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EXISTENCE AND UNIQUENESS OF FIXED POINT FOR NEW CONTRACTIONS IN RECTANGULAR b-METRIC SPACES

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Abstract: In this article, we give some new examples of rectangular b-metric spaces which are neither rectangular metric space nor metric space. After that we prove existence and uniqueness of new fixed points for some new contractions in rectangular b-metric spaces. Then we validate these results with suitable, appropriate and innovative examples.

Keywords and Phrases: Rectangular b-metric space, rectangular metric space, b-metric space, fixed point, contraction mapping.

2020 Mathematics Subject Classification: 54H25, 47H10.

1. Introduction

The basic result of fixed point theory of Banach contraction principle was given by Banach in 1922, which was extended in many ways. After this fundamental contraction principle, several generalized forms of metric spaces were introduced by various mathematicians (see [2], [3], [6], [7], [9]-[12]).

In 1989, Bakhtin [1] introduced b-metric space and Czerwik, S. [5] presented some generalizations of well known Banach's fixed point theorem in so-called b-metric spaces. He proved the following result: Let (X, d) be a complete b-metric space and let $T : X \to X$ satisfy

$$d(T\phi, T\psi) < \rho d(\phi, \psi), \phi, \psi \in X$$

where $\rho : R^+ \to R^+$ is increasing function such that $\lim_{n\to\infty} \rho^n(t) = 0$ for each $t > 0$. Then T has exactly one fixed point u and $\lim_{n\to\infty} d(T^n(\phi), u) = 0$.

In 2000, Brancieri [4] introduced the concept of rectangular metric space. After that George et al. [8] presented the notion of rectangular b-metric space in 2015.

In this paper we extend some well known fixed point theorems which are also valid in rectangular b-metric space. An analogue result of Vildan Öztürk [13] will be proved.

2. Preliminaries

Definition 2.1. [1] *Let X be a non empty set and $s \geq 1$ be a fixed real number. Suppose that a function $d : X \times X \to R^+$ satisfies the subsequent conditions:*

(bm_1) $d(u, v) = 0$ iff $u = v$ for all $u, v \in X$.

(bm_2) $d(u, v) = d(v, u)$ iff $u = v$ for all $u, v \in X$.

(bm_3) $d(u, v) \leq s[d(u, w) + d(w, v)]$ for all distinct points $u, v, w \in X$.

Then the pair (X, d) is termed a b-metric space.

Definition 2.2. [4] *Let X be a non empty set. Suppose that a function $d : X \times X \to R^+$ satisfies the subsequent conditions:*

(rm_1) $d(u, v) = 0$ iff $u = v$ for all $u, v \in X$.

(rm_2) $d(u, v) = d(v, u)$ iff $u = v$ for all $u, v \in X$.

(rm_3) $d(u, v) \leq d(u, w) + d(w, p) + d(p, v)$ for all distinct points $u, v, w, p \in X$.

Then the pair (X, d) is termed a rectangular metric space or generalized metric spaces.

Definition 2.3. [8] *Let X be a non empty set and $s \geq 1$ be a fixed real number. Suppose that a function $d : X \times X \to R^+$ satisfies the subsequent conditions:*

(rbm_1) $d(u, v) = 0$ iff $u = v$ for all $u, v \in X$.

(rbm_2) $d(u, v) = d(v, u)$ iff $u = v$ for all $u, v \in X$.

(rbm_3) $d(u, v) \leq s[d(u, w) + d(w, p) + d(p, v)]$ for all distinct points $u, v, w, p \in X$.

Then the pair (X, d) is termed a rectangular b-metric space.

It is worth noting that every metric space may be a b-metric space, but its converse is not always true.Also every metric space may be a rectangular metric space and every rectangular metric space may be a rectangular b-metric space (with coefficient $s = 1$). However the converse is not always true.

Example 2.4. Let $X = N$, define $d : X \times X \to R^+$ by

$$d(u,v)=\begin{cases}0, \text{ if } u=v\\ c\lambda, \text{ if } (u,v)\in\{4,5\} \text{ and } u\neq v\\ \lambda, \text{ if } u \text{ and } v \text{ do not belong to } \{4,5\} \text{ and } u\neq v,\end{cases}$$

where $\lambda > 0$ and $c > 3$.

