South East Asian J. of Mathematics and Mathematical Sciences
Vol. 18, No. 3 (2022), pp. 161-170

DOI: 10.56827/SEAJMMS.2022.1803.13 **ISSN (Online): 2582-0850**

ISSN (Print): 0972-7752

ON RECURRENT LIGHTLIKE HYPERSURFACE OF KENMOTSU MANIFOLD

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(**Received:** *Aug. 30, 2021* **Accepted:** *Nov. 28, 2022* **Published:** *Dec. 30, 2022)*

Abstract: The object of present paper is to study the properties of recurrent lightlike hypersurfaces of Kenmotsu manifold with $(\ell, m)-$type connection.

Keywords and Phrases: Hypersurfaces, Kenmotsu manifold, Recurrent lightlike hypersurfaces.

2020 Mathematics Subject Classification: 53C15, 53C25.

1. Introduction

A linear connection $\overline{\nabla}$ on a semi-Riemannian manifold $(\overline{M}, \overline{g})$ is called an $(\ell, m)-$type connection [7] if $\overline{\nabla}$ and its torsion tensor $\overline{T}$ satisfy

$$(\overline{\nabla}_{\overline{X}}\overline{g})(\overline{Y}, \overline{Z}) = \ell\{\theta(\overline{Y})\overline{g}(\overline{X}, \overline{Z}) + \theta(\overline{Z})\overline{g}(\overline{X}, \overline{Y}) - m\{\theta(\overline{Y})\overline{g}(J\overline{X}, \overline{Z}) + \theta(\overline{Z})\overline{g}(J\overline{X}, \overline{Y}) \tag{1.1}$$

and

$$\overline{T}(\overline{X}, \overline{Y}) = \ell\{\theta(\overline{Y})\overline{X} - \theta(\overline{X})Y\} + m\{\theta(\overline{Y})J\overline{X} - \theta(\overline{X})J\overline{Y}\}, \tag{1.2}$$

where ℓ and m are two smooth functions on $\overline{M}$, J is a tensor field of type $(1,1)$ and θ is a 1-form associated with a smooth unit vector field ζ which is called the characteristic vector field of $\overline{M}$, given by $\theta(\overline{X}) = \overline{g}(\overline{X}, \zeta)$. By direct calculation it can be easily seen that a linear connection$\overline{\nabla}$ on M is an (ℓ, m)−−type connection if and only if $\overline{\nabla}$ satisfies

$$\overline{\nabla}_{\overline{X}}\overline{Y} = \nabla_{\overline{X}}\overline{Y} + \theta(\overline{Y})\{\ell\overline{X} + mJ\overline{X}\}, \tag{1.3}$$

where ∇ is the Levi-Civita connection of a semi-Riemannian manifold $(\overline{M}, \overline{g})$ with respect to $\overline{g}$.
In case $(\ell, m) = (1, 0)$: The above connection $\overline{\nabla}$ turns into a semi-symmetric non-metric connection . The notion of semisymmetric non-metric connection on a Riemannian manifold was introduced by Ageshe-Chafle [1, 2] and later, studied by several authors [8, 9]. In case $(\ell, m) = (0, 1)$: The above connection $\overline{\nabla}$ becomes a non-metric ϕ-symmetric connection such that

$$\phi(\overline{X}, \overline{Y}) = \overline{g}(J\overline{X}, \overline{Y}).$$

The notion of the non-metric ϕ symmetric connection was introduced by Jin [10, 11, 12].
In case $(\ell, m) = (1, 0)$ in (1.1) and $(\ell, m) = (0, 1)$ in (1.2): The above connection $\overline{\nabla}$ reduces to a quarter-symmetric non-metric connection. The notion of quarter-symmetric non-metric connection was introduced by Golab [5] and then, studied by Sengupta-Biswas [4] and Ahmad-Haseeb [3]. In case $(\ell, m) = (0, 0)$ in (1.1) and $(\ell, m) = (0, 1)$ in (1.2): The above connection $\overline{\nabla}$ will be a quarter-symmetric metric connection. The notion of quarter-symmetric metric connection was introduced Yano-Imai [14]. In case $(\ell, m) = (0, 0)$ in (1.1) and $(\ell, m) = (1, 0)$ in (1.2): The above connection $\overline{\nabla}$ will be a semi-symmetric metric connection. The notion of semi-symmetric metric connection was introduced by Hayden [6].

