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EXTENDED GENERALIZED τ-GAUSS' HYPERGEOMETRIC FUNCTIONS AND THEIR APPLICATIONS

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Abstract: In this article, by means of the extended beta function, we introduce new extension of the generalized τ-Gauss' hypergeometric functions and present some new integral and series representations (including the one obtained by adopting the well-known Ramanujan's Master Theorem). We also consider some new and known results as consequences of our proposed extension of the generalized τ-Gauss hypergeometric function.

Keywords and Phrases: Extended Beta function, Generalized τ-Gauss' hypergeometric function, Ramanujan's Master theorem.

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1. Introduction

New extensions of some of the well-known special functions (e.g. gamma function, beta function, Gauss hypergeometric function, etc.) have been extensively studied in the recent past. By inserting a regularization factor $e^{-pt^{-1}}$, Chaudhry et.al. [4] have introduced the following extension of the gamma function:

$$\Gamma_p(x) = \int_0^\infty t^{x-1} exp(-t - \frac{p}{t}) dt \qquad (\Re(p) > 0), \tag{1.1}$$

and Chaudhry et al. [2] considered the extension of Euler's beta function in the following form:

$$B_p(x,y) = \int_0^{\infty} t^{x-1}(1-t)^{y-1} exp\left(\frac{-p}{t(1-t)}\right) dt \qquad (\Re(p) > 0). \tag{1.2}$$

Later, Chaudhry et al. [3] used $B_p(x,y)$ to extend the Gauss hypergeometric function given by

$$F_p(\alpha,\beta,\gamma,z) = \sum_{n=0}^{\infty} (\alpha)_n \frac{B_p(\beta+n,\gamma-\beta)}{B(\beta,\gamma-\beta)} \frac{z^n}{n!} \tag{1.3}$$

$$(p \geqslant 0, |z| < 1, \Re(\gamma) > \Re(\beta) > 0),$$

where $(\alpha)_n$ denotes the Pochhammer symbol defined in terms of gamma functions by

$$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}.$$

For p = 0, the function (1.3) reduces to the usual Gauss hypergeometric function.

Definition 1.1. ([10], p. 243) *Let a function $\Theta(\{k_l\}_{l\in N_0}; z)$ be analytic within the disk $|z| < R (0 < R < 1)$ and let its Taylor-Maclaurin coefficients be explicitly denoted by the sequence $\{k_l\}_{l\in N_0}$ ([4]). Suppose also that the function $\Theta(\{k_l\}_{l\in N_0}; z)$ can be continued analytically in the right half-plane $\Re(z) > 0$ with the asymptotic property given as follows:*

$$\Theta(\{k_l\}_{l\in N_0}; z)$$

$$= \begin{cases} \sum_{l=0}^{\infty} \{k_l\} \frac{z^l}{l!} & (|z| < R; 0 < R < \infty; k_0 = 1), \\ M_0 z^w \exp(z)[1 + O(\frac{1}{z})] & (\Re(z) \to \infty; M_0 > 0; w \in \mathbb{C}), \end{cases} \tag{1.4}$$

for some suitable constants M_0 and w depending essentially on the sequence $\{k_l\}_{l\in N_0}$.
Srivastava et.al. also defined extended Gamma function $\Gamma_p^{\{k_l\}}(z)$ and the extended beta function respectively as (see [4, Equations (2.2) and (2.3)])

$$\Gamma_p^{\{k_l\}}(z) = \int_0^{\infty} t^{z-1} \Theta(\{k_l\}; -t - \frac{p}{t}) dt \qquad (\Re(p) \geqslant 0, \Re(z) > 0), \tag{1.5}$$

and

$$B_p^{\{k_l\}}(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} \Theta\left(\{k_l\}; \frac{-p}{t(1-t)}\right) dt, \tag{1.6}$$

$$(\Re(p) \geqslant 0, min(\Re(\alpha), \Re(\beta)) > 0).$$

We shall also make use of the following definition of a two-parameter extension of (1.6) due to Srivastava et al. [4, p.256, Eqn. (6.1)] (see also [4, Section 6] for other related two-parameter definitions):

$$B_{(p,q)}^{\{k_l\}}(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\Theta\left(\{k_l\}; -\frac{p}{t} - \frac{q}{(1-t)}\right) dt, \tag{1.7}$$

$$(min(\Re(p), \Re(q)) > 0,\ min(\Re(\alpha), \Re(\beta)) > 0).$$

In this paper, we introduce some extended forms of the generalized τ-Gauss' hypergeometric functions by means of (1.7). Section 2 gives the extensions of τ-Gauss' hypergeometric functions and Section 3 treats extensions of the generalized τ-Gauss' hypergeometric functions together with some of their fundamental properties. Section 4 gives the Mellin transformation and Mellin-Barnes type integral representations by the application of the well-known Ramanujan's Master Theorem.

