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CERTAIN SUBCLASS OF BI-UNIVALENT FUNCTIONS RELATED TO HORADAM POLYNOMIALS ASSOCIATED WITH q-DERIVATIVE

P. Nandini and S. Latha

Department of Mathematics,
JSS Academy of Technical Education,
Srinivaspura, Bengaluru - 560060, INDIA

E-mail : pnandinimaths@gmail.com, drlatha@gmail.com

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Abstract: In this paper, by making use of q-derivative, we define a new subclass of analytic and bi-univalent functions related to Horadam polynomials. For functions belonging to this class, we derive coefficient inequalities and the Fekete-Szegö inequalities. We also provide relevant connections of our results with those considered in earlier investigations.

Keywords and Phrases: Univalent and Bi-univalent functions, Fekete-Szegö inequality, Horadam polynomials and q-derivative.

2020 Mathematics Subject Classification: 52A40.

1. Introduction

We indicate by $\mathcal{A}$ the collection of functions, which are analytic in the open unit disc given by

$$\mathbb{D} = \{z \in \mathbb{C}; |z| < 1\}$$

and have the following normalized form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

We denote by $\mathcal{S}$ the sub-collection of the set $\mathcal{A}$ consisting of functions, which are also univalent in $\mathbb{D}$. According to the Koebe's one-quarter theorem [3], every function $f \in \mathcal{S}$ has an inverse f^{-1} defined by

$$f^{-1}(f(z)) = z \qquad (z \in \mathbb{D})$$

and

$$f(f^{-1}(w)) = w \qquad \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4}\right),$$

where

$$g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + ... \tag{1.2}$$

We say that a function $f \in \mathcal{A}$ is bi-univalent in $\mathbb{D}$ if both f and its inverse f^{-1} are univalent in $\mathbb{D}$. Let Σ stand for the family of bi-univalent functions in $\mathbb{D}$ given by (1.1).

Next, we recall the definition of subordination between analytic functions. For two functions $f, g \in \mathcal{A}$, We say that f is subordinate to g in $\mathbb{D}$, written as $f \prec g$ provided there is an analytic function w in $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$. It follows from Schwarz Lemma that

$$f(z) \prec g(z) \iff f(0) = g(0) \quad and \quad f(\mathbb{D}) \subset g(\mathbb{D}), \quad z \in \mathbb{D}.$$

Jackson [6] introduced the q-derivative operator D_q of a function f as follows:

$$D_qf(z) = \frac{f(z) - f(qz)}{(1-q)z} \quad (0 < q < 1, z \neq 0).$$

For a function $f \in \mathcal{A}$ defined by (1.1), we have

$$D_qf(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1},$$

where $[n]_q = \dfrac{1-q^n}{1-q}$.

Recently, Hörzum and Kocer [5] studied the Horadam polynomials $h_n(x)$, which are given by the following recurrence relation [4]:

$$h_n(x) = \rho x h_{n-1}(x) + \sigma h_{n-2}(x) \quad (x \in \mathbb{R}; \quad n \in \mathbb{N} = \{1, 2, 3, ...\}) \tag{1.3}$$

with

$$h_1(x) = a \quad and \quad h_2(x) = bx,$$

for some real constants a, b, ρ and σ. Moreover, the generating function of the Horadam polynomials $h_n(x)$ is given by

$$\Pi(x, z) = \sum_{n=1}^{\infty} h_n(x) z^{n-1} = \frac{a + (b - a\rho)xz}{1 - \rho xz - \sigma z^2}. \tag{1.4}$$

Remark 1.1. *We record here some special cases of the Horadam polynomials* $h_n(x)$ *by appropriately choosing the parameters* a, b, ρ *and* σ.

(i) Taking $a = b = \rho = \sigma = 1$, *we obtain the Fibonacci polynomials* $F_n(x)$.

(ii) Taking $a = 2, b = \rho = \sigma = 1$, *we get the Lucas polynomials* $L_n(x)$.

(iii) Taking $a = \sigma = 1$ *and* $b = \rho = 2$, *we have the Pell polynomials* $P_n(x)$.

(iv) Taking $a = b = \rho = 2$ *and* $\sigma = 1$, *we find the Pell-Lucas polynomials* $Q_n(x)$.

