South East Asian J. of Mathematics and Mathematical Sciences
Vol. 18, No. 3 (2022), pp. 123-138

DOI: 10.56827/SEAJMMS.2022.1803.10 **ISSN (Online): 2582-0850**

ISSN (Print): 0972-7752

SOME PROPERTIES OF FRACTIONAL HARTLEY TRANSFORM

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(**Received:** *May 23, 2021* **Accepted:** *Dec. 15, 2022* **Published:** *Dec. 30, 2022)*

Abstract: This paper is motivated by the ideas of fractional Fourier transform and Hartley transform. Looking towards the practicality and demanding attention of fractional Hartley transform we take keen interest into it. In this paper, we deal with inverse theorem of FRHT and some important properties of fractional Hartley transform like exponential rule, multiplication rule, transform of derivative and derivative of transform, which play a very crucial role in the development of fractional Hartley transform.

Keywords and Phrases: Fourier transform, fractional Fourier transform (FRFT), Hartley transform, fractional Hartley transform (FRHT) and derivative.

2020 Mathematics Subject Classification: 44A05, 42A38, 42B10, 46F12.

1. Introduction and Preliminaries

The fractional Hartley transform is an extension of classical Hartley transform. Moreover, fractional Hartley transform is very closely related to fractional Fourier transform. In the year 1980, the fractional Fourier transform was introduced by V. Namias to solve some type of ordinary and partial differential equation arising in quantum mechanics [10]. The fractional Fourier transform is used in optical propagation problems [2], time-frequency representations [3], along with this it has numerous uses in science [4, 5, 8, 10, 11]. In 1998, a new definition of fractional Hartley transform was introduced by S. C. Pei, C. C. Tseng, M. H. Yeh and J. J. Ding, which obeys additive property and precisely makes a desirable impact on this paper [12]. Image encryption technology is a very significant research topic in the area of information security [16]. A recently developed method for image encryption is established based on two-dimensional generalization of one-dimensional fractional Hartley transform [6]. It is seen that, fractional Hartley transform has many applications in the field of phase image encryption [15], nonlinear optical double image encryption [14], and optical image encryption [6, 7, 18]. In many applications of engineering and science the concept of fractional operator and measure have been investigated comprehensively. In [17], a new idea of fractional quantum calculus is defined. A fractional calculus approach is used [9] to study the analytic solution for oxygen diffusion from capillary to tissues involving external force effects. Because of plenty amount of applications, image formula of fractional calculus operators have impressed not only statisticians and mathematicians with multiple research interest but also biologist, psychologist, electrical engineers etc. [1]. For pattern recognition and classification, the fractional dimension is being used extensively. However, for image compression and adaptive filtering various unitary transforms have been commonly applied except the Fourier transform. Some common ones are cosine transform, sine transform, and Hartley transform etc. So far, the fractional version of these transforms has been generalized. The fractional Hartley transform is applicable in the various technical fields like electrical power system, oceanography, and electronic communication. Preliminaries part deals with definition and some properties of fractional Fourier transform as it forms a concrete base for fractional Hartley transform. Further, the definition of fractional Hartley transform along with relation between kernel of fractional Hartley transform and kernel of fractional Fourier transform is stated. Main result contends with inverse fractional Hartley transform, some properties of fractional Hartley transform like exponential rule, multiplication rule, fractional Hartley transform of derivative and derivative of fractional Hartley transform.

Definition 1.1. [2] *If $t,\ v,\ \alpha \in \mathbb{R}$, with α is a constant and $h \in \mathrm{L}^1(\mathbb{R}) \cap \mathrm{C}^1(\mathbb{R})$, then fractional Fourier transform are denoted by $R^\alpha[h(t)](v)$ or $F_\alpha(v)$ or $g_F^\alpha(v)$ and is defined by*

$$R^\alpha[h(t)](v) = F_\alpha(v) = g_F^\alpha(v) = \int_{-\infty}^{\infty} h(t) K_F^\alpha(t,\ v) dt,$$

where

$$K_F^{\alpha}(t,\ v) = \sqrt{\frac{1 - i\cot\psi}{2\pi}}\ \exp\left\{i\left[\frac{1}{2}(t^2+v^2)\cot\psi - tv\csc\psi\right]\right\},$$

with $\psi = \frac{\alpha\pi}{2}$ *and* $\psi \neq \pi n$, *for all* $n = 0, 1, 2, \ldots$.

Theorem 1.2. [4, 5] *If* $R^{\alpha}[h(t)](v) = F_{\alpha}(v) = g_F^{\alpha}(v)$ *is the fractional Fourier transform of* $h(t)$, *then* $h(t)$ *is given by*

$$h(t) = \int_{-\infty}^{\infty} \overline{K_F^{\alpha}(t,\ v)} g_F^{\alpha}(v) dv,$$

where

$$\overline{K_F^{\alpha}(t,\ v)} = \sqrt{\frac{1 + i\cot\psi}{2\pi}}\ \exp\left\{-i\left[\frac{1}{2}(t^2+v^2)\cot\psi - tv\csc\psi\right]\right\},$$

with $\psi = \frac{\alpha\pi}{2}$ *and* $\psi \neq \pi n$, *for all* $n = 0, 1, 2, \ldots$.