2. τ-Gauss' Hypergeometric Functions

Using the extended beta function $B_{(p,q)}^{\{k_l\}}(\alpha, \beta)$ defined by (1.7), we can easily form another series representation of the τ-Gauss' hypergeometric function

$${}_2R_1(\alpha_1, \alpha_2, \beta_1, \tau, z) = \sum_{n=0}^{\infty} (\alpha_1)_n \frac{(\alpha_2)_{n\tau}}{(\beta_1)_{n\tau}} \frac{z^n}{n!}, \tag{2.1}$$

$$(\tau > 0,\ |z| < 1,\ \Re(\beta_1) > \Re(\alpha_2) > 0 \ when \ |z| = 1).$$

Let us replace

$$\frac{(\alpha_2)_{n\tau}}{(\beta_1)_{n\tau}} \to \frac{B_{(p,q)}^{\{k_l\}}(\alpha_2 + n\tau, \beta_1 - \alpha_2)}{B(\alpha_2, \beta_1 - \alpha_2)}$$

in (2.1), then we obtain the extended form of the τ-Gauss' hypergeometric function using the extended beta function in the following form:

Definition 2.1. [6] *The extended τ-Gauss' hypergeometric function ${}_2R_1^{\{k_l\}}$ is defined as :*

$${}_2R_1^{\{k_l\}}(\alpha_1, \alpha_2, \beta_1, \tau; z, p, q) = \sum_{n=0}^{\infty} (\alpha_1)_n \frac{B_{(p,q)}^{\{k_l\}}(\alpha_2 + n\tau, \beta_1 - \alpha_2)}{B(\alpha_2, \beta_1 - \alpha_2)} \frac{z^n}{n!}, \tag{2.2}$$

$$(\tau > 0,\ |z| < 1,\ \Re(\beta_1) > \Re(\alpha_2) > 0 \ when \ |z| = 1, min(\Re(p), \Re(q)) > 0).$$

Theorem 2.2. *The integral representation of extended τ-Gauss' hypergeometric function is defined as :*

$$
\begin{aligned}
&{}_2R_1^{\{k_l\}}(\alpha_1,\alpha_2,\beta_1,\tau;z,p,q)\\
&=\frac{1}{B(\alpha_2,\beta_1-\alpha_2)}\int_0^1 t^{\alpha_2-1}(1-t)^{\beta_1-\alpha_2-1}(1-zt^{\tau})^{-\alpha_1}\Theta\left(\{k_l\};-\frac{p}{t}-\frac{q}{(1-t)}\right)dt,
\end{aligned}
\tag{2.3}
$$

$$(\Re(\beta_1)>\Re(\alpha_2)>0,\Re(p)>\Re(q)>0,\ b=d=0,\ |arg(1-z)|<\pi).$$

Proof. Replacing the extended beta function $B_{(p,q)}^{\{k_l\}}(\alpha_2+n\tau,\beta_1-\alpha_2)$ in (2.2) by its integral representation given by (1.7) and then interchanging the order of summation and integration (which can be justified due to the absolute convergence of the integral and the series involved), the integral representation (2.3) follows immediately after some necessary simplification.

In terms of the extended beta function $B_{(p,q)}^{\{k_l\}}(\alpha,\beta)$ defined in (1.7), we can construct a suitable extension of τ-Gauss' hypergeometric function. Consideration of the following cases is required :

1. For $u=v+1$, the coefficients of ${}_uR_v(\alpha_1,\alpha_2,...\alpha_u,\beta_1,\beta_2,...,\beta_v,\tau;z)$ can be written as :

$$(\alpha_1)_n\prod_{j=1}^{v}\frac{\alpha_{(j+1)}}{\beta_j}=(\alpha_1)_n\prod_{j=1}^{v}\frac{B(\alpha_{j+1}+n\tau,\beta_j-\alpha_{j+1})}{B(\alpha_{j+1},\beta_j-\alpha_{j+1})},\qquad(n\in\mathbb{N}_0).$$

