta(X), \tag{2.1}$$

$$\eta(\xi) = 1, \phi\xi = 0, \eta(\phi X) = 0, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.3}$$

$$g(\phi X, Y) = -g(X, \phi Y), \tag{2.4}$$

for any vector fields $X, Y \in \chi(M)$.
The κ-nullity distribution of a Riemannian manifold (M, g) for a real number κ is a distribution given by

$$N(\kappa) : p \mapsto N_p(\kappa) = \{Z \in \chi_p(M) : R(X,Y)Z = \kappa(g(Y,Z)X - g(X,Z)Y)\} \tag{2.5}$$

for any $X, Y, Z \in \chi_p(M)$, where R denotes the Riemannian curvature tensor and $\chi_p(M)$ denotes the tangent vector space of M at any point $p \in M$. If the characteristic vector field of a contact metric manifold belongs to the κ nullity distribution, then the relation

$$R(X,Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) \tag{2.6}$$

holds. A contact metric manifold with $\xi \in N(\kappa)$ is called a $N(\kappa)$-contact metric manifold. In an $N(\kappa)$-contact metric manifold M the following relations hold [4], [3]:

$$\nabla_X \xi = -\phi X - \phi h X, \tag{2.7}$$

$$(\nabla_X \phi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX), \tag{2.8}$$

$$(\nabla_X \eta)Y = g(X + hX, \phi Y), \tag{2.9}$$

$$R(\xi, X)Y = \kappa(g(X, Y)\xi - \eta(Y)X), \tag{2.10}$$

$$R(X, \xi)Y = \kappa(\eta(Y)X - \eta(X)Y), \tag{2.11}$$

$$S(X, Y) = 2(n-1)g(X, Y) + 2(n-1)g(hX, Y) + 2[n\kappa - (n-1)]\eta(X)\eta(Y), n \geq 1 \tag{2.12}$$

$$S(X, \xi) = 2n\kappa\eta(X), \tag{2.13}$$

$$\eta(R(X, Y)Z) = \kappa[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)], \tag{2.14}$$

$$(\nabla_X h)(Y) = [(1-\kappa)g(X, \phi Y) + g(X, h\phi Y)]\xi + \eta(Y)[h(\phi X + \phi h X)], \tag{2.15}$$

where S and r are the Ricci tensor and scalar curvature with respect to Levi-civita connection respectively.

The generalized Tanaka Webster connection $\tilde{\nabla}$ on a contact metric manifold M is defined by [6].

$$\tilde{\nabla}_X Y = \nabla_X Y - \eta(Y)\nabla_X \xi + (\nabla_X \eta)(Y)\xi - \eta(X)\phi Y$$

for any vector fields X, Y on M.
With the help of (2.7) the above equation takes the form,

$$\tilde{\nabla}_X Y = \nabla_X Y + g(X + hX, \phi Y)\xi + \eta(Y)(\phi X + \phi h X) - \eta(X)\phi Y. \tag{2.16}$$

Putting $Y = \xi$ in (2.16) we have,

$$\tilde{\nabla}_X \xi = 0. \tag{2.17}$$

The Riemannian curvature tensor $\tilde{R}$ with respect to generalized Tanaka-Webster connection is given by

$$\tilde{R}(X,Y)Z = \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]} Z. \tag{2.18}$$

Using (2.16) in (2.18), we obtain

$$\begin{aligned}\tilde{R}(X,Y)Z = R(X,Y)Z + \kappa[g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi + \eta(Y)\eta(Z)X \\ -\eta(X)\eta(Z)Y] + g(X + hX, \phi Z)(\phi Y + \phi hY) - g(X + hX, \phi Y) \\ \phi Z - g(Y + hY, \phi Z)(\phi X + \phi hX) + g(Y + hY, \phi X)\phi Z,\end{aligned} \tag{2.19}$$

where R and $\tilde{R}$ denote curvature tensors with respect to ∇ and $\tilde{\nabla}$ respectively. From (2.19), we obtain

$$\tilde{S}(Y,Z) = S(Y,Z) - 2g(Y,Z) + (2n\kappa + 2)\eta(Y)\eta(Z), \tag{2.20}$$

$$\tilde{r} = r + 2n\kappa - 4n. \tag{2.21}$$

From (2.19) we have the following:

$$\eta(\tilde{R}(X,Y)Z) = 0, \tag{2.22}$$

$$\tilde{R}(X,Y)\xi = 2\kappa(\eta(Y)X - \eta(X)Y), \tag{2.23}$$

$$\tilde{S}(X,\xi) = 4n\kappa\eta(X), \tag{2.24}$$

where S and $\tilde{S}$ are the Ricci tensors of M with respect to ∇ and $\tilde{\nabla}$ respectively and r and $\tilde{r}$ denote the scalar curvatures of M with respect to ∇ and $\tilde{\nabla}$ respectively.

