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ON SUBMANIFOLDS OF A MANIFOLD ADMITTING $f_a(2\nu + 3, -1)$ - STRUCTURE

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Abstract: Psomopoulou defined and studied the Invariant submanifolds of a manifold with $f(2\nu + 3, -1)$-structure. In this paper $f_a(2\nu + 3, -1)$ structure has been defined and submanifolds, of a manifold with such a structure have been studied. Some interesting results have been stated and proved in this paper.

Keywords and Phrases: Riemannian Manifold, projection operator, invariant submanifold, integrability conditions.

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1. Introduction and Preliminaries

Let V_n be an n-dimensional C^∞ manifold imbedded differentiabily in an m-dimensional C^∞ Riemannian manifold $W_m (m > n)$ by an imbedding map $b : V_n \to W_m$. If B=db, B is a mapping $T(V_n) \to T(W_m)$ such that a vector field X of $T(V_n)$ correspond to a vector field $BX \in T(W_m); T(V_n); T(W_m)$ denote the tangent bundles of V_n and W_m respectively. If $T(b(V_n))$ is the set of all vectors tangent to the submanifold $b(V_n)$ then $B : T(V_n) \to T(b(V_n))$ is an isomorphism. Let $\tilde{X}, \tilde{Y}$ be C^∞ vector fields, defined along $b(V_n)$ tangent to $b(V_n)$ and let $\tilde{X}$ and $\tilde{Y}$

be local C^∞ extensions of $\bar{X}$ and $\bar{Y}$ respectively. Then $[\bar{X}, \bar{Y}]$ is tangent to $b(V_n)$. If $X, Y \in J_0^1(V_n)$ where $J_0^1(V_n)$ denote the set of all vector fields, in V_n then

$$B[X, Y] = [BX, BY] \tag{1.1}$$

Let $\tilde{g}$ be the Riemannian metric tensor on the enveloping manifold W_m. Then the submanifold V_n also has the induced metric g such that

$$\tilde{g}(BX, BY) = g(X, Y), \text{for all} \quad X, Y \in J_0^1(V_n) \tag{1.2}$$

Let $\bar{\nabla}$ be the Riemannian connection in W_m determined by $\tilde{g}$ and $\tilde{\nabla}$ the induced connection in $b(V_n)$ defined by

$$\bar{\nabla}_{\bar{X}}\bar{Y} = \tilde{\nabla}_{\tilde{X}}\tilde{Y}/b(V_n)$$

where $\bar{X}$,$\bar{Y}$ are C^∞ vector fields defined along $b(V_n)$. and $\tilde{X}$,$\tilde{Y}$ are their C^∞ extensions in W_m. Finally $\bar{\nabla}_{\bar{X}}\bar{Y} = [\bar{X}, \bar{Y}]$ for the manifold $b(V_n)$.
Suppose now that in the manifold W_m, there exists a tensor field $\bar{f}$ of type (1,1) of class C^∞ and rank r (r¡m) such that

$$\bar{f}^{2\nu+3} - a^2\bar{f} = 0 \tag{1.3}$$

where μ is a positive integer and a is a complex number not equal to zero. In such W_m, let us put

$$(i)\ \bar{l} = \frac{\bar{f}^{2\nu+2}}{a^2} \text{ and } (ii)\ \bar{m} = I - \frac{\bar{f}^{2\nu+2}}{a^2} \tag{1.4}$$

I denote the unit tensor field. Then it can be easily proved that

$$\bar{l}^2 = \bar{l}\ ,\ \bar{m}^2 = \bar{m}\ ,\ \bar{l} + \bar{m} = 0, \text{ and } \bar{l}\bar{m} = \bar{m}\bar{l} = 0. \tag{1.5}$$

Thus the operator $\tilde{l}$ and $\tilde{m}$ when applied to the tangent space of W_m at a point are complementary projection operators. Let $\tilde{L}$ and $\tilde{M}$ be distributions corresponding to the complementary projection operators $\tilde{l}$ and $\tilde{m}$ respectively. Let us call such a structure on W_m as a $f_a(2\nu + 3, -1)$ structure of rank r.