The proof of properties 1.3 – 1.6 becomes obvious using the results from [2], [3], [8], [11], [12] and [13]. In the subsequent work we shall use following properties of fractional Fourier transform.

Property 1.3. Exponential rule

If $t,\ v, \alpha, b \in \mathbb{R}$, *with* α *and* b *are constants and* $h \in \mathrm{L}^1(\mathbb{R}) \cap \mathrm{C}^1(\mathbb{R})$, *then the exponential rule for fractional Fourier transform is given by*

$$F_{\alpha}[e^{ibt}h(t)](v) = e^{ib\cos\alpha\left(v - \frac{b\sin\alpha}{2}\right)} F_{\alpha}[h(t)](v - b\sin\alpha).$$

Property 1.4. Multiplication Rule

If $t,\ v,\ \alpha \in \mathbb{R}$, *with* α *is a constant and* $h \in \mathrm{L}^1(\mathbb{R}) \cap \mathrm{C}^1(\mathbb{R})$, *then the multiplication rule for fractional Fourier transform is given by*

$$F_{\alpha}[t^m h(t)](v) = \left(v\cos\alpha + i\sin\alpha\frac{d}{dv}\right)^m F_{\alpha}[h(t)](v),$$

where m *is an positive integer.*

Property 1.5. Transform of derivative

If $t,\ v,\ \alpha \in \mathbb{R}$, *with* α *is a constant and* $h \in \mathrm{L}^1(\mathbb{R}) \cap \mathrm{C}^1(\mathbb{R})$, *then the differentiation rule for fractional Fourier transform is given by*

$$F_{\alpha}\left[\frac{d^m}{dt^m}\Big(h(t)\Big)\right](v) = \left(iv\sin\alpha + \cos\alpha\frac{d}{dv}\right)^m F_{\alpha}[h(t)](v),$$

where m is an positive integer and $h(t)$ is differentiable and vanishes at $t \to \pm\infty$.

Property 1.6. Derivative of the Transform
If $t,\ v,\ \alpha \in \mathbb{R}$, with α is a constant and $h \in \mathrm{L}^1(\mathbb{R}) \cap \mathrm{C}^1(\mathbb{R})$, then the derivative of the fractional Fourier transform is obtained as follows

$$\frac{d}{dv}\Big\{ g_F^\alpha[h(t)](v) \Big\} = iv\cot\psi\ g_F^\alpha[h(t)](v) + \left(\frac{1+e^{i\psi}}{2}\right) g_F^\alpha[-itv\csc\psi h(t)](v),$$

where $g_F^\alpha[h(t)](v)$ is the fractional Fourier transform of $h(t)$ and $\psi = \frac{\alpha\pi}{2}$ and $\psi \neq \pi n$, for all $n = 0, 1, 2, \ldots$.

Definition 1.7. [8] *If $t,\ v,\ \alpha \in \mathbb{R}$, with α is a constant and $h \in \mathrm{L}^1(\mathbb{R}) \cap \mathrm{C}^1(\mathbb{R})$, then the FRHT of $h(t)$ is denoted by $g_H^\alpha[h(t)](v)$ or $g_H^\alpha(v)$ and defined as*

$$g_H^\alpha[h(t)](v) = g_H^\alpha(v) = \int_{-\infty}^{\infty} K_H^\alpha(t,v)h(t)dt, \tag{1}$$

where

$$K_H^\alpha(t,\,v) = \sqrt{\frac{1-i\cot\psi}{2\pi}}\ e^{i\frac{1}{2}(t^2+v^2)\cot\psi}\big[\cos(tv\csc\psi) + e^{i(\psi-\frac{\pi}{2})}\sin(tv\csc\psi)\big]$$

and $\psi = \frac{\alpha\pi}{2}$ if $\psi \neq \pi n$; for all $n = 0, 1, 2, \ldots$

Note 1.8. *If $\psi = \frac{\pi}{2}$, then the extended transform defined in definition 1.7 reduces to Hartley transform.*

Result 1.9. [12] *The correlation between kernel of FRHT and kernel of FRFT is written by*

$$K_H^\alpha(t,v) = \left[\frac{1+e^{\frac{i\alpha\pi}{2}}}{2}\right] K_F^\alpha(t,v) + \left[\frac{1-e^{\frac{i\alpha\pi}{2}}}{2}\right] K_F^\alpha(t,-v). \tag{2}$$

2. Main Results

Result 2.1. *If a kernel of FRFT is denoted by $K_F^\alpha(t,v)$ and a kernel of FRHT is denoted by $K_H^\alpha(t,v)$, then*

$$K_F^\alpha(t,v) = \left[\frac{1+e^{\frac{-i\alpha\pi}{2}}}{2}\right] K_H^\alpha(t,v) + \left[\frac{1-e^{\frac{-i\alpha\pi}{2}}}{2}\right] K_H^\alpha(t,-v).$$

The proof becomes obvious by using definition of kernel of fractional Fourier transform and kernel of fractional Hartley transform.