3. $N(\kappa)$- contact metric manifolds admitting generalized Tanaka-Webster connection belonging to class C_i, $(i = 1, 2, ..., 9)$.

In this section we consider different types of flat, semi-symmetric and pseudo symmetric conditions in a $(2n + 1)$-dimensional $N(\kappa)$-contact metric manifold admitting generalized Tanaka-Webster connection belonging to classes C_i $(i = 1, 2, ..., 9)$.

Theorem 3.1. *A $(2n+1)$ dimensional $N(\kappa)$-contact metric manifold admitting generalized Tanaka Webster connection belonging to class C_9 is an η-Einstein manifold.*

Proof. Case (i): Suppose $\tilde{P}(X,Y)Z = 0$. Then using (1.2), we have

$$\tilde{R}(X,Y)Z = \frac{1}{2n}[\tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y]. \tag{3.1}$$

Taking inner product of (3.1) with ξ, we have

$$g(\tilde{R}(X,Y)Z,\xi) = \frac{1}{2n}[\tilde{S}(Y,Z)\eta(X) - \tilde{S}(X,Z)\eta(Y)]. \tag{3.2}$$

Putting $Y = \xi$ in (3.2), then from (2.20) and (2.22), we have

$$S(X,Z) = (4n\kappa + 2)g(X,Z) - (2n\kappa + 2)\eta(X)\eta(Z). \tag{3.3}$$

Case (ii): Next we consider $\tilde{W}(X,Y)Z = 0$. From (1.3), we have

$$\begin{aligned}\tilde{R}(X,Y)Z = \frac{1}{2n-1}[\tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y + g(Y,Z)\tilde{Q}X - g(X,Z)\tilde{Q}Y]\\ -\frac{\tilde{r}}{2n(2n-1)}[g(Y,Z)X - g(X,Z)Y].\end{aligned} \tag{3.4}$$

Taking inner product of the above with ξ, we obtain

$$\begin{aligned}\eta(\tilde{R}(X,Y)Z) = \frac{1}{2n-1}[\tilde{S}(Y,Z)\eta(X) - \tilde{S}(X,Z)\eta(Y) + g(Y,Z)\eta(\tilde{Q}X)\\ - g(X,Z)\eta(\tilde{Q}Y)] - \frac{\tilde{r}}{2n(2n-1)}[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)].\end{aligned} \tag{3.5}$$

Putting $Y = \xi$ and using (2.19), (2.20) and (2.24), we get

$$S(X,Z) = -(4n\kappa - \kappa - \frac{r}{2n})g(X,Z) + (6n\kappa - \kappa - \frac{r}{2n})\eta(X)\eta(Z). \tag{3.6}$$

Hence Theorem (3.1) concludes from (3.3) and (3.6).

Theorem 3.2. *Let M be a $(2n+1)$ dimensional $N(\kappa)$-contact metric manifold admitting generalized Tanaka Webster connection belonging either to class C_1 or to class C_3. Then M is an η-Einstein manifold. Further M is a manifold with constant scalar curvature.*

Proof. Case(i): Suppose $(\tilde{R}(X,Y)\cdot\tilde{P})(U,V)W = 0$. i.e.

$$\tilde{R}(X,Y)\tilde{P}(U,V)W - \tilde{P}(\tilde{R}(X,Y)U,V)W - \tilde{P}(U,\tilde{R}(X,Y)V)W - \tilde{P}(U,V)\tilde{R}(X,Y)W = 0. \tag{3.7}$$

Taking $X = U = \xi$ in (3.7), we get

$$\tilde{R}(\xi, Y)\tilde{P}(\xi, V)W - \tilde{P}(\tilde{R}(\xi, Y)\xi, V)W - \tilde{P}(\xi, \tilde{R}(\xi, Y)V)W - \tilde{P}(\xi, V)\tilde{R}(\xi, Y)W = 0. \tag{3.8}$$