(v) Taking $a = b = 1, \rho = 2$ *and* $\sigma = -1$, *we obtain Chebyshev polynomials* $T_n(x)$ *of first kind.*

(vi) Taking $a = 1, b = \rho = 2$ *and* $\sigma = -1$, *we have Chebyshev polynomials* $U_n(x)$ *of second kind.*

Definition 1.2. *A function* $f \in \mathcal{A}$ *of the form (1.1) belongs to the class* $\mathcal{S}^*_\Sigma(\alpha, x, q)$ *for* $0 \leq \alpha \leq 1$ *and* $z, w \in \mathbb{D}$, *if the following conditions are satisfied.*

$$\alpha \left(\frac{D_q(zD_qf(z))}{D_qf(z)} \right) + (1 - \alpha)D_qf(z) \prec \Pi(x, z) + 1 - a$$

and

$$\alpha \left(\frac{D_q(wD_qg(w))}{D_qg(w)} \right) + (1 - \alpha)D_qg(w) \prec \Pi(x, w) + 1 - a,$$

where a *is real constant and the function* $g = f^{-1}$ *is given by (1.2).*

Remark 1.3. *The family* $\mathcal{S}^*_\Sigma(\alpha, x, q)$ *generalizes the following known families of bi-univalent functions.*

(i) $\mathcal{S}^*_\Sigma(\alpha, x, 1) = \mathcal{G}^*(\alpha, x)$, *the class of bi-univalent functions established by Orhan* [7].

(ii) $\mathcal{S}^*_\Sigma(0, x, 1) = \Sigma'(x)$, *the class of bi-univalent functions studied by Al-Amoush* [2].

(iii) $\mathcal{S}^*_\Sigma(1, x, 1) = \mathcal{K}_\sigma(x)$, *the class of bi-univalent functions investigated by Abirami* [1].

Theorem 1.4. *Let $f \in \mathcal{A}$ be in the class $\mathcal{S}^*_\Sigma(\alpha, x, q)$. Then*

$$|a_2| \leq \frac{bx\sqrt{bx}}{\sqrt{\left|\left[[3]_q(\alpha([3]_q-2)+1)-[2]_q^2\alpha([2]_q-1)\right]b^2x^2-[2]_q^2(\alpha([2]_q-2)+1)^2(\rho bx^2+\sigma a)\right|}} \tag{1.5}$$

and

$$|a_3| \leq \frac{|bx|}{[3]_q|\alpha([3]_q-2)+1|}+\frac{b^2x^2}{[2]_q^2[\alpha([2]_q-2)+1]^2}. \tag{1.6}$$

Proof. Let$f \in \mathcal{S}^*_\Sigma(\alpha, x, q)$. Then there are two analytic functions $u, v : \mathbb{D} \to \mathbb{D}$ given by

$$u(z) = u_1z + u_2z^2 + u_3z^3 + ...(z \in \mathbb{D}) \tag{1.7}$$

and

$$v(w) = v_1w + v_2w^2 + v_3w^3 + ...(w \in \mathbb{D}) \tag{1.8}$$

with $u(0) = v(0) = 0$ and $\max\{|u(z)|, |v(w)|\} < 1 \quad (z, w \in \mathbb{D})$, such that

$$\alpha\left(\frac{D_q(zD_qf(z))}{D_qf(z)}\right)+(1-\alpha)D_qf(z) \prec \Pi(x, u(z)) + 1 - a$$

$$\alpha\left(\frac{D_q(wD_qg(w))}{D_qg(w)}\right)+(1-\alpha)D_qg(w) \prec \Pi(x, v(w)) + 1 - a.$$

Equivalently

$$\begin{aligned}&\alpha\left(\frac{D_q(zD_qf(z))}{D_qf(z)}\right)+(1-\alpha)D_qf(z)\\ &= h_1(x)+h_2(x)u(z)+h_3(x)(u(z))^2+...+1-a\end{aligned} \tag{1.9}$$

and

$$\begin{aligned}&\alpha\left(\frac{D_q(wD_qg(w))}{D_qg(w)}\right)+(1-\alpha)D_qg(w)\\ &= h_1(x)+h_2(x)v(w)+h_3(x)(v(w))^2+...+1-a.\end{aligned} \tag{1.10}$$

Combining (1.7)-(1.10) yields the following relation:

$$\begin{aligned}&\alpha\left(\frac{D_q(zD_qf(z))}{D_qf(z)}\right)+(1-\alpha)D_qf(z)\\ &= 1+h_2(x)u_1z+[h_2(x)u_2+h_3(x)u_1^2]z^2+...\end{aligned} \tag{1.11}$$

and

$$\alpha\left(\frac{D_q(wD_qg(w))}{D_qg(w)}\right)+(1-\alpha)D_qg(w) = 1+h_2(x)v_1w+[h_2(x)v_2+h_3(x)v_1^2]w^2+.... \tag{1.12}$$