Result 2.2. *If $h \in \mathrm{L}^1(\mathbb{R}) \cap \mathrm{C}^1(\mathbb{R})$, the FRFT of $h(t)$ is denoted by $g_F^\alpha(v)$ and FRHT is denoted by $g_H^\alpha(v)$, then*

$$g_H^\alpha(v) = \left[\frac{1 + e^{\frac{i\alpha\pi}{2}}}{2}\right] g_F^\alpha(v) + \left[\frac{1 - e^{\frac{i\alpha\pi}{2}}}{2}\right] g_F^\alpha(-v).$$

The proof is obtained by using Result 1.9 and linearity of integration.

Result 2.3. *If $h \in \mathrm{L}^1(\mathbb{R}) \cap \mathrm{C}^1(\mathbb{R})$, the FRFT of $h(t)$ is denoted by $g_F^\alpha(v)$ and FRHT is denoted by $g_H^\alpha(v)$, then*

$$g_F^\alpha(v) = \left[\frac{1 + e^{\frac{-i\alpha\pi}{2}}}{2}\right] g_H^\alpha(v) + \left[\frac{1 - e^{\frac{-i\alpha\pi}{2}}}{2}\right] g_H^\alpha(-v).$$

The proof is obtained by using Result 2.1 and linearity of integration.

Theorem 2.4. Inverse fractional Hartley transform.
If $h \in \mathrm{L}^1(\mathbb{R}) \cap \mathrm{C}^1(\mathbb{R})$; $t,\ v,\ \alpha \in \mathbb{R}^n$, where α is a constant, $g_H^\alpha(v)$ is the FRHT of $h(t)$, then $h(t)$ is as follows

$$h(t) = \int_{-\infty}^{\infty} \overline{K_H^\alpha(t,\ v)} g_H^\alpha(v) dt,$$

where

$$\overline{K_H^\alpha(t,\ v)} = \sqrt{\frac{1 + i\cot\psi}{2\pi}}\ e^{-i\frac{t^2+v^2}{2}\cot\psi}\left[\cos(tv\csc\psi) + e^{-i(\psi - \frac{\pi}{2})}\sin(tv\csc\psi)\right]$$

and $\psi = \frac{\alpha\pi}{2}$ if $\psi \neq \pi n$; for all $n = 0, 1, 2, \ldots$.

Proof. By using Theorem 1.2 and Result 2.3, we have

$$\begin{aligned}
h(t) &= \int_{-\infty}^{\infty} \overline{K_F^\alpha(t,\ v)} g_F^\alpha(v) dv \\
&= \int_{-\infty}^{\infty} \overline{K_F^\alpha(t,\ v)} \left[\left(\frac{1 + e^{\frac{-i\alpha\pi}{2}}}{2}\right) g_H^\alpha(v) + \left(\frac{1 - e^{\frac{-i\alpha\pi}{2}}}{2}\right) g_H^\alpha(-v)\right] dv \\
&= \int_{-\infty}^{\infty} \overline{K_F^\alpha(t,\ v)} \left(\frac{1 + e^{\frac{-i\alpha\pi}{2}}}{2}\right) g_H^\alpha(v) dv \\
&\quad + \int_{-\infty}^{\infty} \overline{K_F^\alpha(t,\ v)} \left(\frac{1 - e^{\frac{-i\alpha\pi}{2}}}{2}\right) g_H^\alpha(-v) dv \\
&= \int_{-\infty}^{\infty} \overline{K_F^\alpha(t,\ v)} \left(\frac{1 + e^{\frac{-i\alpha\pi}{2}}}{2}\right) g_H^\alpha(v) dv \\
&\quad + \int_{-\infty}^{\infty} \overline{K_F^\alpha(t,\ -v)} \left(\frac{1 - e^{\frac{-i\alpha\pi}{2}}}{2}\right) g_H^\alpha(v) dv.
\end{aligned}$$

That is

$$h(t) = \int_{-\infty}^{\infty} \left[\overline{K_F^\alpha(t,\ v)} \left(\frac{1 + e^{\frac{-i\alpha\pi}{2}}}{2} \right) + \overline{K_F^\alpha(t,\ -v)} \left(\frac{1 - e^{\frac{-i\alpha\pi}{2}}}{2} \right) \right] g_H^\alpha(v) dv.$$