   By substituting the extended beta function (1.7) for each $B(\alpha_{j+1}+n\tau,\beta_j-\alpha_{j+1})$, we get the coefficients as :

$$(\alpha_1)_n\prod_{j=1}^{v}\frac{B_{(p,q)}^{\{k_l\}}(\alpha_{j+1}+n\tau,\beta_j-\alpha_{j+1})}{B(\alpha_{j+1},\beta_j-\alpha_{j+1})},\qquad(n\in\mathbb{N}_0).$$

2. For $u=v$, the coefficients of our extension are simply :

$$\prod_{j=1}^{v}\frac{B_{(p,q)}^{\{k_l\}}(\alpha_j+n\tau,\beta_j-\alpha_j)}{B(\alpha_j,\beta_j-\alpha_j)},\qquad(n\in\mathbb{N}_0).$$

3. For $u<v$, the only reasonable construction of the coefficients is :

$$\prod_{i=1}^{r}\frac{1}{(\beta_i)_{n\tau}}\prod_{j=1}^{u}\frac{B_{(p,q)}^{\{k_l\}}(\alpha_j+n\tau,\beta_{j+r}-\alpha_j)}{B(\alpha_j,\beta_{j+r}-\alpha_j)},\qquad(n\in\mathbb{N}_0).$$

In the third section, we extend the generalized hypergeometric function by using the extended beta function $B^{\{k_l\}}_{(p,q)}(\alpha,\beta)$.

3. Extended Generalized τ-Gauss' Hypergeometric Functions

The generalized hypergeometric function with u numerator and v denominator parameters is defined as :

$$ {}_uR_v(\alpha_1,\alpha_2,...\alpha_u,\beta_1,\beta_2,...,\beta_v,\tau;z)=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n(\alpha_2)_{n\tau},...,(\alpha_u)_{n\tau}}{(\beta_1)_{n\tau},(\beta_2)_{n\tau},...,(\beta_3)_{n\tau}}\frac{z^n}{n!}, \tag{3.1} $$

$$ (\alpha_l,\beta_j\in\mathbb{C},\ \beta_j\neq 0,-1,-2,...,l=1,...,u,j=1,..,v) $$

which is absolutely convergent for all values of z $\in\mathbb{C}$, if $u\leqslant v$. When $u=v+1$, the series is absolutely convergent for $|z|<1$ *and for* $|z|=1$,when $\Re(\sum_{j=1}^{v}\beta_j-\sum_{l=1}^{u}\alpha_l)>0$, while it is conditionally convergent for $|z|=1(|z|\neq 1)$ if $-1<\Re(\sum_{j=1}^{v}\beta_j-\sum_{l=1}^{u}\alpha_l)\leqslant 0$.

We can now give the formal definition to our extended generalized τ-Gauss' hypergeometric function as follows :

Definition 3.1. *For suitably constrained (real or complex) parameters α_j, $j=1,2,$ .., u; β_i , i=1,2,..,v, we define the extended generalized τ-Gauss' hypergeometric function by*

$$ {}_uR^{\{k_l\}}_v(\alpha_1,\alpha_2,...\alpha_u,\beta_1,\beta_2,...,\beta_v,\tau;z;p,q) $$

$$ =\begin{cases}\sum_{n=0}^{\infty}(\alpha_1)_n\prod_{j=1}^{v}\frac{B^{\{k_l\}}_{(p,q)}(\alpha_{j+1}+n\tau,\beta_j-\alpha_{j+1})}{B(\alpha_{j+1},\beta_j-\alpha_{j+1})}\frac{z^n}{n!},\\ \qquad (|z|<1,u=v+1,\Re(\beta_j)>\Re(\alpha_{j+1})>0)\\ \\ \sum_{n=0}^{\infty}\prod_{j=1}^{v}\frac{B^{\{k_l\}}_{(p,q)}(\alpha_j+n\tau,\beta_j-\alpha_j)}{B(\alpha_j,\beta_j-\alpha_j)}\frac{z^n}{n!},\\ \qquad (z\in\mathbb{C},u=v,\Re(\beta_j)>\Re(\alpha_j)>0)\\ \\ \sum_{n=0}^{\infty}\prod_{i=1}^{r}\frac{1}{(\beta_i)_{n\tau}}\prod_{j=1}^{u}\frac{B^{\{k_l\}}_{(p,q)}(\alpha_j+n\tau,\beta_{j+r}-\alpha_j)}{B(\alpha_j,\beta_{j+r}-\alpha_j)}\frac{z^n}{n!},\\ \quad (z\in\mathbb{C},r=v-u,u<v,\Re(\beta_{r+j}>\Re(\alpha_j)>0)).\end{cases} \tag{3.2} $$

The following theorem demonstrates that the form of the Euler-type integral representation of ${}_uR^{\{k_l\}}_v$ is very similar to that of the Euler-type integral representation of ${}_uR_v$.