Hence (X,d) is a rectangular b-metric space with coefficient $s = c/3 > 1$. But (X,d) is not a rectangular metric space and hence not a metric space, as

$$d(4,5) = c\lambda > 3\lambda = d(4,3) + d(3,2) + d(2,5).$$

Example 2.5. Let $X = N$, define $d : X \times X \to R^+$ by

$$d(u,v)=\begin{cases}0, \text{ if } u=v\\ 8\lambda, \text{ if } u=1,\ v=4\\ 3\lambda, \text{ if } (u,v)\in\{1,2,3\} \text{ and } u\neq v\\ \lambda, otherwise,\end{cases}$$

where $\lambda > 0$ is a constant.

Then (X,d) is a rectangular b-metric space with coefficient $s = 8/7 > 1$. But (X,d) is not a rectangular metric space and hence not a metric space, as

$$d(1,4) = 8\lambda > 7\beta = d(1,2) + d(2,3) + d(3,4).$$

3. Main Results

In this section, firstly we prove common fixed point theorem in complete rectangular b - metric space.

Theorem 3.1. *Let (X,d) be a complete rectangular b- metric space with $s > 1$, and let f, g be two self maps define onto itself such that*

$$d(fu, gv) \le \alpha M(u,v) + \beta N(u,v) \tag{3.1}$$

for all $u, v \in X$, where $\alpha \in [0, 1/s)$, $\beta \ge 0$ and

$M(u,v) = max\{d(u,v), d(v,fu), d(v,gv)\}$

$N(u,v) = min\{d(u,v), d(u,fu), d(u,gv), d(v,fu), d(v,gv)\}$

then f and g have a unique common fixed point.

Proof. Let u_0 be an arbitrary point in X. Define the sequence $\{u_n\}$ in X as $u_{2n+1} = fu_{2n}$ and $u_{2n+2} = gu_{2n+1}$ for $n \ge 1$.

Suppose that there is some $n \ge 1$ such that $u_n = u_{n+1}$.

If $n = 2k$, then $u_{2k} = u_{2k+1}$ and from (3.1),

$$d(u_{2k+1}, u_{2k+2}) = d(fu_{2k}, gu_{2k+1}) \le \alpha M(u_{2k}, u_{2k+1}) + \beta N(u_{2k}, u_{2k+1}) \tag{3.2}$$

where

$$\begin{aligned} M(u_{2k}, u_{2k+1}) &= max\{d(u_{2k}, u_{2k+1}), d(u_{2k+1}, fu_{2k}), d(u_{2k+1}, \mathsf{g}u_{2k+1})\} \\ &= max\{d(u_{2k}, u_{2k+1}), d(u_{2k+1}, u_{2k+1}), d(u_{2k+1}, u_{2k+2})\} \\ &= max\{0, 0, d(u_{2k+1}, u_{2k+2})\} \\ &= d(u_{2k+1}, u_{2k+2}) \end{aligned}$$

$$\begin{aligned} N(u_{2k}, u_{2k+1}) &= min\{d(u_{2k}, u_{2k+1}), d(u_{2k}, fu_{2k}), d(u_{2k}, \\ &\qquad \mathsf{g}u_{2k+1}), d(u_{2k+1}, fu_{2k}), d(u_{2k+1}, \mathsf{g}u_{2k+1})\} \\ &= min\{d(u_{2k}, u_{2k+1}), d(u_{2k}, u_{2k+1}), d(u_{2k}, u_{2k+2}), \\ &\qquad d(u_{2k+1}, u_{2k+1}), d(u_{2k+1}, u_{2k+2})\} \\ &= min\{0, 0, d(u_{2k}, u_{2k+2}), 0, d(u_{2k+1}, u_{2k+2})\} \\ &= 0. \end{aligned}$$

Thus we have from (3.2)

$$d(u_{2k+1}, u_{2k+2}) \leq \alpha d(u_{2k+1}, u_{2k+2}).$$

$$(1-\alpha) d(u_{2k+1}, u_{2k+2}) \leq 0.$$

but $1-\alpha \neq 0$ since $\alpha \in [0, 1/s)$
therefore

$$d(u_{2k+1}, u_{2k+2}) = 0.$$

Hence

$$u_{2k+1} = u_{2k+2} = u_{2k}.$$

$$u_{2k} = fu_{2k} = \mathsf{g}u_{2k}.$$

Hence u_{2k} is a common fixed point of f and g .