Taking inner product of the above with ξ and using (1.2), (2.23) and (2.19), we obtain

$$8n\kappa^2\eta(Y)\eta(V) - 2\kappa\tilde{S}(V, Y) = 0, \tag{3.9}$$

which implies that

$$S(V, Y) = 2g(V, Y) + (2n\kappa - 2)\eta(Y)\eta(V). \tag{3.10}$$

From which we derive

$$r = 2n(\kappa + 2). \tag{3.11}$$

Case(ii): We now consider $(\tilde{C}(X, Y) \cdot \tilde{P})(Z, U)V = 0$. i.e.

$$\tilde{C}(X, Y)\tilde{P}(Z, U)V - \tilde{P}(\tilde{C}(X, Y)Z, U)V - \tilde{P}(Z, \tilde{C}(X, Y)U)V - \tilde{P}(Z, U)\tilde{C}(X, Y)V. \tag{3.12}$$

We take $X = \xi$ in (3.12) to get

$$\tilde{C}(\xi, Y)\tilde{P}(Z, U)V - \tilde{P}(\tilde{C}(\xi, Y)Z, U)V - \tilde{P}(Z, \tilde{C}(\xi, Y)U)V - \tilde{P}(Z, U)\tilde{C}(\xi, Y)V. \tag{3.13}$$

Using (1.1), (2.23) in (3.13), we obtain

$$\begin{aligned}
&[2\kappa - \frac{\tilde{r}}{2n(2n+1)}][\eta(\tilde{P}(Z, U)V)Y + \eta(Z)\tilde{P}(Y, U)V + \eta(V)\tilde{P}(Z, Y)V + \eta(V)\tilde{P}(Z, U)Y] \\
&+ 2\kappa[\eta(Y)\eta(\tilde{P}(Z, U)V)\xi - \eta(Y)\eta(Z)\tilde{P}(\xi, U)V - \eta(Y)\eta(U)\tilde{P}(Z, \xi)V - \eta(Y)\eta(V)\tilde{P}(Z, U)\xi] \\
&- \frac{\tilde{r}}{2n(2n+1)}[g(Y, \tilde{P}(Z, U)V)\xi - g(Y, Z)\tilde{P}(\xi, U)V \\
&- g(Y, U)\tilde{P}(Z, \xi)V + g(Y, V)\tilde{P}(Z, U)\xi] = 0.
\end{aligned} \tag{3.14}$$

Taking the inner product of the above with ξ, setting $U = \xi$ in the resulting equation, we have by using (1.2) and (2.22)

$$\begin{aligned}
&[2\kappa - \frac{\tilde{r}}{2n(2n+1)}][\frac{1}{2n}[\tilde{S}(Z, V)\eta(Y) - \tilde{S}(Y, V)\eta(Z)] - 2\kappa\eta(Z)\eta(V)\eta(Y) \\
&+ \eta(\tilde{P}(Z, Y)V) + \frac{1}{2n}\tilde{S}(Z, Y)\eta(V)] = 0.
\end{aligned} \tag{3.15}$$

Next if we take $V = \xi$ in (3.15), then we have either $r = 2n(4n\kappa + \kappa + 2)$ or

$$S(Z, Y) = 2g(Z, Y) + (2n\kappa - 2)\eta(Y)\eta(Z).$$

This completes the proof.

Theorem 3.3. *Let M be a $(2n+1)$ dimensional $N(\kappa)$-contact metric manifold admitting generalized Tanaka Webster connection belonging to class C_2 or to class C_6. Then M is an η-Einstein manifold or is a manifold with constant scalar curvature with respect to Levi-civita connection.*

Proof. Case(i): Suppose $\tilde{C} \cdot \tilde{C} = L_Q(g, \tilde{C})$ holds. Then

$$(\tilde{C}(X,Y) \cdot \tilde{C})(Z,U)V = L_{\tilde{C}}[((X \wedge_g Y)\tilde{C})(Z,U)V]. \tag{3.16}$$

Consider LHS of (3.16) and take $X = \xi$. Then we have

$$\begin{aligned}(\tilde{C}(\xi,Y) \cdot \tilde{C})(Z,U)V = \tilde{C}(\xi,Y)\tilde{C}(Z,U)V - \tilde{C}(\tilde{C}(\xi,Y)Z,U)V \\ - \tilde{C}(Z,\tilde{C}(\xi,Y)U)V - \tilde{C}(Z,U)\tilde{C}(\xi,Y)V.\end{aligned} \tag{3.17}$$