Theorem 3.2. *For the extended generalized τ-Gauss' hypergeometric function defined by (3.2), we have the following integral representation:*

$$_{u}R_{v}^{\{k_l\}}(\alpha_1, \alpha_2, ...\alpha_u, \beta_1, \beta_2, ..., \beta_v, \tau; z; p, q)$$

$$= \frac{\Gamma(\beta_v)}{\Gamma(\alpha_u)\Gamma(\beta_v-\alpha_u)} \cdot \int_0^1 t^{\alpha_u-1}(1-t)^{\beta_v-\alpha_u-1} \cdot {}_{u-1}R_{v-1}^{\{k_l\}}\left(\begin{cases}\alpha_1, \alpha_2, ..\alpha_{u-1}, \\ \beta_1, \beta_2, ..\beta_{v-1}\end{cases}, \tau; zt; p, q\right) dt. \tag{3.3}$$

$$(\Re(\beta_v) > \Re(\alpha_u) > 0, min(\Re(p), \Re(q)) > 0, p = q = 0, arg|1-z| < \pi)$$

Proof. We need to verify that the formula (3.3) holds for three different expressions of $_{u}R_{v}^{\{k_l\}}(\alpha_1, \alpha_2, ...\alpha_u, \beta_1, \beta_2, ..., \beta_v, \tau; z; p, q)$ given in (3.2) respectively. Consider the case $u = v+1$, in view of the representation that

$$\frac{B_{(p,q)}^{\{k_l\}}(\alpha_{v+1}+n\tau, \beta_v-\alpha_{v+1})}{B(\alpha_{v+1}, \beta_v-\alpha_{v+1})}$$

$$= \frac{\Gamma(\beta_v)}{\Gamma(\alpha_{v+1})\Gamma(\beta_v-\alpha_{v+1})} \cdot \int_0^1 t^{\alpha_{v+1}+n-1}(1-t)^{\beta_v-\alpha_{v+1}-1}\Theta\left(\{k_l\}; -\frac{p}{t}-\frac{q}{(1-t)}\right) dt, \tag{3.4}$$

$$(m \in \mathbb{N}_0, min(\Re(p), \Re(q)) > 0, \Re(\beta_v) > \Re(\alpha_{v+1}) > 0)$$

we find that

$$_{v+1}R_{v}^{\{k_l\}}(\alpha_1, \alpha_2, ...\alpha_{v+1}, \beta_1, \beta_2, ..., \beta_v, \tau; z; p, q)$$

$$= \frac{\Gamma(\beta_v)}{\Gamma(\alpha_{v+1})\Gamma(\beta_v-\alpha_{v+1})} \int_0^1 t^{\alpha_{v+1}-1}(1-t)^{\beta_v-\alpha_{v+1}-1}$$

$$\cdot \sum_{n=0}^{\infty}(\alpha_1)_n \prod_{j=1}^{v-1} \frac{B_{(p,q)}^{\{k_l\}}(\alpha_{j+1}+n\tau, \beta_j-\alpha_{j+1})}{B(\alpha_{j+1}, \beta_j-\alpha_{j+1})} \frac{(zt)^n}{n!} dt$$

$$= \frac{\Gamma(\beta_v)}{\Gamma(\alpha_{v+1})\Gamma(\beta_v-\alpha_{v+1})} \int_0^1 t^{\alpha_{v+1}-1}(1-t)^{\beta_v-\alpha_{v+1}-1} \cdot {}_{v}R_{v-1}^{\{k_l\}}\left(\begin{cases}\alpha_1, \alpha_2, ..\alpha_v, \\ \beta_1, \beta_2, ..\beta_{v-1}\end{cases}, \tau; zt; p, q\right) dt$$

After putting the value $u = v+1$, we get

$$= \frac{\Gamma(\beta_v)}{\Gamma(\alpha_u)\Gamma(\beta_v-\alpha_u)} \int_0^1 t^{\alpha_u-1}(1-t)^{\beta_v-\alpha_u-1} \cdot {}_{u-1}R_{v-1}^{\{k_l\}}\left(\begin{cases}\alpha_1, \alpha_2, ..\alpha_{u-1}, \\ \beta_1, \beta_2, ..\beta_{v-1}\end{cases}, \tau; zt; p, q\right) dt. \tag{3.5}$$

It is clear that the relation (3.3) is also valid for the $u \leqslant v$ and this completes the proof.