2. Preliminaries

Let M be an almost contact manifold equipped with an almost contact metric structure (J, ζ, θ, $\overline{g}$) consisting of a (1,1) tensor field J, a vector field ζ, a 1-form θ and a compatible Riemannian metric $\overline{g}$ satisfying

$$J^2\overline{X} = -\overline{X} + \theta(\overline{X})\zeta, \quad \overline{g}(J\overline{X}, J\overline{Y}) = \overline{g}(\overline{X}, \overline{Y}) - \epsilon\theta(\overline{X})\theta(\overline{Y}), \theta(\zeta) = 1, \tag{2.1}$$

where $\epsilon = 1$ or -1 according as ζ is spacelike or timelike, respectively.
From this, we also have

$$J\zeta = 0, \theta o J = 0, \overline{g}(J\overline{X}, \overline{Y}) = -\overline{g}(\overline{X}, J\overline{Y}), \theta(\overline{X}) = \epsilon\overline{g}(\overline{X}, \zeta), \qquad \forall X, Y \in \chi(M).$$

In the entire discussion of this article, we shall assume that the structure vector field ζ to be unit spacelike one, i.e., $\epsilon = 1$, without loss of generality.

An almost contact metric manifold M is a Kenmotsu manifold [13] if and only if it satisfies

$$(\overline{\nabla}_{\overline{X}}J)\overline{Y} = \overline{g}(J\overline{X}, \overline{Y})\zeta - \theta(\overline{Y})J\overline{X}, \qquad X, Y \in \chi(M).$$

With the above equation and (1.3), (2.1) and $\theta(JY) = 0$, it follows that

$$(\overline{\nabla}_{\overline{X}}J)\overline{Y} = \overline{g}(J\overline{X}, \overline{Y})\zeta - \theta(\overline{Y})J\overline{X} - \theta(\overline{Y})\{\ell J\overline{X} - m\overline{X} + m\theta(\overline{X})\zeta\}. \tag{2.2}$$

Taking $\overline{Y} = \zeta$ and using J$\zeta = 0$ with $\theta(\overline{\nabla}_X\zeta) = \ell\theta(X)$, we have

$$(\overline{\nabla}_{\overline{X}}\zeta) = mJ\overline{X} + (\ell + 1)\overline{X} - \theta(\overline{X})\zeta. \tag{2.3}$$

Let (M, g) be a lightlike hypersurface of $\overline{M}$. The normal bundle TM$^{\perp}$ of M is a subbundle of the tangent bundle TM of M, of rank 1, and coincides with the radical distribution Rad(TM) =$TM\cap$ TM$^{\perp}$. Denote by $F(M)$ the algebra of smooth functions on M and by $T(E)$ the $F(M)$ module of smooth sections of any vector bundle E over M.

A complementary vector bundle $S(TM)$ of Rad(TM) in TM is non-degenerate distribution on M, which is called a screen distribution on M, such that
$TM = \text{Rad}(TM) \oplus_{orth} \ S(TM)$,
where $\oplus_{orth}$ denotes the orthogonal direct sum. For any null section ξ of Rad(TM), there exists a unique null section N of a unique lightlike vector bundle tr(TM) in the orthogonal complement $S(TM)^{\perp}$ of $S(TM)$ satisfying
$\overline{g}(\xi, N) = 1$, $\overline{g}(N, N) = \overline{g}(N, X) = 0; \ \ \forall \ X \in T(S(TM))$.
We call tr(TM) and N the transversal vector bundle and the null transversal vector field of M with respect to the screen distribution $S(TM)$, respectively.
The tangent bundle $T\overline{M}$ of $\overline{M}$ is decomposed as follow:
$T\overline{M} = TM \oplus tr(TM) = \{Rad(TM) \oplus tr(TM)\} \oplus_{orth} S(TM)$.
In the sequel, let X, Y, Z and W be the vector fields on M, unless otherwise specified. Let P be the projection morphism of TM on $S(TM)$. Then the local Gauss and Weingartan formulas of M and $S(TM)$ are given respectively by [7]

$$\overline{\nabla}_X Y = \nabla_X Y + B(X, Y)N, \tag{2.4}$$

$$\overline{\nabla}_X N = -A_N X + \tau(X)N, \tag{2.5}$$

$$\nabla_X PY = \nabla^{\star}_X PY + C(X, PY)\xi, \tag{2.6}$$

$$\nabla_X \xi = -A^{\star}_{\xi} X - \sigma(X)\xi. \tag{2.7}$$

where ∇ and $\nabla^{\star}$ are the induced linear connections on TM and $S(TM)$ respectively, B and C are the local second fundamental forms on TM and $S(TM)$ respectively,