Consider RHS of (3.16) and take $X = \xi$. We have

$$\begin{aligned}L_{\tilde{C}}[((\xi \wedge_g Y)\tilde{C})(Z,U)V] = L_{\tilde{C}}[(\xi \wedge_g Y)\tilde{C}(Z,U)V - \tilde{C}((\xi \wedge_g Y)Z,U)V \\ - \tilde{C}(Z,(\xi \wedge_g Y)U)V - \tilde{C}(Z,U)(\xi \wedge_g Y)V.\end{aligned} \tag{3.18}$$

From (3.16), (3.17) and (3.18), we get

$$\begin{aligned}&-2\kappa[\eta(Y)\eta(Z)\eta(\tilde{C}(\xi,U)V) - \eta(Z)\eta(\tilde{C}(Y,U)V) + \eta(Y)\eta(U)\eta(\tilde{C}(Z,\xi)V) \\ &-\eta(U)\eta(\tilde{C}(Z,Y)V) + \eta(Y)\eta(V)\eta(\tilde{C}(Z,U)\xi) - \eta(V)\eta(\tilde{C}(Z,U)Y)] \\ &-[L_{\tilde{C}} - \frac{\tilde{r}}{2n(2n+1)}[g(Y,\tilde{C}(Z,U)V) - \eta(\tilde{C}(Z,U)V)\eta(Y) - g(Y,Z) \\ &\eta(\tilde{C}(\xi,U)V) + \eta(Z)\eta(\tilde{C}(Y,U)V) - g(Y,U)\eta(\tilde{C}(Z,\xi)V) + \eta(U) \\ &\eta(\tilde{C}(Z,Y)V) - g(Y,V)\eta(\tilde{C}(Z,U)\xi) + \eta(V)\eta(\tilde{C}(Z,U)Y) = 0.\end{aligned} \tag{3.19}$$

Taking $Y = Z = e_i$ in (3.19) and taking summation over $i = 1, ..., 2n+1$, we get either $r = 4n - 2n\kappa$ or

$$S(U,V) = \{2 - \frac{(2n-1)(r - 4n + 2n\kappa)}{2n(2n+1)}\}g(U,V) - (2n\kappa + 2)\eta(U)\eta(V). \tag{3.20}$$

Case (ii): Next we assume $(\tilde{R} \cdot \tilde{W}) = L_{\tilde{W}}\tilde{Q}(g, \tilde{W})$ holds. then

$$(\tilde{R}(X,Y) \cdot \tilde{W})(Z,U)V = L_{\tilde{W}}[((X \wedge_g Y)\tilde{W})(Z,U)V]. \tag{3.21}$$

Taking $X = \xi$ in the LHS of (3.21), we get

$$\begin{aligned}(\tilde{R}(\xi, Y) \cdot \tilde{W})(Z, U)V = \tilde{R}(\xi, Y)\tilde{W}(Z, U)V - \tilde{W}(\tilde{R}(\xi, Y)Z, U)V \\ - \tilde{W}(Z, \tilde{R}(\xi, Y)U)V - \tilde{W}(Z, U)\tilde{R}(\xi, Y)V.\end{aligned} \tag{3.22}$$

Taking $X = \xi$ in the RHS of (3.21), we get

$$\begin{aligned}L_{\tilde{W}}[((\xi \wedge_g Y)\tilde{W})(Z, U)V] = L_{\tilde{W}}[(\xi \wedge_g Y)\tilde{W}(Z, U)V - \tilde{W}((\xi \wedge_g Y)Z, U)V \\ - \tilde{W}(Z, (\xi \wedge_g Y)U)V - \tilde{W}(Z, U)(\xi \wedge_g Y)V.\end{aligned} \tag{3.23}$$