2. Invariant Submanifolds

Let V_n be an n dimensional C^∞ manifold imbedded differentiabily in the m dimensional manifold $W_m(n < m)$. Suppose the enveloping manifold W_m is equipped with $f_a(2\nu + 3, -1)$- structure. Let b be the imbedding and B=db. Then the V_n is said to be the invariant submanifold of W_m of the tangent space $T_p(b(V_n))$ of the manifold $b(V_n)$ is invariant by the mapping f at each point p of $b(V_n)$. Thus for each X $\in J_0^1(V_n)$ where $J_0^1(V_n)$ is the set of vector fields tangents to

$$\tilde{f}(BX) = BY \text{ for some } Y \in J_0^1(V_n) \tag{2.1}$$

Thus the (1, 1) tensor field f on V_n defined by $f(X) = Y$ satisfies the relation

$$\tilde{f}(BX) = BfX \text{ for any } X \in J_0^1(V_n) \tag{2.2}$$

Now we consider the following two cases depending on the fact that distribution $\tilde{M}$ is tangent or not to the submanifold $b(V_n)$.
Let us now suppose that the distribution $\tilde{M}$ is not tangent to the submanifold $b(V_n)$. Therefore any vector field of the form $\tilde{m}\bar{X}$, where $\bar{X}$ is the vector field tangent to $b(V_n)$ is independent of any vector field of the form $X \in J_0^1(V_n)$.

$$\begin{aligned}\tilde{m}(BX) &= (I - \frac{\tilde{f}^{2\nu+2}}{a^2})BX\\ &= BX - B\frac{\tilde{f}^{2\nu+2}X}{a^2}\\ \text{or } \tilde{m}(BX) &= B(I - \frac{\tilde{f}^{2\nu+2}X}{a^2})X\end{aligned}$$

But $\tilde{m}(BX) = 0$, Thus $I - \frac{f^{2\tilde{\nu}+2}}{a^2} = 0$

$$\text{or } (\tilde{f}^{(\nu+1)})^2 = a^2 I$$

Thus the invariant submanifold $b(V_n)$ admits GF-structure.
Hence we have.

Theorem 2.1. *An invariant submanifold V_n imbedded in an $\tilde{f}_a(2\nu+3,-1)$-struture manifold W_m such that the distribution $\tilde{M}$ is not tangent to $b(V_n)$ is a GF-structure manifold and the induced GF-structure is defined by the tensor field $\tilde{f}^{\nu+1}$ of type (1,1).*

Let $\tilde{g}$ be the Riemannian metric in W_m defined as follows [3].

$$\begin{aligned}\tilde{g}(\tilde{X},\tilde{Y}) = \frac{1}{2(\nu+1)a^4}[&h(\tilde{X},\tilde{Y}) + h(\tilde{f}\tilde{X},\tilde{f}\tilde{Y}) + h(\tilde{f}^2\tilde{X},\tilde{f}^2\tilde{Y}) + ...\\ &+ ....h(\tilde{f}^{(2\nu+1)}\tilde{X},\tilde{f}^{(2\nu+1)}\tilde{Y}) - m^*(\tilde{X},\tilde{Y})]\end{aligned} \tag{2.3}$$

where

$$m^*(\tilde{X},\tilde{Y}) = h(\tilde{m}\tilde{X},\tilde{m}\tilde{Y}) \tag{2.4}$$

and h is globally defined positive Riemannian metric. Obviously

$$h(\frac{\tilde{f}^{(2\nu+2)}\tilde{X}}{a^2}, \frac{\tilde{f}^{(2\nu+2)}\tilde{Y}}{a^2}) = h(\tilde{X}, \tilde{Y}) - m^*(\tilde{X}, \tilde{Y}) \tag{2.5}$$

and

$$\begin{aligned} &(i) m^*(\tilde{f}\tilde{X}, \tilde{f}\tilde{Y}) = 0 \\ &(ii) h(\tilde{X}, \tilde{m}\tilde{Y}) = h(\tilde{m}\tilde{X}, \tilde{Y}) = m^*(\tilde{X}, \tilde{Y}) \\ &(iii) m^*(\tilde{X}, \tilde{m}\tilde{Y}) = m^*(\tilde{X}, \tilde{Y}) \end{aligned} \tag{2.6}$$

It can be easily proved that the $\tilde{g}$ satisfies the following relations

$$\tilde{g}(\tilde{X}, \tilde{Y}) = \tilde{g}(\tilde{f}\tilde{X}, \tilde{f}\tilde{Y}) \tag{2.7}$$

Hence the Riemannian metric $\tilde{g}$ on W_m is the Hermitian metric on W_m.
Now we prove the following theorem on the invariant submanifold V_n imbedded in W_m.