A_N and $A_N^\star$ are the shape operators on TM and $S(TM)$ respectively, and τ and σ are 1-forms on M.
For a lightlike hypersurface M of Kenmotsu manifold ($\overline{M}$, $\overline{g}$), it is known [3] that $J(\text{Rad}(TM))$ and $J(\text{tr}(TM))$ are subbundles of $S(TM)$, of rank 1 such that $J(\text{Rad}(TM)) \cap J(tr(TM)) = 0$. Thus there exist two non-degenerate almost complex distributions D_0 and D on M with respect to J, i.e., $J(D_0) = D_0$ and $J(D) = D$, such that

$$
\begin{aligned}
S(TM) &= J(Rad(TM)) \oplus J(tr(TM)) \oplus_{orth} D_0, \\
D &= \{Rad(TM) \oplus_{orth} J(Rad(TM))\} \oplus_{orth} D_0, \\
TM &= D \oplus J(tr(TM)).
\end{aligned}
$$

Consider two null vector fields U and V, and two 1-forms u and v such that

$$U = -JN, V = -J\xi, u(X) = g(X, V); \quad v(X) = g(X, U). \tag{2.8}$$

Denote by S the projection morphism of TM on D. Any vector field X of M is expressed as $X = SX + u(X)U$. Applying J to this form, we have

$$JX = FX + u(X)N, \tag{2.9}$$

where F is a tensor field of type $(1, 1)$ globally defined on M by $F = JoS$. Applying J to (2.9) and using (1.2) , (1.3) and (2.8), we have

$$F^2X = -X + u(X)U + \theta(X)\zeta. \tag{2.10}$$

As $u(U) = 1$ and $FU = 0$, the set (F, u, U) defines an indefinite almost contact structure on M and F is called the structure tensor field of M.

3. $(\ell, m)-$type Connections

Using (1.1),(1.2), (1.3), (2.4) and (2.9), we obtain

$$
\begin{aligned}
(\nabla_X g)(Y, Z) &= B(X, Y)\eta(Z) + B(X, Z)\eta(Y) \\
&\quad - \ell\{\theta(Y)g(X, Z) + \theta(Z)g(X, Y)\} \\
&\quad - m\{\theta(Y)g(JX, Z) + \theta(Z)g(JX, Y)\},
\end{aligned} \tag{3.1}
$$

$$T(X, Y) = \ell\{\theta(Y)X - \theta(X)Y\} + m\{\theta(Y)FX - \theta(X)FY\}, \tag{3.2}$$

$$B(X, Y) - B(Y, X) = m\{\theta(Y)u(X) - \theta(X)u(Y)\}, \tag{3.3}$$

where T is the torsion tensor with respect to ∇ and η is a 1-form such that $\eta(X) = \overline{g}(X, N)$.

Proposition 3.1. *Let M be a lightlike hypersurface of Kenmotsu manifold $\overline{M}$ with an $(\ell, m)-$ type connection such that ζ is tangent to M. Then if $m = 0$, then B is symmetric and conversely if B is symmetric then $m = 0$.*

Proof. If $m = 0$, then B is symmetric by (3.3). Conversely, if B is symmetric,then replacing X by ζ and Y by U, we get $m = 0$.