Consider

$$\begin{aligned}
&\overline{K_F^\alpha(t,\ v)} \left(\frac{1 + e^{\frac{-i\alpha\pi}{2}}}{2} \right) + \overline{K_F^\alpha(t,\ -v)} \left(\frac{1 - e^{\frac{-i\alpha\pi}{2}}}{2} \right) \\
=& \left(\frac{1 + e^{\frac{-i\alpha\pi}{2}}}{2} \right) \sqrt{\frac{1 + i \cot \psi}{2\pi}} \ \exp\left\{ -i \left[\frac{1}{2}(t^2 + v^2) \cot \psi - tv \csc \psi \right] \right\} \\
&\quad + \left(\frac{1 - e^{\frac{-i\alpha\pi}{2}}}{2} \right) \sqrt{\frac{1 + i \cot \psi}{2\pi}} \ \exp\left\{ -i \left[\frac{1}{2}(t^2 + v^2) \cot \psi + tv \csc \psi \right] \right\} \\
=& \sqrt{\frac{1 + i \cot \psi}{2\pi}} \ e^{-i\frac{t^2+v^2}{2} \cot \psi} \left[\left(\frac{e^{itv \csc \psi} + e^{-itv \csc \psi}}{2} \right) \right. \\
&\quad \left. + ie^{\frac{-i\alpha\pi}{2}} \left(\frac{e^{itv \csc \psi} - e^{-itv \csc \psi}}{2i} \right) \right] \\
=& \sqrt{\frac{1 + i \cot \psi}{2\pi}} \ e^{-i\frac{t^2+v^2}{2} \cot \psi} \left[\cos(tv \csc \psi) + e^{\frac{i\pi}{2}} e^{-i\psi} \sin(tv \csc \psi) \right] \\
=& \sqrt{\frac{1 + i \cot \psi}{2\pi}} \ e^{-i\frac{t^2+v^2}{2} \cot \psi} \left[\cos(tv \csc \psi) + e^{-i(\psi - \frac{\pi}{2})} \sin(tv \csc \psi) \right].
\end{aligned}$$

Therefore

$$\begin{aligned}
h(t) = \int_{-\infty}^{\infty} & \sqrt{\frac{1 + i \cot \psi}{2\pi}} \ e^{-i\frac{t^2+v^2}{2} \cot \psi} \left[\cos(tv \csc \psi) + e^{-i(\psi - \frac{\pi}{2})} \sin(tv \csc \psi) \right] \\
& g_H^\alpha(v) dv.
\end{aligned}$$

Hence

$$h(t) = \int_{-\infty}^{\infty} \overline{K_H^\alpha(t,\ v)} g_H^\alpha(v) dv,$$

where

$$\overline{K_H^\alpha(t,\ v)} = \sqrt{\frac{1 + i \cot \psi}{2\pi}} \ e^{-i\frac{t^2+v^2}{2} \cot \psi} \left[\cos(tv \csc \psi) + e^{-i(\psi - \frac{\pi}{2})} \sin(tv \csc \psi) \right]$$

and $\psi = \frac{\alpha\pi}{2}$ if $\alpha \neq \pi n$; for all $n = 0, 1, 2, \ldots$.

Property 2.5. Exponential Rule.
If $t,\ v, \alpha, b \in \mathbb{R}$, with α and b are constants and $h \in \mathrm{L}^1(\mathbb{R}) \cap \mathrm{C}^1(\mathbb{R})$, then the exponential rule for the fractional Hartley transform is written as

$$\begin{aligned}
&g_H^\alpha[e^{ibt}h(t)](v)\\
&= e^{ib\cos\psi\left(v-\frac{b\sin\psi}{2}\right)}\left\{\left(\frac{1+\cos\psi}{2}\right) g_H^\alpha[h(t)](v-b\sin\psi)+\frac{i}{2}\sin\psi\times\right.\\
&\left.g_H^\alpha[h(t)](-v+b\sin\psi)\right\}+e^{ib\cos\psi\left(-v-\frac{b\sin\psi}{2}\right)}\left\{-\frac{i}{2}\sin\psi\ g_H^\alpha[h(t)](-v-b\sin\psi)\right.\\
&\left.+\left(\frac{1-\cos\psi}{2}\right)g_H^\alpha[h(t)](v+b\sin\psi)\right\},
\end{aligned}$$

where $g_H^\alpha(v)$ is the fractional Hartley transform of $h(t)$ and $\psi = \frac{\alpha\pi}{2}$ if $\psi \neq \pi n$; for all $n = 0, 1, 2, \ldots$.

Proof. By definition of fractional Hartley transform, we have

$$g_H^\alpha[e^{ibt}h(t)](v) = \int_{-\infty}^{\infty} K_H^\alpha(t,v)e^{ibt}h(t)dt$$

By using relation between fractional Hartley transform and fractional Fourier transform and also by shifting property of fractional Fourier transform, we have

$$\begin{aligned}
&g_H^\alpha[e^{ibt}h(t)](v)\\
=&\left(\frac{1+e^{i\psi}}{2}\right)g_F^\alpha[e^{ibt}h(t)](v)+\left(\frac{1-e^{i\psi}}{2}\right)g_F^\alpha[e^{ibt}h(t)](-v)\\
=&\left(\frac{1+e^{i\psi}}{2}\right)e^{ib\cos\psi\left(v-\frac{b\sin\psi}{2}\right)}g_F^\alpha[h(t)](v-b\sin\psi)\\
&\quad+\left(\frac{1-e^{i\psi}}{2}\right)e^{ib\cos\psi\left(-v-\frac{b\sin\psi}{2}\right)}g_F^\alpha[h(t)](-v-b\sin\psi).
\end{aligned}$$