If $n = 2k+1$, then using same argument, it can be shown that u_{2k+1} is a common fixed point of f and g.

Now suppose that $u_n \neq u_{n+1}$ for all $n \geq 1$ and from (3.1),

$$d(u_{2n+1}, u_{2n+2}) = d(fu_{2n}, \mathsf{g}u_{2n+1}) \leq \alpha M(u_{2n}, u_{2n+1}) + \beta N(u_{2n}, u_{2n+1}). \tag{3.3}$$

where

$$\begin{aligned} M(u_{2n}, u_{2n+1}) &= max\{d(u_{2n}, u_{2n+1}), d(u_{2n+1}, fu_{2n}), d(u_{2n+1}, \mathsf{g}u_{2n+1})\} \\ &= max\{d(u_{2n}, u_{2n+1}), d(u_{2n+1}, u_{2n+1}), d(u_{2n+1}, u_{2n+2})\} \\ &= max\{d(u_{2n}, u_{2n+1}), 0, d(u_{2n+1}, u_{2n+2})\} \\ &= max\{d(u_{2n}, u_{2n+1}), d(u_{2n+1}, u_{2n+2})\} \end{aligned}$$

$$\begin{aligned}N(u_{2n},u_{2n+1}) &= min\{d(u_{2n},u_{2n+1}),d(u_{2n},fu_{2n}),d(u_{2n},\mathrm{g}u_{2n+1}),\\ &\qquad d(u_{2n+1},fu_{2n}),d(u_{2n+1},\mathrm{g}u_{2n+1})\}\\ &= min\{d(u_{2n},u_{2n+1}),d(u_{2n},u_{2n+1}),d(u_{2n},u_{2n+2}),\\ &\qquad d(u_{2n+1},u_{2n+1}),d(u_{2n+1},u_{2n+2})\}\\ &= min\{d(u_{2n},u_{2n+1}),d(u_{2n},u_{2n+2}),0,d(u_{2n+1},u_{2n+2})\}\\ &= 0.\end{aligned}$$

Case -(i): If $M= d(u_{2n+1},u_{2n+2})$ then from (3.3)

$$d(u_{2n+1},u_{2n+2})\leq \alpha d(u_{2n+1},u_{2n+2}).$$

$$(1-\alpha)d(u_{2n+1},u_{2n+2}) \leq 0$$

but $1-\alpha \neq 0$. Therefore $d(u_{2n+1},u_{2n+2}) \leq 0$
Case -(ii): If $M= d(u_{2n},u_{2n+1})$. Then we have from (3.3)

$$d(u_{2n+1},u_{2n+2})\leq \alpha d(u_{2n},u_{2n+1}).$$

In this way we extend and get

$$d(u_{2n+3},u_{2n+2})\leq \alpha d(u_{2n+2},u_{2n+1}).$$

So, for all $n \geq 1$

$$d(u_{n+1},u_n)\leq \alpha d(u_n,u_{n-1})\leq\alpha^2 d(u_{n-1},u_{n-2}) \leq ... \leq \alpha^n d(u_1,u_0)$$

Similarly we can show that $d(u_{n+2},u_n) \leq \alpha^n\ d(u_2,u_0)$ for all $n \geq 1$.

Now, we show that $\{u_n\}$ is a rectangular b-Cauchy sequence.