As $B(X, Y) = \overline{g}(\overline{\nabla}_X Y, \xi)$, so B is independent of the choice of $S(TM)$ and satisfies

$$B(X, \xi) = 0, \qquad B(\xi, X) = 0. \tag{3.4}$$

Local second fundamental forms are related to their shape operators by

$$B(X, Y) = g(A^{\star}_{\xi} X, Y) + m\ u(X)\theta(Y), \tag{3.5}$$

$$C(X, PY) = g(A_N X, PY) + \{\ell\eta(X) + m\ u(X)\}\theta(PY), \tag{3.6}$$

$$\overline{g}(A^{\star}_{\xi} X, N) = 0, \overline{g}(A_N X, N) = 0, \sigma = \tau. \tag{3.7}$$

$S(TM)$ is non-degenerate, so using (3.4), (3.5), we have

$$A^{\star}_{\xi}\xi = 0, \qquad \overline{\nabla}_x \xi = -A^{\star}_{\xi} X - \tau(X)\xi. \tag{3.8}$$

Taking $\overline{\nabla}_x$ to $\overline{g}(\zeta, \xi) = 0$ and $\overline{g}(\zeta, N) = 0$ and using (1.1), (2.3), (2.5), (3.5), (3.6) and (3.8), we have

$$g(A^{\star}_{\xi} X, \zeta) = 0, \quad B(X, \zeta) = m\ u(X), \tag{3.9}$$

$$g(A_N X, \zeta) = \eta(X), \qquad C(X, \zeta) = (\ell + 1)\eta(X) + mv(X). \tag{3.10}$$

By (2.9), (2.3) and (2.4), we have

$$\nabla_X \zeta = m\ FX + (\ell + 1)X - \theta(X)\zeta. \tag{3.11}$$

Applying $\overline{\nabla}_x$ to (2.8) and (2.9) and using (2.2), (2.4), (2.5), (2.9), (2.10), (3.1), (3.6), (3.8) with $\theta(U) = \theta(V) = 0$, we have

$$B(X, U) = C(X, V), \tag{3.12}$$

$$\nabla_X U = F(A_N X) + \tau(X)U - V(X)\zeta, \tag{3.13}$$

$$\nabla_X V = F(A^{\star}_{\xi} X) - \tau(X)V - u(X)\zeta, \tag{3.14}$$

$$\begin{aligned}(\nabla_X F)Y = {} & u(Y)A_N X - B(X,Y)U \\ & + \{\overline{g}(JX,Y) - m\ \theta(X)\theta(Y)\}\zeta \\ & + m\ \theta(Y)X - (\ell+1)\theta(Y)FX, \end{aligned} \tag{3.15}$$

$$(\nabla_X u)Y = -u(Y)\tau(X) - B(X,FY) - (\ell+1)\theta(Y)u(X), \tag{3.16}$$

$$(\nabla_X u)Y = v(Y)\tau(X) - g(A_N X, FY) - (\ell+1)\theta(Y)v(X) + m\ \theta(Y)\eta(X). \tag{3.17}$$

4. Recurrent Hypersurfaces

Structure tensor field F of M is said to be recurrent [10] if there exists a non-zero 1-form ω on TM such that $(\nabla_X F)Y = w(X)FY$.
A lightlike hypersurface M of a Kenmotsu manifold $\overline{M}$ is called recurrent if it admits a recurrent structure tensor field F.

Theorem 4.1. *There exist no recurrent lightlike hypersurface of Kenmotsu manifold with an $(\ell, m)-$type connection such that ζ is tangent to M and F is recurrent.*
Proof. As M is recurrent so by definition and (3.15), we have

$$\begin{aligned} w(X)FY = {} & u(Y)A_N X - B(X,Y)U \\ & + \{\overline{g}(JX,Y) - m\theta(X)\theta(Y)\}\zeta \\ & + m\theta(Y)X - (\ell+1)\theta(Y)FX. \end{aligned} \tag{4.1}$$

Taking $Y = \xi$ and using (3.4) with $F\xi = -V$, we have

$$w(X)V + u(X)\zeta = 0.$$

Taking scalar product with U, we get $w = 0$.
Hence F is parallel to ∇ .
Replacing Y by ξ and using (3.9), we get

$$m\{X - u(X)U - \theta(X)\zeta\} = \ell FX.$$

Replacing X by V, we get $mVC = \ell\xi$, which implies $m = 0$ and $\ell = 0$.
Taking scalar product with ζ to (4.1) and using (3.10), we get

$$u(X)v(Y) - u(Y)v(X) = 0.$$

Hence $m = 0$, which is a contradiction that $(\ell, m) \neq (0, 0)$. Hence the theorem follows.