Special Cases

1. When $u = 2$, and $v = 1$; (3.3) reduces to the following extended τ-Gauss' hypergeometric function:

$$ {}_2R_1^{\{k_l\}}(\alpha_1, \alpha_2, \beta_1; \tau; z, p, q) $$

$$ = \frac{1}{B(\alpha_2, \beta_1 - \alpha_2)} \int_0^1 t^{\alpha_2 - 1}(1-t)^{\beta_1 - \alpha_2 - 1} \cdot {}_1R_0^{\{k_l\}}(\alpha_1, \_, zt^{\tau}, p, q)dt. $$

2. When $\tau = 1$, $u = 2$, and $v = 1$, $p = q = 0$; (3.3) reduces to the following Gauss hypergeometric function:

$$ {}_2R_1^{\{k_l\}}(\alpha_1, \alpha_2; \beta_1; z) = \frac{1}{B(\alpha_2, \beta_1 - \alpha_2)} \int_0^1 t^{\alpha_2 - 1}(1-t)^{\beta_1 - \alpha_2 - 1} \, {}_1R_0^{\{k_l\}}(\alpha_1, \_, zt)dt. $$

Remark. A multidimensional case of the Euler-type integral representation of (3.5) is given as:

$$ {}_{v+1}R_v^{\{k_l\}}(\alpha_1, \alpha_2, ...\alpha_{v+1}, \beta_1, \beta_2, ..., \beta_v, \tau; z; p, q) = \prod_{j=1}^{v} \frac{\Gamma(\beta_v)}{\Gamma(\alpha_{v+1})\Gamma(\beta_v - \alpha_{v+1})} $$

$$ \int_0^1 .. \int_0^1 \prod_{j=1}^{v} t_j^{\alpha_{j+1}}(1-t_j)^{\beta_j - \alpha_{j+1} - 1}\Theta\left(\{k_l\}; -\frac{p}{t_j} - \frac{q}{(1-t_j)}\right)(1 - t_1 t_2 .. t_v z)^{-\alpha_1} dt_1 .. dt_v, $$

which follows from the repeated application of the functional equation (3.5).

Special Case

When $\tau = 1$; $v = 1$; equation (3.5) reduces to an extended Gauss hypergeometric function [9]

$$ {}_2R_1^{\{k_l\}}(\alpha_1, \alpha_2, \beta_1; z, p, q) $$

$$ = \frac{\Gamma(\beta_1)}{\Gamma(\alpha_2)\Gamma(\beta_1 - \alpha_2)} \int_0^1 t^{\alpha_2 - 1}(1-t)^{\beta_1 - \alpha_2 - 1}(1 - zt)^{-\alpha_1}.\Theta(k_l, -\frac{p}{t} - \frac{q}{1-t})dt $$

$$ = \frac{1}{B(\alpha_2, \beta_1 - \alpha_2)} \int_0^1 t^{\alpha_2 - 1}(1-t)^{\beta_1 - \alpha_2 - 1}(1 - zt)^{-\alpha_1}.\Theta(k_l, -\frac{p}{t} - \frac{q}{1-t})dt. $$

Theorem 3.3. *The following derivative formula holds for* $u \leqslant v+1$

$$\frac{d^n}{dz^n}\{{}_uR_v^{\{k_l\}}(\alpha_1,\alpha_2,...\alpha_u,\beta_1,\beta_2,...,\beta_v,\tau;z;p,q)\}$$

$$=\frac{(\alpha_1)_n...(\alpha_u)_{n\tau}}{(\beta_1)_{n\tau}...(\beta_v)_{n\tau}}. {}_uR_v^{\{k_l\}}(\alpha_1+n,...\alpha_u+n\tau,\beta_1+n\tau,...,\beta_v+n\tau,\tau;z;p,q)\ (n\in N_0). \tag{3.6}$$