Using (3.21) and (3.22) in (3.23), we obtain

$$\begin{aligned}&2\kappa[\eta(Y)\eta(\tilde{W}(Z, U)V) - \eta(Y)\eta(Z)\eta(\tilde{W}(\xi, U)V) - \eta(Y)\eta(U)\eta(Z, \xi)V) \\ &- \eta(Y)\eta(V)\eta(\tilde{W}(Z, U)\xi)] - L_{\tilde{W}}[g(Y, \tilde{W}(Z, U)V) - g(Y, Z)\eta(\tilde{W}(\xi, U)V) \\ &- g(Y, U)\eta(\tilde{W}(Z, \xi)V) - g(Y, V)\eta(\tilde{W}(Z, U)\xi)] + (L_{\tilde{W}} - 2\kappa)[\eta(\tilde{W}(Z, U)V) \\ &\eta(Y) - \eta(Z)\eta(\tilde{W}(Y, U)V) - \eta(U)\eta(\tilde{W}(Z, Y)V) - \eta(V)\eta(\tilde{W}(Z, U)Y)] = 0.\end{aligned} \tag{3.24}$$

Taking $Y = Z = e_i$ in (3.24) and taking summation over i, we get either $r = 2n\kappa - 2\kappa + 2n + \dfrac{1}{2n}$ or

$$S(U, V) = [2 - 8n^2\kappa + \tilde{r}]g(U, V) + [2n(8n\kappa) - \tilde{r} - 2n\kappa - 2]\eta(U)\eta(V). \tag{3.25}$$

Theorem 3.4. *Let M be a $(2n + 1)$ dimensional $N(\kappa)$-contact metric manifold admitting generalized Tanaka Webster connection belonging to class C_5. Then M is of constant scalar curvature with respect to Levi-civita connection.*
Proof. Consider $(\tilde{R}(X, Y) \cdot \tilde{W})(Z, U)V = 0$. i.e.

$$\tilde{R}(X, Y)\tilde{W}(Z, U)V - \tilde{W}(\tilde{R}(X, Y)Z, U)V - \tilde{W}(Z, \tilde{R}(X, Y)U)V - \tilde{W}(Z, U)\tilde{R}(X, Y)V = 0. \tag{3.26}$$

Setting $X = \xi$ in (3.26), we obtain

$$\tilde{R}(\xi, Y)\tilde{W}(Z, U)V - \tilde{W}(\tilde{R}(\xi, Y)Z, U)V - \tilde{W}(Z, \tilde{R}(\xi, Y)U)V - \tilde{W}(Z, U)\tilde{R}(\xi, Y)V = 0. \tag{3.27}$$

Simplifying (3.27) using (2.10), and taking inner product of ξ with resulting equation, we have

$$\begin{aligned}&- 2\kappa[\eta(Y)\eta(Z)\eta(\tilde{W}(\xi, U)V) - \eta(Z)\eta(\tilde{W}(Y, U)V) + \eta(Y)\eta(U)\eta(\tilde{W}(Z, \xi)V) \\ &- \eta(U)\eta(\tilde{W}(Z, Y)V) + \eta(Y)\eta(V)\eta(\tilde{W}(Z, U)\xi) - \eta(V)\eta(\tilde{W}(Z, U)Y)] = 0.\end{aligned} \tag{3.28}$$

On plugging $Y = Z = e_i$ in (3.28) and taking summation over i, we obtain

$$r = 6n\kappa + 2n - 2\kappa - \{\frac{2\kappa - 1}{2n}\}. \tag{3.29}$$

Theorem 3.5. *Let M be a $(2n+1)$ dimensional $N(\kappa)$-contact metric manifold admitting generalized Tanaka Webster connection belonging to class C_7. Then either manifold M belonging to class C_1 or M is of constant scalar curvature with respect to Levi-civita connection.*

Proof. Suppose $(\tilde{R}(X,Y) \cdot \tilde{R}) = L_{\tilde{R}}((X \wedge_g Y) \cdot \tilde{R})$. Then

$$(\tilde{R}(X,Y) \cdot \tilde{R})(U,V)W = L_{\tilde{R}}((X \wedge_g Y) \cdot \tilde{R}(U,V)W, \tag{3.30}$$

where $L_{\tilde{R}}$ is a function on M. From (3.30) we have

$$\begin{aligned} &\tilde{R}(X,Y)\tilde{R}(U,V)W - \tilde{R}(\tilde{R}(X,Y)U,V)W - \tilde{R}(U,\tilde{R}(X,Y)V)W \\ &- \tilde{R}(U,V)\tilde{R}(X,Y)W = L_{\tilde{R}}[(X \wedge_g Y)\tilde{R}(U,V)W - \tilde{R}((X \wedge_g Y)U,V)W \\ &- \tilde{R}(U,(X \wedge_g Y)V)W - \tilde{R}(U,V)(X \wedge_g Y)W]. \end{aligned} \tag{3.31}$$