Corollary 4.1. *There exist no recurrent lightlike hypersurface of Kenmotsu manifold with an $(\ell, m)-$type connection such that ζ is tangent to M and F is parallel with respect to connection ∇ of M.*

5. Lie Recurrent Hypersurfaces

Structure tensor field F of M is said to be Lie recurrent [10] if there exists a non-zero 1-form v on TM such that $(L_X F)Y = v(X)FY$,
where L_X denote the Lie derivative on M with respect to X.
Structure tensor field F is called Lie parallel if $L_X F = 0$. A lightlike hypersurface M of Kenmotsu manifold $\overline{M}$ is called Lie recurrent if it admits a Lie recurrent structure tensor field F.

Theorem 5.1. *Let M be a Lie recurrent lightlike hypersurface of Kenmotsu manifold $\overline{M}$ with an $(\ell, m)-$type connection such that ζ is tangent to M and F is Lie recurrent. Then*
(1) F is Lie parallel,
(2) 1-form τ satisfies $\tau = 0$ and
(3) Shape operator $A^{\star}_{\xi}$ satisfies $A^{\star}_{\xi} U = A^{\star}_{\xi} V = 0$.
Proof. (1) By definition of Lie recurrent, (2.9), (2.10), (3.2) and (3.15), we have

$$\begin{aligned} \nu(X)FY = & -\nabla_{FY}X + F\nabla_Y X + u(Y)A_N X \\ & - \{B(X,Y) - m\theta(Y)u(X)\}U \\ & - \theta(Y)(FX) + \overline{g}(JX,Y)\zeta. \end{aligned} \tag{5.1}$$

Replacing Y by ξ and using (3.4), we have

$$-\nu(X)V = \nabla_V X + F\nabla_\xi X + u(X)\zeta. \tag{5.2}$$

Taking the scalar product with V and ξ, we get

$$u(\nabla_V X) = 0, \qquad \theta(\nabla_V X) + u(X) = 0. \tag{5.3}$$

Taking $Y = V$ in (5.1) and using $\theta(V) = 0$, we get

$$-\nu(X)\xi = -\nabla_\xi X + F\nabla_V X - B(X,V)U. \tag{5.4}$$

Applying F and using (2.10) and (5.3), we have

$$\nu(X)V = \nabla_V X + F\nabla_\xi X + u(X)\zeta.$$

Comparing this with (5.2), we have $\nu = 0$. Hence F is Lie parallel.
(2) Taking scalar product with N to (5.1) and using (3.7), we have

$$-\overline{g}(\nabla_{FY}X, N) + \overline{g}(\nabla_Y X, U) = 0. \tag{5.5}$$

Taking $X = \xi$ and using (2.7) with (3.5), we get

$$B(X, U) = \tau(FX). \tag{5.6}$$

Taking $X = U$ and using (3.12) with $FU = 0$, we have

$$C(U, V) = B(U, U) = 0. \tag{5.7}$$

Taking $X = V$ in (5.5) and using (3.5) with (3.14), we have

$$B(FY, U) = -\tau(Y).$$

Taking $Y = U$ and $Y = \zeta$ with the fact $FU = F\zeta = 0$, we get

$$\tau(U) = 0, \qquad \tau(\xi) = 0. \tag{5.8}$$

Taking $X = U$ to (5.1) and using (3.3), (3.10), (3.12), (3.13), we get
$u(Y)A_N U - F(A_N FY) - A_N Y - \tau(FY)U + \eta(Y)\zeta = 0$.
Taking scalar product with V and using (3.6),(3.12) and (5.7), we get
$B(X, U) = -\tau(FX)$.
Comparing with (5.6), we have $\tau(FX) = 0$.
Replacing X by FY and using (2.10) with (5.8), we have
$\tau = 0$.
(3)Taking $X = U$ in (3.3) and using (5.6) with $\tau = 0$, we have

$$B(U, X) = m\ \theta(X). \tag{5.9}$$

Taking $X = U$ in (3.5) and using (5.9), we have $g(A^{\star}_{\xi}U, X) = 0$. Hence $A^{\star}_{\xi}U = 0$.
Replacing X by ξ in (4.3) and using (3.8) with $\tau = 0$, we have $A_N V = 0$.

Acknowledgement

Author is thankful to the referee for his helpful suggestions towards improvement of the paper. This work is financially supported by Research and Development Grant of Higher Education Department, Government of Uattar Pradesh. Letter No.: 89/2022/1585/70-4-2-22/001-4-32-2022 dated 10^{th} November, 2022.

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