Applying rectangular b-inequality and $u_n \neq x_{n+1}$ for all $n \geq 1$ and $d_n = d(u_n,u_{n+1})$, $d_n^* = d(u_n,u_{n+2})$
For the sequence $\{u_n\}$, we consider $d(u_n,u_{n+p})$ in two cases.
If p is odd say $p = 2m+1$ then

$$\begin{aligned}d(u_n,u_{n+2m+1}) &\leq s[d(u_n,u_{n+1})+d(u_{n+1},u_{n+2})+d(u_{n+2},u_{n+2m+1})]\\ &\leq s[d_n+d_{n+1}]+s^2[d_{n+2}+d_{n+3}]+...+s^m d_{n+2m}\\ &\leq s[\alpha^n d_0+\alpha^{n+1}d_0]+s^2[\alpha^{n+2}d_0+\alpha^{n+3}d_0]+...+s^m\alpha^{n+2m}d_0\\ &\leq s\alpha^n(1+s\alpha^2+s^2\alpha^4+...)d_0+s\alpha^{n+1}(1+s\alpha^2+s^2\alpha^4+...)d_0\\ &= \frac{1+\alpha}{1-s\alpha^2}s\alpha^n d_0(s\alpha^2<1).\end{aligned}$$

Hence

$$d(x_n, x_{n+2m+1}) \leq \frac{1+\alpha}{1-s\alpha^2} s\alpha^n d_0. \tag{3.4}$$

In the same manner, we can show that when p is even

$$d(u_n, u_{n+2m}) \leq [(1+\alpha)/(1-s\alpha^2)]s\alpha^n d_0 + \alpha^{n-2} d_0 * . \tag{3.5}$$

On taking limit $n \to \infty$ of (3.4) and (3.5) we obtain
$d(u_n, u_{n+p})$=0 for all $p = 1, 2, 3$ ...
Hence $\{u_n\}$ is a rectangular b-Cauchy sequence in (X, d).
By completeness of (X, d), there exists $r \in X$ such that
$u_n = fu_{n-1} \to r$ as $n \to \infty$.

Now we show that $fr = r$.
By rectangular b-inequality

$$d(fr, r) \leq s[d(fr, \mathrm{g}u_n) + d(\mathrm{g}u_n, u_n) + d(u_n, r)]$$

$$\frac{1}{s} d(fr, r) \leq \alpha M(r, u_n) + \beta N(r, u_n) + d(u_{n+1}, u_n) + d(u_n, r) \tag{3.6}$$

where

$$\begin{aligned} M(r, u_n) &= max\{d(r, u_n), d(u_n, fr), d(u_n, \mathrm{g}u_n)\} \\ &= max\{d(r, u_n), d(u_n, fr), d(u_n, u_{n+2})\} \\ &= max\{0, d(u_n, fr), 0)\} \\ &= d(u_n, fr) \end{aligned}$$

$$\begin{aligned} N(r, u_n) &= min\{d(r, u_n), d(r, fr), d(u_n, \mathrm{g}u_n), d(r, \mathrm{g}u_n), d(u_n, fr)\} \\ &= min\{d(r, r), d(r, fr), d(u_n, \mathrm{g}r), d(r, \mathrm{g}r), d(u_n, fr)\} \\ &= 0. \end{aligned}$$

Putting these values in (3.6) we get

$$\begin{aligned} \frac{1}{s} d(fr, r) &\leq \alpha d(r, fr) \\ d(fr, r) &\leq s\alpha d(r, fr) \\ (1 - s\alpha) d(fr, r) &\leq 0. \end{aligned}$$

Since $1 - s\alpha \neq 0$ therefore $fr = r$. Hence r is a fixed point of f.

Now we show that r is a fixed point of g. Suppose $\mathsf{g}r \neq r$.

$$d(r, \mathsf{g}r) = d(fr, \mathsf{g}r) \leq \alpha M(r, r) + \beta N(r, r) \tag{3.7}$$

where

$$\begin{aligned} M(r, r) &= max\{d(r, r), d(r, fr), d(r, \mathsf{g}r)\} \\ &= max\{0, 0, d(r, gr)\} \\ &= d(r, gr) \end{aligned}$$

$$\begin{aligned} N(r, r) &= min\{d(r, r), d(r, fr), d(r, \mathsf{g}r), d(r, fr), d(r, \mathsf{g}r)\} \\ &= min\{0, 0, d(r, \mathsf{g}r), 0, d(r, \mathsf{g}r)\} \\ &= 0. \end{aligned}$$