Now, by using relation between fractional Fourier transform and fractional Hartley transform, we get

$$\begin{aligned}
&g_H^\alpha[e^{ibt}h(t)](v)\\
&=\left(\frac{1+e^{i\psi}}{2}\right)e^{ib\cos\psi\left(v-\frac{b\sin\psi}{2}\right)}\left\{\left(\frac{1+e^{-i\psi}}{2}\right)g_H^\alpha[h(t)](v-b\sin\psi))\right.\\
&\quad\left.+\left(\frac{1-e^{-i\psi}}{2}\right)g_H^\alpha[h(t)](-v+b\sin\psi)\right\}+\left(\frac{1-e^{i\psi}}{2}\right)e^{ib\cos\psi\left(-v-\frac{b\sin\psi}{2}\right)}\times
\end{aligned}$$

$$\left\{\left(\frac{1+e^{-i\psi}}{2}\right)g_H^\alpha[h(t)](-v-b\sin\psi)+\left(\frac{1-e^{-i\psi}}{2}\right)g_H^\alpha[h(t)](v+b\sin\psi)\right\}$$

$$=e^{ib\cos\psi\left(v-\frac{b\sin\psi}{2}\right)}\left\{\left(\frac{1+\cos\psi}{2}\right)g_H^\alpha[h(t)](v-b\sin\psi)+\frac{i}{2}\sin\psi\times\right.$$

$$\left.g_H^\alpha[h(t)](-v+b\sin\psi)\right\}+e^{ib\cos\psi\left(-v-\frac{b\sin\psi}{2}\right)}\left\{-\frac{i}{2}\sin\psi\ g_H^\alpha[h(t)](-v-b\sin\psi)\right.$$

$$\left.+\left(\frac{1-\cos\psi}{2}\right)g_H^\alpha[h(t)](v+b\sin\psi)\right\}.$$

Property 2.6. Multiplication Rule.
If $t,\ v,\ \alpha\in\mathbb{R}$, with α is a constant, $h\in \mathrm{L}^1(\mathbb{R})\cap \mathrm{C}^1(\mathbb{R})$, and m is any positive integer, then multiplication rule for fractional Hartley transform is given by

$$g_H^\alpha[t^m h(t)](u)=\begin{cases}\cos\psi\left(u\cos\psi+i\sin\psi\frac{d}{du}\right)^m g_H^\alpha[h(t)](u) & \\ +i\sin\psi\left(u\cos\psi+i\sin\psi\frac{d}{du}\right)^m g_H^\alpha[h(t)](-u); & \text{if } m \text{ is odd}\\ \left(u\cos\psi+i\sin\psi\frac{d}{du}\right)^m g_H^\alpha[h(t)](u); & \text{if } m \text{ is even,}\end{cases}$$

where $g_H^\alpha[h(t)](v)$ is the fractional Hartley transform of $h(t)$ and $\psi=\frac{\alpha\pi}{2}$ if $\psi\neq\pi n$; for all $n=0,1,2,\ldots$.

Proof. By using relation between fractional Hartley transform and fractional Fourier transform and also by multiplication rule of fractional Fourier transform, we have

$$\begin{aligned}g_H^\alpha[th(t)](u)=&\left(\frac{1+e^{i\psi}}{2}\right)g_F^\alpha[th(t)](u)+\left(\frac{1-e^{i\psi}}{2}\right)g_F^\alpha[th(t)](-u)\\ =&\left(\frac{1+e^{i\psi}}{2}\right)\left(u\cos\psi+i\sin\psi\frac{d}{du}\right)g_F^\alpha[h(t)](u)\\ &+\left(\frac{1-e^{i\psi}}{2}\right)(-1)\left(u\cos\psi+i\sin\psi\frac{d}{du}\right)g_F^\alpha[h(t)](-u).\end{aligned}$$

Now, by using relation between fractional Fourier transform and fractional Hartley transform, we get

$$
\begin{aligned}
& g_H^\alpha[th(t)](u) \\
=& \left(\frac{1+e^{i\psi}}{2}\right)\left(u\cos\psi + i\sin\psi\frac{d}{du}\right)\left\{\left(\frac{1+e^{-i\psi}}{2}\right)g_H^\alpha[h(t)](u) + \left(\frac{1-e^{-i\psi}}{2}\right)\times\right. \\
& \left. g_H^\alpha[h(t)](-u)\right\} + \left(\frac{1-e^{i\psi}}{2}\right)(-1)\left(u\cos\psi + i\sin\psi\frac{d}{du}\right)\left\{\left(\frac{1+e^{-i\psi}}{2}\right)\times\right. \\
& \left. g_H^\alpha[h(t)](-u) + \left(\frac{1-e^{-i\psi}}{2}\right)g_H^\alpha[h(t)](u)\right\} \\
=& \left(\frac{e^{i\psi}+e^{-i\psi}}{2}\right)\left(u\cos\psi + i\sin\psi\frac{d}{du}\right)g_H^\alpha[h(t)](u) \\
& + \left(i\frac{e^{i\psi}-e^{-i\psi}}{2i}\right)\left(u\cos\psi + i\sin\psi\frac{d}{du}\right)g_H^\alpha[h(t)](-u) \\
=& \cos\psi\left(u\cos\psi + i\sin\psi\frac{d}{du}\right)g_H^\alpha[h(t)](u) + \\
& i\sin\psi\left(u\cos\psi + i\sin\psi\frac{d}{du}\right)g_H^\alpha[h(t)](-u).
\end{aligned}
$$