Proof. Differentiating ${}_{v+1}R_v^{\{k_l\}}$ with respect to z, we obtain

$$\frac{d}{dz}\{{}_{v+1}R_v^{\{k_l\}}(\alpha_1,...\alpha_u,\beta_1,...\beta_v,\tau,z,p,q)\}$$

$$=\sum_{n=1}^{\infty}(\alpha_1)_n\prod_{j=1}^{v}\frac{B_{(p,q)}^{\{k_l\}}(\alpha_{j+1}+n\tau,\beta_j-\alpha_{j+1})}{B(\alpha_{j+1},\beta_j-\alpha_{j+1})}\frac{z^{n-1}}{(n-1)!}. \tag{3.7}$$

Replacing n by n+1 in the right-hand side of (3.7), we are lead to

$$\frac{d}{dz}\{{}_{v+1}R_v^{\{k_l\}}(\alpha_1,..\alpha_{v+1},\beta_1,..\beta_v,\tau;z;p,q\}$$

$$=(\alpha_1)\cdot\frac{\prod_{j=1}^{v}\alpha_{j+1}}{\prod_{j=1}^{v}\beta_j}\cdot\{{}_{v+1}R_v^{\{k_l\}}(\alpha_1+1,..\alpha_{v+1}+\tau,\beta_1+\tau\cdots\beta_v+\tau,\tau,z,p,q\}. \tag{3.8}$$

Recursive application of this procedure n-times gives us the general form (3.6).

Similarly, we can prove the result for the case $u \leqslant v$.

For $u = 2$ and $v = 1$; we at once get

$$\frac{d^n}{dz^n}\,{}_2R_1^{\{k_l\}}(\alpha_1,\alpha_2,\beta_1,\tau;z;p,q)=\frac{(\alpha_1)_n(\alpha_2)_{n\tau}}{(\beta_1)_{n\tau}}\,{}_2R_1^{\{k_l\}}(\alpha_1+n,\alpha_2+n\tau,\beta_1+n\tau,\tau;z;p,q), \tag{3.9}$$

which corresponds to the known result [6, Theorem 3].

Next, We derive the Mellin transformation and Mellin Barnes type contour integral representation of the function (3.2). We need the following well-known theorem which is widely used to evaluate definite integrals and infinite series.

4. Mellin transform representation and Mellin Barnes type integral representation of the Extended Generalized τ-Gauss' Hypergeometric Functions

Theorem 4.1. (Ramanujan's Master theorem [1]) *Assume f admits an expansion of the form :*

$$f(x)=\sum_{k=0}^{\infty}\frac{\lambda(k)(-x)^k}{k!}\qquad(\lambda(0)\neq 0).$$

Then the Mellin transform of f is given by

$$F(s) = \int_0^\infty x^{s-1} f(x) dx = \Gamma(s)\lambda(-s).$$

By means of Ramanujan's master theorem, we obtain the following Mellin transformation and Mellin Barnes type integral representation.

Theorem 4.2. *For the extended generalized τ-Gauss' hypergeometric function, we have the following Mellin transformation representation of the first kind for the function (3.2) as :*

$$M\{{}_uR_v^{\{k_l\}}(\alpha_1, \alpha_2, .., \alpha_u, \beta_1, \beta_2, .., \beta_v, \tau; z; p, q)\}$$

$$= \Gamma(s)(\alpha_1)_s\{{}_uR_v^{\{k_l\}}(\alpha_1 + s, \alpha_2, .., \alpha_u, \beta_1, \beta_2, .., \beta_v, \tau; z; p, q)\}. \tag{4.1}$$

Proof. To obtain the Mellin Transform, we multiply both sides of (3.2) by t^{s-1} and then integrate with respect to t over integral $[0,\infty)$ as follows :

$$= \int_0^\infty t^{s-1} \left((\alpha_1)_n \prod_{j=1}^{v} \frac{B_{(p,q)}^{\{k_l\}}(\alpha_{j+1} + n\tau, \beta_j - \alpha_{j+1})}{B(\alpha_{j+1}, \beta_j - \alpha_{j+1})} \frac{z^n}{n!} \right) dt$$

$$= \prod_{j=1}^{v} \frac{B_{(p,q)}^{\{k_l\}}(\alpha_{j+1} + n\tau, \beta_j - \alpha_{j+1})}{B(\alpha_{j+1}, \beta_j - \alpha_{j+1})} \frac{z^n}{n!} \int_0^\infty t^{s-1} (\alpha_1)_n dt, \tag{4.2}$$