Now putting these values in (3.7) we have

$$\begin{aligned} d(r, \mathsf{g}r) &\leq \alpha d(r, \mathsf{g}r) \\ (1 - \alpha) d(r, \mathsf{g}r) &\leq 0. \end{aligned}$$

But $(1 - \alpha) \nleq 0$ since $\alpha \in [0, 1/s)$. Thus gr = r. Hence

$$fr = \mathsf{g}r = r.$$

Therefore f and g have a common fixed point of X.
Now suppose p and q are two common fixed points.
Then clearly $p = fp$ and $q = \mathsf{g}q$, then

$$d(p, q) = d(fp, \mathsf{g}q) \leq \alpha M(p, q) + \beta N(p, q) \tag{3.8}$$

where

$$\begin{aligned} M(p, q) &= max\{d(p, q), d(q, fp), d(q, \mathsf{g}q)\} \\ &= max\{d(p, q), d(q, p), d(q, q)\} \\ &= d(p, q) \end{aligned}$$

$$\begin{aligned} N(p, q) &= min\{d(p, q), d(p, fp), d(p, \mathsf{g}q), d(q, fp), d(q, \mathsf{g}q)\} \\ &= min\{d(p, q), d(p, p), d(p, q), d(q, p), d(q, q)\} \\ &= 0. \end{aligned}$$

On putting these values in (3.8) we get

$$d(p,g) \leq \alpha d(p,q)$$
$$(1-\alpha)d(p,q) \leq 0$$
$$\text{but } (1-\alpha) \neq 0 \text{ thus } d(p,q) = 0$$
$$\text{Hence } p = q.$$

Corollary 3.1. *Let (X,d) be a complete rectangular b-metric space with $s > 1$, and let $f : X \to X$, $g : X \to X$ be two self maps satisfying*

$$d(fu, gv) \leq \alpha M(u,v) + \beta N(u,v)$$

for all $u, v \in X$, where $\alpha \in [0, 1/s)$ and $\beta \geq 0$ and
$M(u,v)=$ max $\{d(u,v), d(v,gv)\}$
$N(u,v)=$ min $\{d(u,v), d(u,fu), d(u,gv)\}$.
Then f and g have a unique common fixed point.

Corollary 3.2. *Let (X,d) be a complete rectangular b-metric space with $s > 1$, and let $f : X \to X$, $g : X \to X$ be two self maps satisfying*

$$d(fu, gv) \leq \alpha M(u,v) + \beta N(u,v)$$

for all $u, v \in X$,where $\alpha \in [0, 1/s)$ and $\beta \geq 0$ and
$M(u,v)=$ max $\{d(u,v), d(v,gv)\}$
$N(u,v)=$ min $\{d(u,fu), d(u,gv)\}$.
Then f and g have a unique common fixed point.
Now, we prove another common fixed point theorem with new contraction mapping in complete rectangular b- metric spaces.

Theorem 3.2. *Let (X,d) be a complete rectangular b-metric space with $s > 1$, and let $f : X \to X$, $g : X \to X$ be two self maps satisfying*

$$d(fu, gv) \leq \alpha M(u,v) + \beta N(u,v) \tag{3.9}$$

for all $u, v \in X$,where $\alpha \in [0, 1/s)$ and $\beta \geq 0$ and
$M(u,v)=$ max $\{d(u,v), \frac{d(u,gv)d(v,fu)}{1+d(fu,gv)}\}$
$N(u,v)=$ min $\{d(u,v), d(v,fu), d(u,gv)\}$.
Then f and g have a unique common fixed point.