Again by using relation between fractional Hartley transform and fractional Fourier transform and also by multiplication rule of fractional Fourier transform, we have

$$
\begin{aligned}
g_H^\alpha[t^2\ h(t)](u) =& \left(\frac{1+e^{i\psi}}{2}\right)\ g_F^\alpha[t^2\ h(t)](u) + \left(\frac{1-e^{i\psi}}{2}\right)\ g_F^\alpha[t^2\ h(t)](-u) \\
=& \left(\frac{1+e^{i\psi}}{2}\right)\left(u\cos\psi + i\sin\psi\frac{d}{du}\right)^2\ g_F^\alpha[h(t)](u) \\
& \quad + \left(\frac{1-e^{i\psi}}{2}\right)(-1)^2\left(u\cos\psi + i\sin\psi\frac{d}{du}\right)^2\ g_F^\alpha[h(t)](-u) \\
=& \left(u\cos\psi + i\sin\psi\frac{d}{du}\right)^2\ g_H^\alpha[h(t)](u)\left\{\left(\frac{1+e^{i\psi}}{2}\right)\ g_F^\alpha[h(t)](u)\right. \\
& \left. + \left(\frac{1-e^{i\psi}}{2}\right)\ g_F^\alpha[h(t)](-u)\right\} \\
=& \left(u\cos\psi + i\sin\psi\frac{d}{du}\right)^2\ g_H^\alpha[h(t)](u).
\end{aligned}
$$

Similarly

$$g_H^\alpha[t^3\ h(t)](u) = \cos\psi\left(u\cos\psi + i\sin\psi\frac{d}{du}\right)^3 g_H^\alpha[h(t)](u) + i\sin\psi\left(u\cos\psi + i\sin\psi\frac{d}{du}\right)^3 g_H^\alpha[h(t)](-u).$$

Therefore by mathematical induction method for any positive integer m, we can obtain

$$g_H^\alpha[t^m h(t)](u) = \begin{cases} \cos\psi\left(u\cos\psi + i\sin\psi\frac{d}{du}\right)^m g_H^\alpha[h(t)](u) & \\ +i\sin\psi\left(u\cos\psi + i\sin\psi\frac{d}{du}\right)^m g_H^\alpha[h(t)](-u); & \text{if } m \text{ is odd} \\ \left(u\cos\psi + i\sin\psi\frac{d}{du}\right)^m g_H^\alpha[h(t)](u); & \text{if } m \text{ is even.} \end{cases}$$

Property 2.7. Transform of the derivative.
If $t,\ v,\ \alpha \in \mathbb{R}$, with α is a constant, $h \in \mathrm{L}^1(\mathbb{R}) \cap \mathrm{C}^1(\mathbb{R})$, m is any positive integer, $h(t)$ is differentiable and vanishes at $t \to \pm\infty$, then Differentiation rule for fractional Hartley transform is given by

$$g_H^\alpha\left[\frac{d^m}{dt^m}\Big(h(t)\Big)\right](u) = \begin{cases} \cos\psi\left(i\ u\sin\psi + \cos\psi\frac{d}{du}\right)^m g_H^\alpha[h(t)](u) & \\ +i\sin\psi\left(i\ u\sin\psi + \cos\psi\frac{d}{du}\right)^m g_H^\alpha[h(t)](-u); & \textit{if } m \textit{ is odd} \\ \left(i\ u\sin\psi + \cos\psi\frac{d}{du}\right)^m g_H^\alpha[h(t)](u); & \textit{if } m \textit{ is even,} \end{cases}$$

where $g_H^\alpha[h(t)](v)$ is the fractional Hartley transform of $h(t)$ and $\psi = \frac{\alpha\pi}{2}$ if $\psi \neq \pi n$; for all $n = 0, 1, 2, \ldots$.

Proof. By using relation between fractional Hartley transform and fractional Fourier transform and also by differentiation rule of fractional Fourier transform, we have

$$\begin{aligned} g_H^\alpha[h'(t)](u) =& \left(\frac{1+e^{i\psi}}{2}\right) g_F^\alpha[h'(t)](u) + \left(\frac{1-e^{i\psi}}{2}\right) g_F^\alpha[h'(t)](-u) \\ =& \left(\frac{1+e^{i\psi}}{2}\right)\left(i\ u\sin\psi + \cos\psi\frac{d}{du}\right) g_F^\alpha[h(t)](u) \\ &+ \left(\frac{1-e^{i\psi}}{2}\right)(-1)\left(i\ u\sin\psi + \cos\psi\frac{d}{du}\right) g_F^\alpha[h(t)](-u). \end{aligned}$$