Replacing X by ξ in (3.31), we get

$$\begin{aligned} &\tilde{R}(\xi,Y)\tilde{R}(U,V)W - \tilde{R}(\tilde{R}(\xi,Y)U,V)W - \tilde{R}(U,\tilde{R}(\xi,Y)V)W \\ &- \tilde{R}(U,V)\tilde{R}(\xi,Y)W = L_{\tilde{R}}[(\xi \wedge_g Y)\tilde{R}(U,V)W - \tilde{R}((\xi \wedge_g Y)U,V)W \\ &- \tilde{R}(U,(\xi \wedge_g Y)V)W - \tilde{R}(U,V)(\xi \wedge_g Y)W]. \end{aligned} \tag{3.32}$$

Contracting the above with ξ, we get

$$\begin{aligned} &2\kappa[\eta(U)\eta(\tilde{R}(Y,V)W) + \eta(V)\eta(\tilde{R}(U,Y)W) + \eta(W)\eta(\tilde{(}R(U,V)Y)] \\ &= L_{\tilde{R}}[g(Y,\tilde{R}(U,V)W) - \eta(\tilde{R}(U,V)W)\eta(Y) + \eta(U)\eta(\tilde{R}(Y,V)W) \\ &+ \eta(\tilde{R}(U,Y)W)\eta(V) + \eta(\tilde{R}(U,V)Y)\eta(W)]. \end{aligned} \tag{3.33}$$

On plugging $Y = U = e_i$ in (3.33) and taking summation over i, we obtain either $L_{\tilde{R}} = 0$ or $r = 2n(2-\kappa)$.

Theorem 3.6. *Let M be a $(2n+1)$ dimensional $N(\kappa)$-contact metric manifold admitting generalized Tanaka Webster connection. The Ricci tensor of M belonging to class C_4 satisfies*

$$S^2(U,V) = (2-4n\kappa)S(U,V) + [\frac{4\tilde{r}(2n\kappa-1) - 3\kappa\tilde{r}}{(2n+1)}]g(U,V) + [2n\kappa + 2 - \frac{2\kappa\tilde{r}}{2n+1}]\eta(U)\eta(V).$$

Proof. Consider $(\tilde{K}(X,Y) \cdot \tilde{C})(Z,U)V = 0$. i.e.

$$\tilde{K}(X,Y)\tilde{C}(Z,U)V - \tilde{C}(\tilde{K}(X,Y)Z,U)V - \tilde{C}(Z,\tilde{K}(X,Y)U)V - \tilde{C}(Z,U)\tilde{K}(X,Y)V = 0. \tag{3.34}$$

Taking $X=\xi$ in (3.34), we get

$$\tilde{K}(\xi,Y)\tilde{C}(Z,U)V-\tilde{C}(\tilde{K}(\xi,Y)Z,U)V-\tilde{C}(Z,\tilde{K}(\xi,Y)U)V-\tilde{C}(Z,U)\tilde{K}(\xi,Y)V=0. \tag{3.35}$$

Taking inner product with ξ, we have

$$\begin{aligned}&\eta(\tilde{K}(\xi,Y)\tilde{C}(Z,U)V)-\eta(\tilde{C}(\tilde{K}(\xi,Y)Z,U)V)-\eta(\tilde{C}(Z,\tilde{K}(\xi,Y)U)V)\\&-\eta(\tilde{C}(Z,U)\tilde{K}(\xi,Y)V)=0.\end{aligned} \tag{3.36}$$

On plugging $Y=Z=e_i$ in (3.36) and taking summation over i, we get

$$S^2(U,V)=(2-4n\kappa)S(U,V)+[\frac{4\tilde{r}(2n\kappa-1)-3\kappa\tilde{r}}{(2n+1)}]g(U,V)+[2n\kappa+2-\frac{2\kappa\tilde{r}}{2n+1}]\eta(U)\eta(V). \tag{3.37}$$