Proof. Taking u_0 be an arbitrary point in X. Define the sequence $\{u_n\}$ in X as $u_{2n+1} = fu_{2n}$ and $u_{2n+2} = gu_{2n+1}$ for $n \geq 1$.
Assume that there is some $n \geq 1$ such that $u_n = u_{n+1}$.
If $n = 2k$, then $u_{2k} = u_{2k+1}$ and from (3.9),

$$d(u_{2k+1}, u_{2k+2}) = d(fu_{2k}, gu_{2k+1}) \leq \alpha M(u_{2k}, u_{2k+1}) + \beta N(u_{2k}, u_{2k+1}) \tag{3.10}$$

where

$$\begin{aligned} M(u_{2k},u_{2k+1}) &= max\{d(u_{2k},u_{2k+1}), \frac{d(u_{2k},\mathrm{g}u_{2k+1})d(u_{2k+1},fu_{2k})}{1+d(fu_{2k},\mathrm{g}u_{2k+1})}\} \\ &= max\{d(u_{2k},u_{2k+1}), \frac{d(u_{2k},u_{2k+2})d(u_{2k+1},u_{2k+1})}{1+d(u_{2k+1},u_{2k+2})}\} \\ &= max\{0, \frac{0}{1+d(u_{2k+1},u_{2k+2})}\} \\ &= 0 \end{aligned}$$

$$\begin{aligned} N(u_{2k},u_{2k+1}) &= min\{d(u_{2k},u_{2k+1}), d(u_{2k+1},fu_{2k}), d(u_{2k},\mathrm{g}u_{2k+1}),\} \\ &= min\{0, d(u_{2k+1},u_{2k+1}), d(u_{2k},u_{2k+2})\} \\ &= min\{0,0,d(u_{2k},u_{2k+2})\} \\ &= 0. \end{aligned}$$

Now putting these values in (3.9)

$$d(u_{2k+1},u_{2k+2}) \leq 0$$

Therefore

$$u_{2k+1} = u_{2k+2}.$$

Thus

$$u_{2k} = u_{2k+1} = u_{2k+2}.$$

$$u_{2k} = fu_{2k} = \mathrm{g}u_{2k}.$$

Hence u_{2k} is a common fixed point of f and g.

Now suppose that $u_n \neq u_{n+1}$ for all $n \geq 1$. Then from (3.9),

$$d(u_{2n+1},u_{2n+2}) = d(fu_{2n},\mathrm{g}u_{2n+1}) \leq \alpha M(u_{2n},u_{2n+1}) + \beta N(u_{2n},u_{2n+1}) \quad (3.11)$$

where

$$\begin{aligned} M(u_{2n},u_{2n+1}) &= max\{d(u_{2n},u_{2n+1}), \frac{d(u_{2n},\mathrm{g}u_{2n+1})d(u_{2n+1},fu_{2n})}{1+d(fu_{2n},\mathrm{g}u_{2n+1})}\} \\ &= max\{d(u_{2n},u_{2n+1}), \frac{d(u_{2n},u_{2n+2})d(u_{2n+1},u_{2n+1})}{1+d(u_{2n+1},u_{2n+2})}\} \\ &= max\{d(u_{2n},u_{2n+1}), 0\} \\ &= d(u_{2n},u_{2n+1}) \end{aligned}$$

$$\begin{aligned} N(u_{2n}, u_{2n+1}) &= min\{d(u_{2n}, u_{2n+1}), d(u_{2n+1}, fu_{2n}), d(u_{2n}, \mathsf{g}u_{2n+1})\} \\ &= min\{d(u_{2n}, u_{2n+1}), d(u_{2n+1}, u_{2n+1}), d(u_{2n}, u_{2n+2})\} \\ &= min\{d(u_{2n}, u_{2n+1}), 0, d(u_{2n}, u_{2n+2})\} \\ &= 0. \end{aligned}$$

Putting these values in (3.11)

$$d(u_{2n+1}, u_{2n+2}) \leq \alpha d(u_{2n}, u_{2n+1}).$$

Similarly

$$d(u_{2n+2}, u_{2n+3}) \leq \alpha d(u_{2n+1}, u_{2n+2}).$$

So, for all $n \geq 1$

$$d(u_{n+1}, u_n) \leq \alpha d(u_n, u_{n-1}) \leq \alpha^2 d(u_{n-1}, u_{n-2}) \leq ... \leq \alpha^n d(u_1, u_0).$$

In the same manner we can show that for all $n \geq 1$

$$d(u_{n+2}, u_n) \leq \alpha^n d(u_2, u_0).$$

Now we say that $\{u_n\}$ is a rectangular b-Cauchy sequence. Applying rectangular b-inequality and $u_n \neq u_{n+1}$ for all $n \geq 1$ and

$$d_n = d(u_n, u_{n+1}), d_n^* = d(u_n, u_{n+2}).$$

For the sequence $\{u_n\}$, we consider $d(u_n, u_{n+p})$ in two cases.