Now, by using relation between fractional Fourier transform and fractional Hartley transform, we get

$$
\begin{aligned}
& g_H^\alpha[h'(t)](u) \\
=& \left(\frac{1+e^{i\psi}}{2}\right)\left(i\,u\sin\psi+\cos\psi\frac{d}{du}\right)\left\{\left(\frac{1+e^{-i\psi}}{2}\right)g_H^\alpha[h(t)](u)+\left(\frac{1-e^{-i\psi}}{2}\right)\times\right. \\
& \left. g_H^\alpha[h(t)](-u)\right\}+\left(\frac{1-e^{i\psi}}{2}\right)(-1)\left(i\,u\sin\psi+\cos\psi\frac{d}{du}\right)\left\{\left(\frac{1+e^{-i\psi}}{2}\right)\times\right. \\
& \left. g_H^\alpha[h(t)](-u)+\left(\frac{1-e^{-i\psi}}{2}\right)g_H^\alpha[h(t)](u)\right\} \\
=& \left(\frac{e^{i\psi}+e^{-i\psi}}{2}\right)\left(i\,u\sin\psi+\cos\psi\frac{d}{du}\right)g_H^\alpha[h(t)](u) \\
& +\left(i\frac{e^{i\psi}-e^{-i\psi}}{2i}\right)\left(i\,u\sin\psi+\cos\psi\frac{d}{du}\right)g_H^\alpha[h(t)](-u) \\
=& \cos\psi\left(i\,u\sin\psi+\cos\psi\frac{d}{du}\right)g_H^\alpha[h(t)](u) \\
& +i\sin\psi\left(i\,u\sin\psi+\cos\psi\frac{d}{du}\right)g_H^\alpha[h(t)](-u).
\end{aligned}
$$

Again by using relation between fractional Hartley transform and fractional Fourier transform and also by differentiation rule of fractional Fourier transform, we have

$$
\begin{aligned}
& g_H^\alpha\left[\frac{d^2}{dt^2}\Big(h(t)\Big)\right](u) \\
=& \left(\frac{1+e^{i\psi}}{2}\right)\ g_F^\alpha\left[\frac{d^2}{dt^2}\Big(h(t)\Big)\right](u)+\left(\frac{1-e^{i\psi}}{2}\right)\ g_F^\alpha\left[\frac{d^2}{dt^2}\Big(h(t)\Big)\right](-u) \\
=& \left(\frac{1+e^{i\psi}}{2}\right)\left(i\,u\sin\psi+\cos\psi\frac{d}{du}\right)^2\ g_F^\alpha[h(t)](u) \\
& \qquad +\left(\frac{1-e^{i\psi}}{2}\right)(-1)^2\left(i\,u\sin\psi+\cos\psi\frac{d}{du}\right)^2\ g_F^\alpha[h(t)](-u) \\
=& \left(i\,u\sin\psi+\cos\psi\frac{d}{du}\right)^2\ g_H^\alpha[h(t)](u)\left\{\left(\frac{1+e^{i\psi}}{2}\right)\ g_F^\alpha[h(t)](u)\right.
\end{aligned}
$$

$$+\left(\frac{1-e^{i\psi}}{2}\right)g_F^{\alpha}[h(t)](-u)\Bigg\}$$
$$=\left(i\,u\sin\psi+\cos\psi\frac{d}{du}\right)^2 g_H^{\alpha}[h(t)](u).$$

Similarly

$$g_H^{\alpha}\left[\frac{d^3}{dt^3}\Big(h(t)\Big)\right](u)=\cos\psi\left(i\,u\sin\psi+\cos\psi\frac{d}{du}\right)^3 g_H^{\alpha}[h(t)](u)$$
$$+\,i\sin\psi\left(i\,u\sin\psi+\cos\psi\frac{d}{du}\right)^3 g_H^{\alpha}[h(t)](-u).$$

Therefore by mathematical induction method for any positive integer m, we can obtain

$$g_H^{\alpha}\left[\frac{d^m}{dt^m}\Big(h(t)\Big)\right](u)=\begin{cases}\cos\psi\left(i\,u\sin\psi+\cos\psi\frac{d}{du}\right)^m g_H^{\alpha}[h(t)](u) & \\ +i\sin\psi\left(i\,u\sin\psi+\cos\psi\frac{d}{du}\right)^m g_H^{\alpha}[h(t)](-u); & \text{if } m \text{ is odd}\\ \left(i\,u\sin\psi+\cos\psi\frac{d}{du}\right)^m g_H^{\alpha}[h(t)](u); & \text{if } m \text{ is even.}\end{cases}$$

Property 2.8. Derivative of the Transform.
If $t,\ v,\ \alpha\in\mathbb{R}$, with α is a constant, $h\in \mathrm{L}^1(\mathbb{R})\cap \mathrm{C}^1(\mathbb{R})$, then the derivative of the fractional Hartley transform is obtained as follows

$$\frac{d}{dv}\Big\{g_H^{\alpha}[h(t)](v)\Big\}$$
$$=iv\cot\psi g_H^{\alpha}[h(t)](v)+\left(\frac{1+\cos\psi}{2}\right)g_H^{\alpha}[-itv\csc\psi h(t)](v)$$
$$+\frac{i}{2}\sin\psi g_H^{\alpha}[-itv\csc\psi h(t)](-v)+\frac{-i}{2}\sin\psi g_H^{\alpha}[itv\csc\psi h(t)](-v)$$
$$+\left(\frac{1-\cos\psi}{2}\right)g_H^{\alpha}[itv\csc\psi h(t)](v),$$

where $g_H^{\alpha}[h(t)](v)$ is the fractional Hartley transform of $h(t)$ and $\psi=\frac{\alpha\pi}{2}$ if $\psi\neq\pi n$; for all $n=0,1,2,\ldots$.