using result $(\alpha_1)_n = \frac{\Gamma(\alpha_1+n)}{\Gamma(\alpha_1)}$ in (4.2), we get

$$\prod_{j=1}^{v} \frac{B_{(p,q)}^{\{k_l\}}(\alpha_{j+1} + n\tau, \beta_j - \alpha_{j+1})}{B(\alpha_{j+1}, \beta_j - \alpha_{j+1})} \frac{z^n}{n!} \int_0^\infty t^{s-1} \frac{\Gamma(\alpha_1 + n)}{\Gamma(\alpha_1)} dt, \tag{4.3}$$

using the result [5, p. 16, eq. (1.110)] given by

$$\int_0^\infty t^{s-1} \Gamma(\alpha_1 + n) dt = \Gamma(\alpha_1 + s + n)\Gamma(s)$$

in (4.3), we get

$$\prod_{j=1}^{v} \frac{B_{(p,q)}^{\{k_l\}}(\alpha_{j+1} + n\tau, \beta_j - \alpha_{j+1})}{B(\alpha_{j+1}, \beta_j - \alpha_{j+1})} \frac{\Gamma(s)(\alpha_1 + s)}{\Gamma(\alpha_1)(\alpha_1 + s)} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1 + s + n) z^n}{n!}$$

$$= \prod_{j=1}^{v} \frac{B_{(p,q)}^{\{k_l\}}(\alpha_{j+1}+n\tau, \beta_j - \alpha_{j+1})}{B(\alpha_{j+1}, \beta_j - \alpha_{j+1})} \Gamma(s)(\alpha_1)_s \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1+s)_n z^n}{n!}$$

$$= \Gamma(s)(\alpha_1)_s \sum_{n=0}^{\infty} (\alpha_1+s)_n \prod_{j=1}^{v} \frac{B_{(p,q)}^{\{k_l\}}(\alpha_{j+1}+n\tau, \beta_j - \alpha_{j+1})}{B(\alpha_{j+1}, \beta_j - \alpha_{j+1})} \frac{z^n}{n!}$$

$$= \Gamma(s)(\alpha_1)_s \{ {}_uR_v^{\{k_l\}}(\alpha_1+s, \alpha_2, .., \alpha_u, \beta_1, \beta_2, .., \beta_v, \tau; z; p, q)\}.$$

Hence the proof is completed.

Theorem 4.3. *The Mellin Barnes type integral representation of the function (3.2) is given by*

$${}_uR_v^{\{k_l\}}(\alpha_1, \alpha_2, .., \alpha_u, \beta_1, \beta_2, .., \beta_v, \tau; z; p, q)$$

$$= \begin{cases} \frac{1}{2\pi\iota} \int_{L_1} \prod_{j=1}^{v} \frac{B_{p,q}^{\{k_l\}}(\alpha_{j+1}-s\tau, \beta_j-\alpha_j)\Gamma(\alpha_1-s)\Gamma(s)(-z)^{-s} ds}{B(\alpha_j, \beta_j-\alpha_j)\Gamma(\alpha_1)}, \\ \qquad (u=v+1, \Re(\beta_j) > \Re(\alpha_{j+1} > 0; j=1,2,.,q; \Re(\alpha_1) > 0) \\ \frac{1}{2\pi\iota} \int_{L_2} \prod_{j=1}^{v} \frac{B_{p,q}^{\{k_l\}}(\alpha_j-s\tau, \beta_j-\alpha_j)\Gamma(s)(-z)^{-s} ds}{B(\alpha_j, \beta_j-\alpha_j)}, \\ \qquad (u=v, \Re(\beta_j) > \Re(\alpha_j > 0; j=1,2,.,q) \\ \frac{1}{2\pi\iota} \int_{L_3} \prod_{i=1}^{r} \frac{\Gamma(\beta_i)}{\Gamma(\beta_i-s\tau)} \prod_{j=1}^{v} \frac{B_{p,q}^{\{k_l\}}(\alpha_j-s\tau, \beta_{j+r}-\alpha_j)\Gamma(s)(-z)^{-s} ds}{B(\alpha_j, \beta_{j+r}-\alpha_j)}, \\ \qquad (r=v-u, u<v, \Re(\beta_{r+j}) > \Re(\alpha_j) > 0; i=1,2,.,r; \Re(\beta_j) > 0) \end{cases} \quad (4.4)$$

where L_i; i=1,2,3 are Mellin Barnes type contours from $-i\infty$ to $+i\infty$ with the usual in-dentations in order to seperate one set of polar from the other set of poles in the integrand.