If p is odd say $p = 2m + 1$ then

$$\begin{aligned} d(u_n, u_{n+2m+1}) &\leq s[d(u_n, u_{n+1}) + d(u_{n+1}, u_{n+2}) + d(u_{n+2}, u_{n+2m+1})] \\ &\leq s[d_n + d_{n+1}] + s^2[d_{n+2} + d_{n+3}] + ... + s^m d_{n+2m} \\ &\leq s[\alpha^n d_0 + \alpha^{n+1} d_0] + s^2[\alpha^{n+2} d_0 + \alpha^{n+3} d_0] + ... + s^m \alpha^{n+2m} d_0 \\ &\leq s\alpha^n (1 + s\alpha^2 + s^2\alpha^4 + ...)d_0 + s\alpha^(n+1)(1 + s\alpha^2 + s^2\alpha^4 + ...)d_0 \\ &= \frac{1+\alpha}{1-s\alpha^2} s\alpha^n d_0 (s\alpha^2 < 1). \end{aligned}$$

Hence $d(u_n, u_{n+2m+1}) \leq \frac{1+\alpha}{1-s\alpha^2} s\alpha^n d_0$.
In the same manner, when p is even then we can show

$$d(u_n, u_{n+2m}) \leq \frac{1+\alpha}{1-s\alpha^2}(s\alpha^n d_0) + \alpha^{n-2} d_0 * .$$

On taking limit $n \to \infty$ we get

$$d(u_n, u_{n+p}) = 0$$

for all $p = 1, 2, 3, ...$ Hence $\{u_n\}$ is a rectangular b-Cauchy sequence in (X, d). By completeness of (X, d), there exists $r \in X$ such that
$u_n = fu_{n-1} \to r$ as $n \to \infty$.

Now we prove that $fr = r$. By rectangular b-inequality,

$$d(fr, r) \leq s[d(fr, \mathrm{g}u_n) + d(\mathrm{g}u_n, u_n) + d(u_n, r)]$$

$$1/sd(fr, r) \leq \alpha M(r, u_n) + \beta N(r, u_n) + d(u_{n+1}, u_n) + d(u_n, r) \tag{3.12}$$

where

$$\begin{aligned} M(r, u_n) &= max\{d(r, u_n), \frac{d(r, \mathrm{g}u_n)d(u_n, fr)}{1 + d(fr, \mathrm{g}u_n)}\} \\ &= max\{d(r, u_n), \frac{d(r, u_{n+1})d(u_n, fr)}{1 + d(fr, u_{n+1})}\} \\ &= max\{0, 0\} \\ &= 0 \end{aligned}$$

since $u_n = fu_{n-1} \to r$ as $n \to \infty$
and
$N(r, u_n) = \min\ \{d(r, u_n), d(u_n, fr), d(r, \mathrm{g}u_n)\} \to 0$ as $n \to \infty$.

Now putting these values in (3.12) we get
$\frac{1}{s}\ d(fr, r) \to 0$ as $n \to \infty$ implies that $fr = r$.
Hence r is a fixed point of f.

Now we have to show that $\mathrm{g}r = r$. Suppose $\mathrm{g}r \neq r$.

$$d(r, \mathrm{g}r) = d(fr, gr) \leq \alpha M(r, r) + \beta N(r, r) \tag{3.13}$$

where

$$\begin{aligned} M(r, r) &= max\{d(r, r), \frac{d(r, \mathrm{g}r)d(r, fr)}{1 + d(fr, \mathrm{g}r)}\} \\ &= max\{0, \frac{d(r, \mathrm{g}r)d(r, r)}{1 + d(fr, \mathrm{g}r)}\} \\ &= 0 \\ N(r, r) &= min\{d(r, r), d(r, fr), d(r, \mathrm{g}r)\} \\ &= min\{0, d(r, fr), d(r, \mathrm{g}r)\} \\ &= 0 \end{aligned}$$

Thus

$$d(r, \mathsf{g}r) \leq 0.$$

Therefore $\mathsf{g}r = r$
Hence $fr = \mathsf{g}r = r$.
Therefore f and g have a common fixed point.