Proof. By definition of fractional Hartley transform, we have

$$g_H^{\alpha}[h(t)](v)=\int_{-\infty}^{\infty}K_H^{\alpha}(t,v)h(t)dt.$$

By using relation between fractional Hartley transform and fractional Fourier transform and also by derivative of fractional Fourier transform, we have

$$\begin{aligned}
&\frac{d}{dv}\left\{g_H^\alpha[h(t)](v)\right\}\\
=&\frac{d}{dv}\left\{\left(\frac{1+e^{i\psi}}{2}\right)g_F^\alpha[h(t)](v)+\left(\frac{1-e^{i\psi}}{2}\right)g_F^\alpha[h(t)](-v)\right\}\\
=&\left(\frac{1+e^{i\psi}}{2}\right)\left\{iv\cot\psi g_F^\alpha[h(t)](v)+g_F^\alpha[-itv\csc\psi h(t)](v)\right\}\\
&\qquad+\left(\frac{1-e^{i\psi}}{2}\right)\left\{iv\cot\psi g_F^\alpha[h(t)](-v)+g_F^\alpha[itv\csc\psi h(t)](-v)\right\}\\
=&iv\cot\psi g_H^\alpha[h(t)](v)+\left(\frac{1+e^{i\psi}}{2}\right)g_F^\alpha[-itv\csc\psi h(t)](v)\\
&\qquad+\left(\frac{1-e^{i\psi}}{2}\right)g_F^\alpha[itv\csc\psi h(t)](-v).
\end{aligned}$$

Now, by using relation between fractional Fourier transform and fractional Hartley transform, we get

$$\begin{aligned}
&\frac{d}{dv}\left\{g_H^\alpha[h(t)](v)\right\}\\
&=iv\cot\psi g_H^\alpha[h(t)](v)+\left(\frac{1+e^{i\psi}}{2}\right)\left\{\left(\frac{1+e^{-i\psi}}{2}\right)g_H^\alpha[-itv\csc\psi h(t)](v)\right.\\
&\quad\left.+\left(\frac{1-e^{-i\psi}}{2}\right)g_H^\alpha[-itv\csc\psi h(t)](-v)\right\}+\left(\frac{1-e^{i\psi}}{2}\right)\left\{\left(\frac{1+e^{-i\psi}}{2}\right)\times\right.\\
&\quad\left.g_H^\alpha[itv\csc\psi h(t)](-v)+\left(\frac{1-e^{-i\psi}}{2}\right)g_H^\alpha[itv\csc\psi h(t)](v)\right\}\\
&=iv\cot\psi g_H^\alpha[h(t)](v)+\left(\frac{1+e^{i\psi}}{2}\right)\left(\frac{1+e^{-i\psi}}{2}\right)g_H^\alpha[-itv\csc\psi h(t)](v)\\
&\quad+\left(\frac{1+e^{i\psi}}{2}\right)\left(\frac{1-e^{-i\psi}}{2}\right)g_H^\alpha[-itv\csc\psi h(t)](-v)+\left(\frac{1-e^{i\psi}}{2}\right)\left(\frac{1+e^{-i\psi}}{2}\right)\times\\
&\quad g_H^\alpha[itv\csc\psi h(t)](-v)+\left(\frac{1-e^{i\psi}}{2}\right)\left(\frac{1-e^{-i\psi}}{2}\right)g_H^\alpha[itv\csc\psi h(t)](v)
\end{aligned}$$

$$
\begin{aligned}
&= iv\cot\psi g_H^\alpha[h(t)](v) + \left(\frac{1+\cos\psi}{2}\right) g_H^\alpha[-itv\csc\psi h(t)](v) \\
&\quad + \frac{i}{2}\sin\psi g_H^\alpha[-itv\csc\psi h(t)](-v) + \frac{-i}{2}\sin\psi g_H^\alpha[itv\csc\psi h(t)](-v) \\
&\quad + \left(\frac{1-\cos\psi}{2}\right) g_H^\alpha[itv\csc\psi h(t)](v).
\end{aligned}
$$

3. Examples

Example 3.1. If $h(t) = 1$, then the FRHT of $h(t)$ with parameter $\alpha \in \mathbb{R}$ is equal to $\sqrt{1 + i\tan\psi}\, e^{-i\frac{1}{2}v^2\tan\psi}$ if $\psi - \frac{\pi}{2}$ is not multiple of π see in [3].

Example 3.2. If $h(t) = e^{\frac{-t^2}{2}}$, then the FRHT of $h(t)$ with parameter $\alpha \in \mathbb{R}$ is equal to $e^{\frac{-v^2}{2}}$.

4. Conclusion

The present work proved inverse theorem of FRHT and some important properties of fractional Hartley transform like exponential rule, multiplication rule, transform of derivative and derivative of transform.

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