Proof. The result follows rather directly upon using the Ramanujan's Master theorem and the inversion Mellin transform.

5. Further Results

Definition 5.1. *For suitably constrained (real or complex) parameters α_j , j=1,2,..,u; β_i , i=1,2,..,v, we defined the extended generalized τ-Gauss' hypergeometric functions by*

$${}_uR_v^{\{k_l\}}((\alpha_1, k_1), ...(\alpha_u, k_u), \beta_1, ..., \beta_v, \tau; z; p, q)$$

$$= \begin{cases} \sum_{n=0}^{\infty}(\alpha_1)_{k_1 n}\prod_{j=1}^{v}\frac{B_{(p,q)}^{\{k_l\}}(\alpha_{j+1}+k_{j+1}n\tau,\beta_j-\alpha_{j+1})}{B(\alpha_{j+1},\beta_j-\alpha_{j+1})}\frac{z^n}{n!}, \\ \quad (|z|<1, u=v+1, \Re(\beta_j)>\Re(\alpha_{j+1})>0) \\ \sum_{n=0}^{\infty}\prod_{j=1}^{v}\frac{B_{(p,q)}^{\{k_l\}}(\alpha_j+k_j n\tau,\beta_j-\alpha_j)}{B(\alpha_j,\beta_j-\alpha_j)}\frac{z^n}{n!}, \\ \quad (z\in\mathbb{C}, u=v, \Re(\beta_j)>\Re(\alpha_j)>0) \\ \sum_{n=0}^{\infty}\prod_{i=1}^{r}\frac{1}{(\beta_i)_{n\tau}}\prod_{j=1}^{u}\frac{B_{(p,q)}^{\{k_l\}}(\alpha_j+k_j n\tau,\beta_{j+r}-\alpha_j)}{B(\alpha_j,\beta_{j+r}-\alpha_j)}\frac{z^n}{n!}, \\ \quad (z\in\mathbb{C}, r=v-u, u<v, \Re(\beta_{r+j}>\Re(\alpha_j)>0)) \end{cases} \tag{5.1}$$

where the new parameters $k_1 \in \{0,1\}, k_j, j = 1,2,..,u$ are non-negative integers. Obviously (5.1) reduces to (3.2), whenever $k_j = 1, j = 1,2,..,u$. To illustrate its advantages, we first consider the following function.

$${}_2R_1^{\{k_l\}}[(\alpha_1,k_1),(\alpha_2,k_2),\beta_1,\tau;z;p,q] = \sum_{n=0}^{\infty}(\alpha_1)_n.\frac{B_{p,q}^{\{k_l\}}(\alpha_2+k_2n\tau,\beta_1-\alpha_2)}{B(\alpha_2,\beta_1-\alpha_2)}\cdot\frac{z^n}{n!}. \tag{5.2}$$

Its integral representation can be written as :

$${}_2R_1^{\{k_l\}}[(\alpha_1,k_1),(\alpha_2,k_2),\beta_1,\tau;z;p,q]$$

$$= \frac{1}{B(\alpha_2,\beta_1-\alpha_2)}\int_0^1 t^{\alpha_2-1}(1-t)^{\beta_1-\alpha_2-1}(1-zt^{k_2\tau})\Theta(\{k_l\},-\frac{p}{t}-\frac{q}{1-t})dt. \tag{5.3}$$

$$(R(\beta_1)>R(\alpha_2)>0; min\{R(p),R(q)\}>0; p=q=0, arg|1-z|<\pi)$$

6. Concluding Remarks

In this paper, the authors have first introduced a new extension of τ-Gauss' hypergeometric function and investigated some properties of these extended functions. Motivated mainly by the results of [8], we are working in establishing closed integral expressions for the Mathieu-type a-series and for the associated alternating versions whose terms contain this newly formed extended τ–Gauss' hypergeometric functions with related contiguous functional relations. Also in the light of techniques used by Parmar and Saxena ([7] and [9]), this study can be further extended in the field of the incomplete generalized τ hypergeometric and second τ- Appell functions.

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