Suppose that r and r_1 are two different common fixed points. Then

$$d(r, r_1) = d(fr, \mathsf{g}r_1) \leq \alpha M(r, r_1) + \beta N(r, r_1) \tag{3.14}$$

where

$$\begin{aligned} M(r, r_1) &= max\{d(r, r_1), \frac{d(r, \mathsf{g}r_1)d(r_1, fr)}{1 + d(fr, \mathsf{g}r_1)}\} \\ &= max\{d(r, r_1), \frac{d(r, r_1)d(r_1, r)}{1 + d(r, r_1)}\} \\ &= d(r, r_1) \\ N(r, r_1) &= min\{d(r, r_1), d(r, \mathsf{g}r_1), d(r_1, fr)\} \\ &= min\{d(r, r_1), d(r, r_1), d(r_1, r)\} \\ &= d(r, r_1). \end{aligned}$$

Now putting these values in (3.14) we get

$$\begin{aligned} d(r, r_1) &\leq \alpha d(r, r_1) + \beta d(r, r_1) \\ &\leq (\alpha + \beta) d(r, r_1) \\ d(r, r_1)(1 - \alpha - \beta) &\leq 0. \end{aligned}$$

Since $1 - \alpha - \beta \neq 0$, therefore $d(r, r_1) = 0$
Hence, $r = r_1$.
Example 3.1. Let X=$[1, 2] \bigcup A$, where $A = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}\}$ and B=$[1, 2]$.
Define $d : X * X \to [0, \infty)$ such that $d(u, v) = d(v, u)$ for all $u, v \in X$ and

$$\begin{cases} d\left(\frac{1}{2}, \frac{1}{3}\right) & = d\left(\frac{1}{4}, \frac{1}{6}\right) = 0.16 \\ d\left(\frac{1}{2}, \frac{1}{6}\right) & = d\left(\frac{1}{3}, \frac{1}{6}\right) = 0.25 \\ d\left(\frac{1}{2}, \frac{1}{4}\right) & = d\left(\frac{1}{3}, \frac{1}{5}\right) = 0.09 \\ d\left(\frac{1}{3}, \frac{1}{4}\right) & = d\left(\frac{1}{5}, \frac{1}{6}\right) = 0.04 \\ d\left(x, y\right) & = |u - v|^2 \qquad \text{otherwise} \end{cases}$$

Now, we see that

$$\begin{aligned} d\Big(1,\frac{1}{5}\Big) &> d\Big(1,\frac{1}{2}\Big)+d\Big(\frac{1}{2},\frac{1}{3}\Big)+d\big(\frac{1}{3},\frac{1}{5}\big)\\ \frac{16}{25} &> \frac{1}{4}+0.16+0.09\\ .64 &> .25+0.25\\ .64 &> .5\\ \therefore\ \ s=1.28 &> 1. \end{aligned}$$

Therefore (X,d) is neither a metric space nor a rectangular metric space. But (X,d) is a rectangular b-metric space with coefficient $s=1.28>1$.

Let $f:X\to X$ and $\mathsf{g}:X\to X$ be defined as

$$f(u)=\begin{cases}\frac{1}{4} & \text{if } u\in A\\ \frac{1}{5} & \text{if } u\in B\end{cases}$$

$$\mathsf{g}(u)=\begin{cases}\frac{1}{4} & \text{if } u\in A\\ \frac{1}{6} & \text{if } u\in B\end{cases}$$

Then f and g satisfy all conditions of Theorem 3.2 with $\alpha\in[0,\frac{1}{s})$ and $\beta\geq 0$ and has a unique fixed point $u=\frac{1}{4}$.

4. Conclusion

In this article we have given two fixed point theorems and some corollaries for existence and uniqueness of new fixed points in rectangular b-metric spaces and we proved them with suitable examples. We have made some new examples which are neither metric spaces nor rectangular metric spaces but they are rectangular b-metric spaces with coefficient $s>1$.

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