It is known that, if

$$\max\{|u(z)|,|v(w)|\}<1 \quad (z,w\in\mathbb{D}),$$

then

$$|u_j|\leq 1 \qquad \textit{and} \qquad |v_j|\leq 1 \qquad (\forall z\in\mathbb{N}). \tag{1.13}$$

Now, by comparing the corresponding coefficients in (1.11) and (1.12), we find that

$$[2]_q\left[\alpha([2]_q-2)+1\right]a_2=h_2(x)u_1 \tag{1.14}$$

$$[3]_q\left[\alpha([3]_q-2)+1\right]a_3-[2]_q^2\alpha([2]_q-1)a_2^2=h_2(x)u_2+h_3(x)u_1^2 \tag{1.15}$$

$$-[2]_q[\alpha([2]_q-2)+1]a_2=h_2(x)v_1 \tag{1.16}$$

$$-[3]_q[\alpha([3]_q-2)+1]a_3+\left\{2[3]_q(\alpha([3]_q-2)+1)-[2]_q^2([2]_q-1)\alpha\right\}a_2^2=h_2(x)v_2+h_3(x)v_1^2. \tag{1.17}$$

It follows from (1.14) and (1.16) that

$$u_1=-v_1, \qquad \textit{provided} \qquad h_2(x)=bx\neq 0 \tag{1.18}$$

and

$$a_2^2=\frac{1}{2[2]_q^2(\alpha([2]_q-2)+1)^2}(h_2(x))^2(u_1^2+v_1^2). \tag{1.19}$$

If we add (1.15) to (1.17), we find that

$$2\left[[3]_q(\alpha([3]_q-2)+1)-[2]_q^2([2]_q-1)\alpha\right]a_2^2=h_2(x)(u_2+v_2)+h_3(x)(u_1^2+v_1^2). \tag{1.20}$$

Upon substituting the value of $u_1^2+v_1^2$ from (1.19) into the right hand side of (1.20), we deduce the following result:

$$a_2^2=\frac{(h_2(x))^3(u_2+v_2)}{2\left\{(h_2(x))^2\left[[3]_q(\alpha([3]_q-2)+1)-[2]_q^2\alpha([2]_q-1)\right]-h_3(x)[2]_q^2(\alpha([2]_q-2)+1)^2\right\}}. \tag{1.21}$$

By further computations using (1.3), (1.13) and (1.21) we obtain

$$|a_2| \leq \frac{bx\sqrt{bx}}{\sqrt{\left|\left[[3]_q(\alpha([3]_q-2)+1)-[2]_q^2\alpha([2]_q-1)\right]b^2x^2-[2]_q^2(\alpha([2]_q-2)+1)^2(\rho bx^2+\sigma a)\right|}}.$$

Next, if we subtract (1.17) from (1.15), we can easily see that

$$2[3]_q\left[\alpha([3]_q-2)+1\right](a_3-a_2^2) = h_2(x)(u_2-v_2)+h_3(x)(u_1^2-v_1^2). \tag{1.22}$$

In the light of (1.18) and (1.19), we conclude from (1.22) that

$$a_3 = \frac{h_2(x)(u_2-v_2)}{2[3]_q(\alpha([3]_q-2)+1)} + \frac{(h_2(x))^2(u_1^2+v_1^2)}{2[2]_q^2(\alpha([2]_q-2)+1)^2}.$$

Thus, by applying (1.3), we obtain the following inequality

$$|a_3| \leq \frac{|bx|}{[3]_q|\alpha([3]_q-2)+1|} + \frac{b^2x^2}{[2]_q^2[\alpha([2]_q-2)+1]^2}.$$

In the next theorem, we present the Fekete-Szegö inequality for $\mathcal{S}^*_\Sigma(\alpha,x,q)$.

Theorem 1.5. *Let $f\in\mathcal{A}$ be in the class $\mathcal{S}^*_\Sigma(\alpha,x,q)$ and $\mu\in\mathbb{R}$, then*

$$|a_3-\mu a_2^2| \leq \begin{cases} \dfrac{|bx|}{[3]_q(\alpha([3]_q-2)+1)}, \\ \left(|\mu-1| \leq \dfrac{\left|\left[[3]_q(\alpha([3]_q-2)+1)-[2]_q^2\alpha([2]_q-1)\right]b^2x^2-[2]_q^2(\alpha([2]_q-2)+1)^2(\rho bx^2+\sigma a)\right|}{b^2x^2[3]_q(\alpha([3]_q-2)+1)}\right) \\ \dfrac{|bx|^3|\mu-1|}{\left|\left[[3]_q(\alpha([3]_q-2)+1)-[2]_q^2\alpha([2]_q-1)\right]b^2x^2-[2]_q^2(\alpha([2]_q-2)+1)^2(\rho bx^2+\sigma a)\right|} \\ \left(|\mu-1| \geq \dfrac{\left|\left[[3]_q(\alpha([3]_q-2)+1)-[2]_q^2\alpha([2]_q-1)\right]b^2x^2-[2]_q^2(\alpha([2]_q-2)+1)^2(\rho bx^2+\sigma a)\right|}{b^2x^2[3]_q(\alpha([3]_q-2)+1)}\right). \end{cases}$$