Class	Curvature condition	M
C_1	$\tilde{R}(X,Y)\cdot\tilde{P}=0$	is η-Einstein
C_2	$\tilde{C}\cdot\tilde{C}=L_{\tilde{C}}\tilde{Q}(g,\tilde{C})$	is η-Einstein or has constant scalar curvature
C_3	$\tilde{C}(X,Y)\cdot\tilde{P}=0$	is η-Einstein or has constant scalar curvature
C_4	$\tilde{K}(X,Y)\cdot\tilde{C}=0$	Ricci tensor has expression in terms of $S^2(U,V)$
C_5	$\tilde{R}(X,Y)\cdot\tilde{W}=0$	is η-Einstein or has constant scalar curvature
C_6	$\tilde{R}\cdot\tilde{W}=L_{\tilde{W}}\tilde{Q}(g,\tilde{W})$	is η-Einstein or has constant scalar curvature
C_7	$\tilde{R}(X,Y)\cdot\tilde{R}=L_{\tilde{R}}((X\wedge_g Y)\cdot\tilde{R})$	is of constant scalar curvature
C_9	$\tilde{P}(X,Y)Z=0$	is η-Einstein
C_9	$\tilde{W}(X,Y)Z=0$	is η-Einstein

Table 1

4. Example

In this section we construct an example of projectively flat and conformaly flat 3-dimensional $N(\kappa)$-contact metric manifold.

We consider 3-dimensional manifold $M=\{(x,y,z)\in R^3\}$, where (x,y,z) are the standard coordinates in R^3. Let e_1, e_2, e_3 be three vector fields in R^3 which satisfy $[e_1, e_2] = (1+a)e_3$, $[e_2, e_3] = 2e_1$, $[e_3, e_1] = (1-a)e_2$, where a is a real number. Let g be a metric defined by $g(e_1, e_2) = g(e_2, e_3) = g(e_1, e_3) = 0$, $g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1$. Let η be the 1-form defined by $\eta(X) = g(X, e_1)$ for any $X \in \chi(M)$. Let ϕ be the $(1,1)$-tensor field defined by $\phi e_1 = 0$, $\phi e_2 = e_3$, $\phi e_3 = -e_2$. Using the linearity of ϕ and g, we have $\eta(e_1) = 1$, $\phi^2 X = -X + \eta(X)\xi$ and $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$, for any $X, Y \in \chi(M)$. Moreover, $he_1 = 0$, $he_2 = ae_2$ and $he_3 = -ae_3$.

The Riemannian connection ∇ of the metric tensor g is given by

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]).$$

Using Koszul's formula, we get the following,

$$\begin{aligned} &\nabla_{e_1} e_1 = 0, \nabla_{e_1} e_2 = 0, \nabla_{e_1} e_3 = 0, \nabla_{e_2} e_1 = -(1+a)e_3, \nabla_{e_2} e_2 = 0, \\ &\nabla_{e_2} e_3 = (1+a)e_1, \nabla_{e_3} e_1 = (1-a)e_2, \nabla_{e_3} e_2 = -(1-a)e_1, \nabla_{e_3} e_3 = 0. \end{aligned} \tag{4.1}$$

In view of the above relations, we have $\nabla_X \xi = -\phi X - \phi h X$ for $e_1 = \xi$. Therefore, the manifold is a contact metric manifold with the contact structure (ϕ, ξ, η, g). Next we find the curvature tensor as follows:

$$\begin{aligned} &R(e_1, e_2)e_2 = (1-a^2)e_1, R(e_3, e_2)e_2 = -(1-a^2)e_3, R(e_1, e_3)e_3 = (1-a^2)e_1, \\ &R(e_2, e_3)e_3 = -(1-a^2)e_2, R(e_2, e_3)e_1 = 0, R(e_1, e_2)e_1 = -(1-a^2)e_2, \\ &R(e_3, e_1)e_1 = (1-a^2)e_3. \end{aligned} \tag{4.2}$$

In view of the expression of the curvature tensor we conclude that the manifold is a $N(1-a^2)$-contact metric manifold. We find the components of Ricci tensor as follows:

$$S(e_1, e_1) = g(R(e_1, e_2)e_2, e_1) + g(R(e_1, e_3)e_3, e_1) = 2(1-a^2). \tag{4.3}$$

Similarly we find $S(e_2, e_2) = 0 = S(e_3, e_3)$. Hence $r = 2(1-a^2)$.