Theorem 2.2. *An invariant submanifold imbedded in an $\tilde{f}_a(2\nu+3,-1)$-structure manifold W_m in such a way that the distribution $\tilde{M}$ is not tangent to $b(V_n)$ is equipped with a Hermitian metric g induced by the Riemannian metric $\tilde{g}$ on W_m given by (2.3).*
Proof. Replacing $\tilde{X}$ by BX , $\tilde{Y}$ by BY in equation (2.7) we get

$$\tilde{g}(\tilde{f}BX, \tilde{f}BY) = \tilde{g}(BX, BY). \tag{2.8}$$

In view of the equation (2.2) the above equation takes the form

$$\tilde{g}(BfX, BfY) = \tilde{g}(BX, BY) \tag{2.9}$$

Where f is (1,1) tensor field induced on the submanifold V_n from the (1,1) tensor field $\tilde{f}$ on the enveloping manifold W_m.
As g is the induced metric on the submanifold V_n, hence the equation (2.9) is equivalent to

$$g(fX, fY) = g(X, Y)$$

Hence g is the Hermition metric on the manifold V_n.
Since the enveloping manifold W_m admits $\tilde{f}_a(2\nu+3,-1)$- structure hence from the equations (1.3) and (2.16) it follows that

$$f^{2\nu+3} - a^2 f = 0$$

Thus the submanifold V_n admits $f_a(2\nu+3,-1)$- structure induced from the enveloping manifold W_m. Thus we have

Theorem 2.3. *An invariant submanifold V_n imbedded in an $\tilde{f}_a(2\nu+3,-1)$-structure manifold W_m in such a way that the distribution $\tilde{M}$ is tangent to $b(V_n)$ is equipped with the similar $f_a(2\nu+3,-1)$-structure.*

Let $\tilde{N}$ and N be the Nijenhuis tensors corresponding to $\tilde{f}$ and f respectively. Then we have
$\tilde{N}(BX,BY)= [\tilde{f}(BX),\tilde{f}(BY)]-\tilde{f}[\tilde{f}(BX),BY]-\tilde{f}[BX,\tilde{f}(BY)]+\tilde{f}^2[BX,BY]$, for $X,Y\in J_0^1(V_n)$

In view of the equation (1.1) and (2.2), the above equation takes the form

$$\tilde{N}(BX,BY)=[B(fX),B(fY)]-\tilde{f}[B(fX),BY]-\tilde{f}[BX,B(fY)]+\tilde{f}^2(B[X,Y])$$

Since $\tilde{f}^2(BX)=B(f^2(X))$, $X\in J_0^1(V_n)$, we have
$\tilde{N}(BX,BY)=B([fX,fY])-B(f[fX,Y])-B([X,fY])+B(f^2[X,Y])$

Let us now suppose that the distribution $\tilde{M}$ is tangent to the submanifold $b(V_n)$. Thus for each $X\in J_0^1(V_n)$).

$$\tilde{m}(BX)=BY \text{for some} Y\in J_0^1(V_n) \tag{2.10}$$

Thus the tensor field m in V_n given by

$$mX=Y \tag{2.11}$$

satisfies the relation

$$\tilde{m}(BX)=B(mX) \tag{2.12}$$

Let us define a (1,1) tensor field 'l' on V_n as

$$l=-\frac{f^{(2\nu+2)}}{a^2} \tag{2.13}$$

Thus in view of the equations (1.4), (2.2) and (2.13) it follows that

$$\tilde{l}(BX)=B(lX), X\in J_0^1(V_n) \tag{2.14}$$

Theorem 2.4. *The (1,1) tensor field 'l' and 'm' defined on the invariant submnifold V_n satisfies the following relations*

$$\begin{aligned}&(i) l+m=0,\\&(ii) lm=ml=0,\\&(iii) l^2=0, m^2=0\end{aligned} \tag{2.15}$$

Proof. Proof follows easily by virtue of equation (1.5), (2.12) and (2.14).
In view of the equation (2.2) we can show that

$$(\tilde{f}^{(2\nu+3)} - a^2\tilde{f})(BX) = B(f^{(2\nu+2)} - a^2 f)X \tag{2.16}$$

Thus

$$\tilde{N}(BX, BY) = BN(X, Y) \tag{2.17}$$

Using the equations (2.12) and (2.14) it can be easily verified that

$$(i)\tilde{N}(\tilde{l}(BX), \tilde{l}(BY)) = BN(lX, lY)$$
$$(ii)\tilde{N}(\tilde{m}(BX), \tilde{m}(BY)) = BN(mX, mY)$$
$$(iii)\tilde{N}(\tilde{l}(BX), \tilde{l}(BY)) = BN(lX, lY)$$
$$(iv)\tilde{m}\tilde{N}(BX, BY) = BmN(X, Y)$$

Let us denote by L, M the complementary distributions of V_n corresponding to the projection operators l and m respectively. Thus according to the integrability conditions, we have the following theorem.

Theorem 2.5. *If the distribution $\bar{L}$ respectively $\bar{M}$ of W_m is integrable then the distribution L respectively M of V_n is also integrable.*

Theorem 2.6. *If the distributions $\bar{L}$ and $\bar{M}$ are both integrable then both the distributions L and M of V_n are also integrable.*

Theorem 2.7. *If the $\tilde{f}_a(2\nu+3, -1)$- structure defined on W_m is integrable then the induced structure $f_a(2\nu+3, -1)$ on V_n is also integrable.*

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