Proof. It follows from (1.21) and (1.22) that

$$a_3-\mu a_2^2 = \frac{h_2(x)(u_2-v_2)}{2[3]_q[\alpha([3]_q-2)+1]} + (1-\mu)a_2^2$$

$$= \frac{h_2(x)(u_2-v_2)}{2[3]_q[\alpha([3]_q-2)+1]}$$

$$+(1-\mu)\frac{(h_2(x))^3(u_2+v_2)}{2\left\{(h_2(x))^2\left[[3]_q(\alpha([3]_q-2)+1)-[2]_q^2\alpha([2]_q-1)\right]-h_3(x)[2]_q^2(\alpha([2]_q-2)+1)^2\right\}}$$

$$= \frac{h_2(x)}{2}\left[\left(\eta(\mu,x)+\frac{1}{[3]_q(\alpha([3]_q-2)+1)}\right)u_2 + \left(\eta(\mu,x)-\frac{1}{[3]_q(\alpha([3]_q-2)+1)}\right)v_2\right]$$

$$\eta(\mu, x) = \frac{(h_2(x))^2(1-\mu)}{(h_2(x))^2\left[[3]_q(\alpha([3]_q-2)+1) - [2]_q^2\alpha([2]_q-1)\right] - h_3(x)[2]_q^2(\alpha([2]_q-2)+1)^2}.$$

Thus, according to(1.3), we have the following inequality:

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{|bx|}{[3]_q(\alpha([3]_q-2)+1)}, & if \quad 0 \leq |\eta(\mu, x)| \leq \dfrac{1}{[3]_q(\alpha([3]_q-2)+1)} \\ |bx|.|\eta(\mu, x)| & if \quad |\eta(\mu, x)| \geq \dfrac{1}{[3]_q(\alpha([3]_q-2)+1)} \end{cases}$$

which, after simple computation, yields the following inequality:

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{|bx|}{[3]_q(\alpha([3]_q-2)+1)}, \\ \left(|\mu - 1| \leq \dfrac{\left|\left[[3]_q(\alpha([3]_q-2)+1) - [2]_q^2\alpha([2]_q-1)\right] b^2x^2 - [2]_q^2(\alpha([2]_q-2)+1)^2(\rho bx^2+\sigma a)\right|}{b^2x^2[3]_q(\alpha([3]_q-2)+1)}\right) \\ \dfrac{|bx|^3|\mu-1|}{\left|\left[[3]_q(\alpha([3]_q-2)+1) - [2]_q^2\alpha([2]_q-1)\right] b^2x^2 - [2]_q^2(\alpha([2]_q-2)+1)^2(\rho bx^2+\sigma a)\right|} \\ \left(|\mu - 1| \geq \dfrac{\left|\left[[3]_q(\alpha([3]_q-2)+1) - [2]_q^2\alpha([2]_q-1)\right] b^2x^2 - [2]_q^2(\alpha([2]_q-2)+1)^2(\rho bx^2+\sigma a)\right|}{b^2x^2[3]_q(\alpha([3]_q-2)+1)}\right) \end{cases}.$$

we have thus completed the proof of Theorem 1.5.

By putting $\mu = 1$ in Theorem 1.5, we are led to the following corollary.

Corollary 1.7. *Let $f \in \mathcal{A}$ be in the class $\mathcal{S}_\Sigma^*(\alpha, x, q)$, then*

$$|a_3 - a_2^2| \leq \frac{|bx|}{[3]_q(\alpha([3]_q-2)+1)}.$$

Remark 1.7.

(i) If we take $q \to 1$ in our Theorems, we have the corresponding results for the family $\mathcal{G}^*(\alpha, x)$ of bi-univalent functions which was considered by Orhan [7].

(ii) If we put $\alpha = 0$ and $q \to 1$ in our Theorems, we have the corresponding results for the family $\Sigma'(x)$,which was studied recently by Al-Amoush [2].

(iii) If we put $\alpha = 1$ and $q \to 1$ in our Theorems, we have the corresponding results for the family $\mathcal{K}_\sigma(x)$, which was discussed by Abirami [1].

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