From (2.16) we have the following:

$$\begin{aligned} &\tilde{\nabla}_{e_1} e_1 = 0, \tilde{\nabla}_{e_1} e_2 = -e_3, \tilde{\nabla}_{e_1} e_3 = e_2, \tilde{\nabla}_{e_2} e_1 = 0, \tilde{\nabla}_{e_2} e_2 = 0, \tilde{\nabla}_{e_2} e_3 = 0, \\ &\tilde{\nabla}_{e_3} e_1 = 0, \tilde{\nabla}_{e_3} e_2 = 0, \tilde{\nabla}_{e_3} e_3 = 0. \end{aligned} \tag{4.4}$$

By the above result we can obtain the components of curvature tensor and Ricci tensor with respect to generalized Tanaka-Webster connection as follows:

$$\begin{aligned} &\tilde{R}(e_1, e_2)e_2 = 0, \tilde{R}(e_3, e_2)e_2 = -2e_3, \tilde{R}(e_1, e_3)e_3 = 0, \\ &\tilde{R}(e_2, e_3)e_3 = -2e_2, \tilde{R}(e_2, e_3)e_1 = 0, \tilde{R}(e_1, e_2)e_1 = 0, \tilde{R}(e_3, e_1)e_1 = 0. \end{aligned} \tag{4.5}$$

and $\tilde{S}(e_1, e_1) = 0$, $\tilde{S}(e_2, e_2) = -2$, $\tilde{S}(e_3, e_3) = -2$. Hence $\tilde{r} = -4$.
Computation of the following components of Ricci tensor
$\tilde{S}(e_1, e_2) = \tilde{S}(e_1, e_3) = \tilde{S}(e_2, e_1) = \tilde{S}(e_2, e_3) = \tilde{S}(e_3, e_1) = \tilde{S}(e_3, e_2) = 0$ lead to the following:

$$\tilde{P}(e_2, e_1)e_1 = \tilde{P}(e_3, e_1)e_1 = \tilde{P}(e_2, e_3)e_1 = 0. \tag{4.6}$$

and

$$\tilde{W}(e_2, e_1)e_1 = \tilde{W}(e_3, e_1)e_1 = \tilde{W}(e_2, e_3)e_1 = 0. \tag{4.7}$$

This is true for other components also. Therefore from (4.6) and (4.7), the manifold is projectively flat and conformaly flat. Hence this example verifies Theorem (3.1).

References

[1] Barman A., Concircular curvature tensor of a semi-symmetric metric connection in a Kenmotsu manifold, Thai J. Math., 13, no. 1 (2015), 245-257.

[2] Blair D. E., Inversion theory and conformal mapping, Student Mathematical Library, 9. American Mathematical Society, Providence, RI., 2000.

[3] Blair D. E., Contact manifolds in Riemannian geometry, Lecture Notes in Mathematics, Vol. 509. Springer-Verlag, Berlin-New York, 1976.

[4] Blair D. E., Riemannian geometry of contact and symplectic manifolds, Progress in Mathematics, 203, Birkhauser Boston, Inc., Boston, MA, xii+260, 2002.

[5] Blair D. E., Koufogiorgos T. and Papantoniou B. J., Contact metric manifolds satisfying a nullity condition., Israel J. Math., 91, No. 1-3 (1995), 189-214.

[6] De U. C. and Ghosh G., On generalized Tanaka-Webster connection in Sasakian manifolds, Bull. Transilv. Univ. Braşov Ser., III 9 (58), No. 2 (2016), 13-23.

[7] Hosseinzadeh A. and Taleshian A., On conformal and quasi-conformal curvature tensors of an $N(k)$-quasi Einstein manifold, Commun. Korean Math. Soc., 27, No. 2 (2012), 317-326.

[8] Majhi P. and De U. C., Classifications of $N(k)$-contact metric manifolds satisfying certain curvature conditions, Acta Math. Univ. Comenian. (N. S.), 84, No. 1 (2015), 167-178.

[9] Shaikh A. A. and Kundu H., Some curvature restricted geometric structures for projective curvature tensors, Int. J. Geom. Methods Mod. Phys., 15, No. 9 (2018), 38.

[10] Shaikh A. A. and Kundu H., On equivalency of various geometric structures, J. Geom., 105, No. 1 (2014), 139-165.

[11] Siddiqui S. A. and Ahsan Z., Conharmonic curvature tensor and space time of general relativity, Differential Geometry-Dynamical systems, Vol.12, (2010), 213-220.

[12] Tanno S., Ricci curvatures of contact Riemannian manifolds, Tohoku Math. J., (2) 40, No. 3 (1998), 441-448.

[13] Tanno S., The automorphism groups of almost contact Riemannian manifolds, Tohoku Math. J., (2) 21, (1969), 21-38.